PHYSICAL REVIEW C VOLUME 24, NUMBER 2 AUGUST 1981

Classical limit of the interacting boson Hamiltonian

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(Received 5 January 1981)

We present a simple method for the derivation of the classical limit of the most general interacting boson model Hamiltonian as well as the Hamiltonians in the three limits of the interacting boson model. Also the problem of higher order terms is considered.

[NUCLEAR STRUCTURE Classical limit of the interacting boson model.]

In recent years, one has seen the confrontation of two different approaches to describe nuclear collective excitations: on the one hand the geometrical approach, essentially based on the quadrupole degrees of freedom (for instance, Refs. 1 – 3) which are extensions of the Bohr-Mottelson vibrational Hamiltonian, and on the other hand the algebraic approach (for instance, Refs. 4 – 8), which exploits symmetry by using group theoretical methods. In this paper, we will discuss the relation between the interacting boson model (IBM) of Arima and Iachello[6–8] and the geometrical models.

To start with, we want to point out a conceptual difference between the IBM and the geometrical models, which is important for the subsequent discussion. The problem of nuclear collective motion is formulated by Bohr and Mottelson *ab initio* in terms of shape variables (for instance, in the case of quadrupole variables, the three Euler angles θ_i, and the intrinsic deformation parameters β and γ). On the other hand, in the IBM one immediately starts from a Hamiltonian which is written in second quantized form in terms of $s^\dagger$ and $d^\dagger$ operators.

Although one might doubt the eligibility of using shape variables in quantum problems with a relative small number of particles, the concept of shape has been extremely fruitful in obtaining at the same time an intuitive and quantitative understanding of nuclear collective motion. For this reason, it is important to bridge the gap between the IBM and other nuclear collective models, which are formulated in terms of shape variables. This has been done in a number of recent papers.[9–12]

Based on the coherent state formalism and making use of results and an algorithm developed by Gilmore and Feng,[13–15] a technique for going from an IBM Hamiltonian to a potential energy surface in the variables β and γ, has been outlined in Ref. 9. Within this coherent state formalism, one can obtain for a system with a finite number of particles N an upper bound E_+ and a lower bound E_- for the expectation value of the Hamiltonian (in general, any operator belonging to a compact Lie algebra). Furthermore, one can show that, in the limit $N \rightarrow \infty$ (classical limit), the upper and lower bounds will coincide, both giving the exact result. Applying this method to the IBM, one can derive for each SU(6) Hamiltonian a potential energy surface $E(\beta,\gamma)$, by calculating the upper bound of the ground-state expectation value of this Hamiltonian. In this paper, we calculate the upper bound which is simply given by the expectation value of the Hamiltonian in the coherent state. The lower bound has a more complicated structure.

In this way, Dieperink *et al.* showed[9] that each of the three limiting cases of the IBM corresponds to a certain shape phase and they also studied the nature of the shape phase transition in the region between the three limits. However, in constructing the classical limit they used simplified versions of the three limiting Hamiltonians originally employed by Arima and Iachello.[2–4] In a recent paper of

Ginocchio and Kirson,[16] the problem of finding the classical limit of the most general IBM Hamiltonian was solved. In this paper we present an alternative simple method for calculating the expectation value of the IBM Hamiltonian in the coherent state. With this method we are able to reproduce the results of Ref. 16, but due to its simplicity, we can also consider the classical limit of cubic terms in the IBM Hamiltonian and its relation to triaxiality.

