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Faddeev wave function decomposition using bipolar harmonics

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The standard partial wave (channel) representation for the Faddeev solution to the Schrödinger equation for the ground state of 3 nucleons is written in terms of functions which couple the interacting pair and spectator angular momenta to give S, P, and D waves. For each such coupling there are three terms, one for each of the three cyclic permutations of the nucleon coordinates. A series of spherical harmonic identities is developed which allows writing the Faddeev solution in terms of a basis set of 5 bipolar harmonics: 1 for S waves; 1 for P waves; and 3 for D waves. The choice of a D-wave basis is largely arbitrary, and specific choices correspond to the decomposition schemes of Derrick and Blatt, Sachs, Gibson and Schiff, and Bolsterli and Jezak. The bipolar harmonic form greatly simplifies applications which utilize the wave function, and we specifically discuss the isoscalar charge (or mass) density and the ^{3}He Coulomb energy.

[NUCLEAR STRUCTURE Three-body problem, classification of states.]

I. INTRODUCTION

Before the widespread use of Faddeev[1] and variational[2] techniques for solving the Schrödinger equation numerically, the phenomenological approach was used. This technique utilizes symmetry principles, including permutation symmetry, to write the most general form for the trinucleon ground state wave function in terms of (irreducible) spin-isospin wave functions (ϕ_i) with total isospin $\frac{1}{2}$ and total (intrinsic) spin $\frac{1}{2}$ and $\frac{3}{2}$. The latter functions are coupled to 5 orbital angular momentum functions (1 S wave, 1 P wave, and 3 D wave) to determine the relative orientation of the intrinsic spin and orbital angular momentum of the composite system. The combination of these isospin-spin-orbital angular momentum functions having total spin and isospin $\frac{1}{2}$ with 16 scalar functions of the appropriate permutation symmetry[3] represents the most general form for the nonrelativistic ground state wave function of ^{3}He and ^{3}H (neglecting the effects of charge-symmetry-breaking interactions).

The phenomenological studies concentrated on parametrizing these 16 scalar functions, which are the 16 dependent variables in the 3-nucleon Schrödinger equation with 3 independent variables. These are listed at the top of Table I, with the spectroscopic notation (total intrinsic spin and orbital

character) along the upper row, and permutation symmetry type along the side: symmetric (S), mixed (M), and antisymmetric (A). There are 3 distinct pairs of mixed symmetry D state scalar functions (v_3,v_4). The specific decomposition is known to be nonunique[3–6] and will be discussed in more detail later. The six corresponding spin-isospin wave functions (ϕ's) are categorized in the middle section. There are 4 doublet (spin $\frac{1}{2}$) functions and 2 quartet (spin $\frac{3}{2}$) functions.

The number of states which can be formed from the 6 ϕ's and 16 u's and v's is 10, according to Derrick and Blatt.[3] There are 3 ^2S states, 3 ^2P states, 1 ^4P state, and 3 ^4D states, which are categorized at the bottom of Table I (the D state in brackets symbolizes 3 distinct components). The symmetry type listed there refers to the overall internal-orbital character. For the P states, the unique P-wave orbital function H, to be defined later, has been introduced. This is coupled in the usual way with the intrinsic spin to a total spin of $\frac{1}{2}$, as are the D-state functions v_3 and v_4. From the top down, the S-wave components of Ψ are denoted by S, S', and S'', which refers to the *internal* (i.e., scalar) function. We correspondingly denote the ^2P states by ^2P, $^2P'$, and $^2P''$, according to their internal character; the single ^4P state is unique. The function H can be defined to be antisymmetric under the interchange of any two nucleon labels. Thus, $u_{P''}$ is antisymmetric, while u_P is symmetric. One might expect the probability $P(^2P'')$ to be small; this will be seen to be true. In the earliest classification scheme by Sachs,[6] the S'' and $^2P''$ states were missing because antisymmetric internal (scalar) components $u_{S''}$ and $u_{P''}$ were not considered. In most applications the ^4P states are left out, as are the S'' states, because these states are believed to be very small in probability. A majority of calculations have used either 1 (S only), 2 (S and S'), or 5 (S,S' and the 3 D states) of the 10 components.