First we will give some general results which can be derived easily. It is well known that, $b_i^\dagger(b_i)$ being boson creation (annihilation) operators and $f(b)$ being a polynomial of $b_1^\dagger, b_2^\dagger, \ldots$ and $b_1, b_2, \ldots$, the boson commutators can be evaluated by means of differentiation, i.e.,

$$[b_i, f(b)] = \frac{\partial}{\partial b_i^\dagger} f(b) \ , \tag{1a}$$

$$[f(b), b_i^\dagger] = \frac{\partial}{\partial b_i} f(b) \ . \tag{1b}$$

Suppose the operator B is a linear combination of various kinds of bosons

$$B = \sum_i \alpha_i b_i \ , \tag{2}$$

and let 1 be the identity operator, A_1 a general one-body operator

$$A_1 = \sum_{ij} \xi_{ij} b_i^\dagger b_j \ , \tag{3a}$$

and A_2 a general two-body operator

$$A_2 = \sum_{ijkl} n_{ijkl} b_i^\dagger b_j^\dagger b_k b_l \ . \tag{3b}$$

With the help of Eqs. (1), the matrix elements of the operators 1, A_1, and A_2 between the n-boson states $|B^{\dagger n}\rangle \equiv (B^\dagger)^n|0\rangle$ can be written down directly:

$$\text{(i)} N_n \equiv \langle B^n|1|B^{\dagger n}\rangle = \left\langle B^{n-1} \left| \sum_i \alpha_i \frac{\partial}{\partial b_i^\dagger} \right| B^{\dagger n} \right\rangle$$

$$= n\alpha^2 N_{n-1} = n!\alpha^{2n} \ , \tag{4a}$$

where we used the abbreviation $\alpha^2 = \sum_i \alpha_i^2$;

$$\text{(ii)} \langle B^n|A_1|B^{\dagger n}\rangle = n^2 N_{n-1} \langle B|A_1|B^\dagger\rangle$$

or

$$\langle A_1\rangle \equiv N_n^{-1}\langle B^n|A_1|B^{\dagger n}\rangle = \frac{n}{\alpha^2}\sum_{ij}\alpha_i\alpha_j\xi_{ij} \ ; \tag{4b}$$

$$\text{(iii)} \langle B^n|A_2|B^{\dagger n}\rangle = \left[\frac{n(n-1)}{2}\right]^2 N_{n-2} \times \langle B^2|A_2|B^{\dagger 2}\rangle$$

or

$$\langle A_2\rangle \equiv N_n^{-1}\langle B^n|A_2|B^{\dagger n}\rangle = \frac{n(n-1)}{\alpha^4}\sum_{ijkl}\alpha_i\alpha_j\alpha_k\alpha_l n_{ijkl} \ . \tag{4c}$$

These results can be extended easily to a general m-body operator.

Applying this to the IBM, where we use as boson operators $s^\dagger, d_{-2}^\dagger, \ldots, d_2^\dagger$, we define

$$B^\dagger = s^\dagger + \sum_m \beta_m d_m^\dagger \ . \tag{5}$$

In the intrinsic frame we have

$$\beta_0 = \beta\cos\gamma, \ \beta_{-2} = \beta_2 = \frac{1}{\sqrt{2}}\beta\sin\gamma, \ \beta_{\pm 1} = 0 \ . \tag{6}$$

For the most general IBM Hamiltonian

$$H = \epsilon_s n_s + \epsilon_d n_d + \sum_l \tfrac{1}{2} c_l (2l+1)^{1/2} [(d^\dagger d^\dagger)^{(l)} (\tilde{d}\,\tilde{d})^{(l)}]^{(0)} + \frac{1}{\sqrt{2}}\tilde{v}_2 [(d^\dagger d^\dagger)^{(2)}(\tilde{d}s)^{(2)} + (s^\dagger d^\dagger)^{(2)}(\tilde{d}\tilde{d})^{(2)}]^{(0)}$$
$$+ \tfrac{1}{2}\tilde{v}_0[(d^\dagger d^\dagger)^{(0)}(ss)^{(0)} + (s^\dagger s^\dagger)^{(0)}(\tilde{d}\tilde{d})^{(0)}]^{(0)} + u_2[(d^\dagger s^\dagger)^{(2)}(\tilde{d}s)^{(2)}]^{(0)} + \tfrac{1}{2}u_0[(s^\dagger s^\dagger)(ss)]^{(0)} \ , \tag{7}$$

we obtain the following classical limit

$$H = \frac{n}{1+\beta^2}(\epsilon_s + \epsilon_d\beta^2) + \frac{n(n-1)}{(1+\beta^2)^2} \times (a_1\beta^4 + a_2\beta^3\cos 3\gamma + a_3\beta^2 + \tfrac{1}{2}u_0) \ , \tag{8a}$$

with

$$a_1 = \tfrac{1}{10}c_0 + \tfrac{1}{7}c_2 + \tfrac{9}{35}c_4 \ ,$$
$$a_2 = -2\left[\frac{1}{35}\right]^{1/2}\tilde{v}_2 \ , \tag{8b}$$
$$a_3 = \frac{1}{\sqrt{5}}(\tilde{v}_0 + u_2) \ .$$

The classical limit of the IBM Hamiltonian in the three different limits can be found by inserting the appropriate values for the constants in Eq. (8). The Hamiltonians and their corresponding classical limit are given by

(I) $SU(6) \supset SU(5) \supset O(5) \supset O(3)$

$$H^{(\mathrm{I})} = \epsilon_d n_d + \sum_l \tfrac{1}{2}(2l+1)^{1/2} \times c_l[(d^\dagger d^\dagger)^{(l)}(\tilde{d}\tilde{d})^{(l)}]^{(0)} \ . \tag{9a}$$

$$E^{(\mathrm{I})}(\beta) \equiv \langle H^{(\mathrm{I})} \rangle = \epsilon_d \frac{n\beta^2}{1+\beta^2} + a_1 n(n-1)\frac{\beta^4}{(1+\beta^2)^2} \ , \tag{9b}$$

with a_1 given by Eq. (8b).

(II) $SU(6) \supset SU(3) \supset O(3)$

$$H^{(\mathrm{II})} = -\kappa_1 Q \cdot Q - \kappa_2 L \cdot L \ . \tag{10a}$$

$$E^{(\mathrm{II})}(\beta,\gamma) = -\kappa_1 \left[\frac{n}{1+\beta^2}\left[5 + \frac{11}{4}\beta^2\right] + \frac{n(n-1)}{(1+\beta^2)^2} \times \left[\frac{\beta^4}{2} + 2\sqrt{2}\beta^3 \cos 3\gamma + 4\beta^2\right]\right] - \kappa_2 \frac{6n\beta^2}{1+\beta^2} \ . \tag{10b}$$

(III) $SU(6) \supset O(6) \supset O(5) \supset O(3)$

$$H^{(\mathrm{III})} = AS_+S_- + BC_5 + CL \cdot L \ , \tag{11a}$$

with

$$S_+ = \tfrac{1}{2}\sum_m (-1)^m d_m^\dagger d_{-m}^\dagger - \tfrac{1}{2}s^\dagger s^\dagger \ ,$$

$$C_5 = -\frac{\sqrt{7}}{3}[(d^\dagger \tilde{d})^{(3)}(d^\dagger \tilde{d})^{(3)}]^{(0)} - \frac{\sqrt{3}}{3}[(d^\dagger \tilde{d})^{(1)}(d^\dagger \tilde{d})^{(1)}]^{(0)} \ . \tag{11b}$$

$$E^{(\mathrm{III})}(\beta) = \kappa_3 \frac{n\beta^2}{1+\beta^2} + \kappa_4 n(n-1)\left[\frac{1-\beta^2}{1+\beta^2}\right]^2 \ ,$$

with $\kappa_3 = \frac{2}{3}B + 6c$ and $\kappa_4 = A/4$.

From Eqs. (9b), (10b), and (11b) we know that there are two independent parameters for each limiting case and also that $E^{(\mathrm{I})}$ and $E^{(\mathrm{III})}$ are γ independent. In Fig. 1 we give a typical example of the potential energy surface in each of the three limits. For the limiting case (I) the equilibrium shape of the nucleus is always spherical ($\beta = 0$). For the limiting cases (II) and (III), there are two competing factors: one favors spherical form and the other favors deformation. We will give an illustration of this phenomenon in both the limiting cases (II) and (III).

In looking for the equilibrium shape in the O(6) limit we have to minimize $E^{(\mathrm{III})}(\beta)$ of Eq. (11b) with respect to β. Analysis shows that a minimum occurs at $\beta = 0$ when $\kappa_3 > 4(n-1)\kappa_4$ and at $\beta = \{[4\kappa_4(n-1) - \kappa_3]/[4\kappa_4(n-1) + \kappa_3]\}^{1/2}$ when $\kappa_3 < 4(n-1)\kappa_4$. For instance, for the nucleus $^{196}Pt_{118}$, the empirical parameters are (Ref. 8)

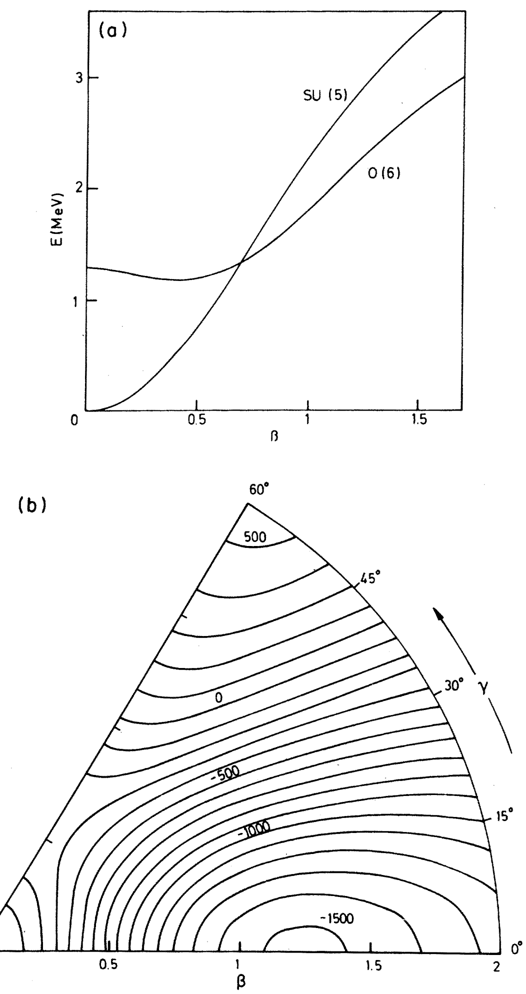

FIG. 1. (a). Energy versus deformation plot in the SU(5) and O(6) limit for the nuclei $^{102}Ru_{58}$ and $^{196}Pt_{118}$, respectively, according to Eqs. (9b) and (11b). The parameters are taken from Refs. 6 and 8. (b) Potential energy surface for the SU(3) nucleus $^{156}Gd_{92}$, according to Eq. (10b). The contour lines are marked in keV. The parameters are taken from Ref. 7.

$A = 171$ keV, $B = 300$ keV, and $C = 10$ keV. Thus, $\kappa_3 = 260$ keV and $\kappa_4 \simeq 43$ keV, which shows that the equilibrium shape of ^{196}Pt is slightly deformed, $\beta = 0.7319$, instead of $\beta = 1$ quoted in Ref. 9, where the first term in Eq. (11b) is ignored. The deformation parameter β_{IBM}, which we obtain here, is connected with the deformation parameter of Bohr and Mottelson β_{BM} by the approximate relation, derived in Ref. 16, $\beta_{\mathrm{BM}} \simeq 1.18\,(2N/A)\beta_{\mathrm{IBM}}$, where A is the number of nucleons and N the number of valence bosons.

In the SU(3) limit, the equilibrum shape can be found by minimizing $E^{(\mathrm{II})}(\beta,\gamma)$ of Eq. (10b) with respect to β and γ. We remark that $E^{(\mathrm{II})}(\beta,\gamma)$ is invariant under the substitutions $(\beta,\gamma) \to (\beta,-\gamma) \to (\beta,\gamma + 120°)$, and $(\beta,\gamma) \to (-\beta,\gamma + 60°)$ (as should be) and consequently, we may restrict ourselves to the region $\beta > 0$ and $0° \le \gamma \le 60°$. From Eq. (10b) it can be seen that for realistic (positive) values of κ_1, $E^{(\mathrm{II})}(\beta,\gamma)$ reaches a minimum for $\gamma = 0°$. Furthermore, it can be shown that the extremum values of $E^{(\mathrm{II})}(\beta,\gamma = 0°)$ with respect to β, are given by $\beta = 0$ and the roots of the equation

$$\beta^3 - D_n\beta^2 - 3\beta - D_n - \frac{7}{\sqrt{2}} = 0 \ , \qquad (12)$$

with

$$D_n = \frac{(6\kappa_2/\kappa_1 - 3n + 3/4)}{(n-1)\sqrt{2}} \ .$$

Again we illustrate this in the case of the nucleus $^{156}\mathrm{Gd}_{92}$, where $\kappa_1 = 7.6$ keV and $\kappa_2 = -7.7$ keV.[7] With these values, we obtain from Eq. (12) three minima for β, two of which are negative and must be discarded. The third minimum at $\beta = 1.2426$ [see Fig. 1(b)] is only slightly different from the value $\beta = \sqrt{2}$, quoted in Ref. 9.

Since the classical limit of the most general IBM Hamiltonian depends only on γ via a term in $\cos 3\gamma$, it is clear that, even for this general potential energy surface, a minimum with respect to γ can only occur for $\gamma = 0°$ or $\gamma = 60°$. In other words, the equilibrium shape of the classical limit of a general IBM Hamiltonian can never be triaxial. Only the inclusion of higher order terms in the IBM Hamiltonian, can lead to triaxial equilibrium shapes. We will illustrate this by considering cubic terms with three creation and three annihilation operators of d bosons. Such cubic terms can be written in general as

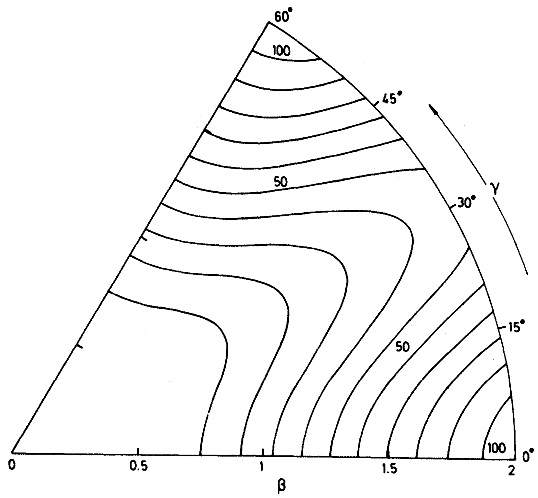

FIG. 2. Potential energy surface of the cubic term $[[d^\dagger d^\dagger]^{(2)} d^\dagger]^{(4)} \cdot [[\tilde{d}\tilde{d}]^{(2)}\tilde{d}]^{(4)}$ according to Eq. (15). The energy units are arbitrary.

$$d(l,k,r) \equiv [[d^\dagger d^\dagger]^{(l)} d^\dagger]^{(r)} \cdot [[\tilde{d}\tilde{d}]^{(k)}\tilde{d}]^{(r)} \ . \qquad (13)$$

There are five linear independent combinations of type (13), which are determined uniquely by the value of $r\,(=0,2,3,4,6)$. In particular, one can always choose $l = k$. Other cubic terms, which differ from (13) by the order of the operators $d^\dagger$ and $\tilde{d}$, can be put into relation with the $d(l,k,r)$ and/or lower order terms. Using the differentiation method, it is easy to show that

TABLE I. The coefficients A and B of Eq. (15) in the expression of the classical limit of the cubic term $[[d^\dagger d^\dagger]^{(l)} d^\dagger]^{(r)} \cdot [[\tilde{d}\tilde{d}]^{(k)}\tilde{d}]^{(r)}$.

r	0	2	3	4	6
A	0	$\frac{1}{5}$	$-\frac{1}{7}$	$\frac{3}{49}$	$\frac{14}{55}$
B	$\frac{2}{35}$	0	$\frac{1}{7}$	$\frac{3}{35}$	$-\frac{8}{385}$

$$\langle d(l,k,r)\rangle = [n(n-1)(n-2)]^2 N_{n-3} N_n{}^{-1} \times \left[\sum_{\mu_i,\nu_i} \langle 2\mu_1 2\mu_2 \,|\, l\mu_1+\mu_2\rangle \langle l\mu_1+\mu_2 2\mu_3 \,|\, r\mu_1+\mu_2+\mu_3\rangle \langle 2\nu_1 2\nu_2 \,|\, k\nu_1+\nu_2\rangle \times \langle k\nu_1+\nu_2 2\nu_3 \,|\, r\nu_1+\nu_2+\nu_3\rangle (-1)^r \beta_{\mu_1}\beta_{\mu_2}\beta_{\mu_3}\beta_{\nu_1}\beta_{\nu_2}\beta_{\nu_3}\delta\left[\sum_i \mu_i, \sum_i \nu_i\right] \right] \quad . \tag{14}$$

For $l=k$, the expression between the curly brackets can be written in a simpler form as

$$\sum_{\mu_{123}} \left[\sum_{\mu_i}{}' \langle 2\mu_1 2\mu_2 \,|\, l\mu_1+\mu_2\rangle \times \langle l\mu_1+\mu_2 2\mu_3 \,|\, r\mu_{123}\rangle \beta_{\mu_1}\beta_{\mu_2}\beta_{\mu_3} \right]^2 ,$$

where the prime in the summation symbol indicates that the sum over μ_1, μ_2, and μ_3 is carried out under fixed μ_{123}. Using Eq. (6), we obtain the final result

$$\langle d(l,k,r)\rangle = n(n-1)(n-2)\frac{\beta^6}{(1+\beta^2)^3} \times (A + B\cos^2 3\gamma) \quad , \tag{15}$$

with the coefficients A and B listed in Table I. In Fig. 2 we illustrate this with a diagram of the potential energy surface in a particular case of Eq. (15).

The extreme simplicity of the method outlined here, can perhaps be best illustrated by exploring the relation between higher order terms and triaxiality in its full generality. Let us consider a general m-tic term (i.e., a term with m creation and m annihilation operators) with $k(\leq 2m)$ creation or annihilation d operators. The classical limit of such a term can immediately be written in the following form

$$m!\begin{bmatrix} n \\ m \end{bmatrix} \frac{\beta^k}{(1+\beta^2)^m} \sum_{i=0}^{k} a_i \sin^i\gamma \cos^{k-i}\gamma \quad , \tag{16}$$

where the coefficients a_i depend on the coupling sequence used for the $d^\dagger$ and $\tilde{d}$ operators. Symmetry arguments require that this potential energy surface only depends on $\cos 3\gamma = \cos^3\gamma - 3\cos\gamma\sin^2\gamma$, and this severely restricts the occurrence of triaxial minima. For instance, for $k \leq 5$ no term in $\cos^2 3\gamma$ can be formed from (16). Thus, the only cubic terms ($m=3$) which can give rise to triaxiality, must have $k=6$, precisely the $d(l,k,r)$ of Eq. (13).

In conclusion, we have given a clear and simple prescription for calculating the classical limit in the IBM. We have derived the classical limit of the most general IBM Hamiltonian as well as the classical limit of the Hamiltonian in the SU(5), SU(3), and O(6) limit, respectively. Also, the problem of the inclusion of higher order terms and its influence on the nuclear shape, has been considered. The fact that the low-energy spectrum can be fairly well accounted for by the interacting boson Hamiltonian (without higher order terms), implies that the triaxiality of nuclei in the ground state, if there is any, is rather small.

We wish to thank Professor F. Iachello and Dr. K. Heyde for stimulating our interest in this subject and for interesting discussions. One of the authors (PVI) acknowledges the Nationaal Fonds voor Wetenschappelijk Onderzoek (NFWO) for financial support.

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