The Faddeev method traditionally has simplified the complete problem of solving 16 coupled differential equations in 3 independent variables by decomposing the nucleon-nucleon interaction into partial waves. The variables which are traditionally used are the Jacobi coordinates ($\vec{x}_1$,$\vec{y}_1$). The lengths x_1 and y_1 in addition to $\mu_1 = \cos\theta_1$ are the three scalar variables shown in Fig. 1. Since the three nucleons form a plane, all vectors between nucleons can be expressed in terms of $\vec{x}_1$ and $\vec{y}_1$, which we will denote by $\vec{x}$ and $\vec{y}$, henceforth. In particular, counterclockwise rotations of the nucleon labels produce successively ($\vec{x}_2$,$\vec{y}_2$) and ($\vec{x}_3$,$\vec{y}_3$), which can be expressed in terms of $\vec{x}$ and $\vec{y}$. The basic structure of the Schrödinger wave function as it is expressed in terms of the Faddeev amplitudes is illustrated by the following schematic example. If we work in L-S coupling, the orbital part of a partial wave component of the Faddeev amplitude with the form

$$\Psi^{l_1 l_2 LM}(\vec{x},\vec{y}) = \psi^{l_1 l_2 L}(x,y)[Y_{l_1}(\hat{x}) \times Y_{l_2}(\hat{y})]_{LM} , \tag{1}$$

may generate the Schrödinger wave function component given by

TABLE I. The 16 scalar trinucleon wave function components (v_3 and v_4 comprise 3 each) are categorized at the top, the 6 spin-isospin wave functions in the middle, and the 10 Schrödinger wave function components (3 from the D states) at the bottom. The column headings give the spectroscopic classification and the row labels give the permutation symmetry classification. For the Schrödinger wave functions the symmetry refers to orbital-internal.

	^2S	^2P	^4P	^4D
(S)	u	u_P		
(M)	v_1, v_2	v_1^P, v_2^P	v_3^P, v_4^P	v_3, v_4
(A)	$u_{S''}$	$u_{P''}$		
(A)	ϕ_a	ϕ_a		
(M)	ϕ_1, ϕ_2	ϕ_1, ϕ_2	ϕ_3, ϕ_4	ϕ_3, ϕ_4
(S)	ϕ_s	ϕ_s		
(S)	$u\phi_a$	$u_{P''}(H \times \phi_a)$		
(M)	$[v_2\phi_1 - v_1\phi_2]$	$[v_2^P(H \times \phi_1) - v_1^P(H \times \phi_2)]$	$[v_4^P(H \times \phi_3) - v_3^P(H \times \phi_4)]$	$[v_4 \times \phi_3 - v_3 \times \phi_4]$
(A)	$u_{S''}\phi_s$	$u_P(H \times \phi_s)$		

$$\Psi = \psi^{l_1 l_2 L}(x,y)[Y_{l_1}(\hat{x}) \times Y_{l_2}(\hat{y})]_{LM} + \psi^{l_1 l_2 L}(x_2,y_2)[Y_{l_1}(\hat{x}_2) \times Y_{l_2}(\hat{y}_2)]_{LM} + \psi^{l_1 l_2 L}(x_3,y_3)[Y_{l_1}(\hat{x}_3) \times Y_{l_2}(\hat{y}_3)]_{LM} \ . \quad (2)$$

That is, the 3 sets of variables we discussed earlier are inserted successively in the same function and added together. The second and third terms will be denoted "permuted terms." The angular factors are bipolar harmonics,[7] a coupled pair of spherical harmonics obtained by coupling together an interacting pair angular momentum l_1 with spectator angular momentum l_2, to form $L = 0,1,2$ (S, P, and D waves) for the trinucleon ground state. These factors have different arguments and in general have no obvious relationships to the 5 angular factors discussed earlier and used in Refs. 3–6.

Since the ground state parity is positive and must equal $(-1)^{l_1+l_2}$, l_1 and l_2 must both be even or odd. Since the maximum intrinsic spin is $\frac{3}{2}$, which limits L to 0,1,2, it is clear that l_1 and l_2 must differ by $0,\pm 2$. In addition, under the interchange of particle labels 2 and 3, one finds $\vec{x} \rightarrow -\vec{x}$ and $\vec{y} \rightarrow \vec{y}$; thus Eq. (1) becomes multiplied by $(-l)^{l_1}$. Thus, the wave function components for which l_1 and l_2 are even can contribute to components whose overall space symmetry (internal-orbital) is symmetric or mixed symmetric in nature, while odd values of l_1 and l_2 can contribute to antisymmetric or mixed symmetric space components. Thus, even partial waves can contribute to the first two rows at the bottom of Table I, while the odd waves contribute to the bottom two rows. Since the p-f partial waves are not utilized in "standard" Faddeev calculations (i.e., 5 channels or less), the S'' and ^2P states are usually not encountered. However, the $^2P''$ state, overlooked by Sachs,[6] is encountered in 5 channel calculations.[1]

Our task in this paper will be to determine the relationships between the angular factors in the permuted terms and those in the Faddeev amplitude, Eq. (1), and the relationships between the angular factors in the Faddeev amplitude and a set of 5 standard orbital functions. This is done in Sec. II, and will allow us to generate the u's and v's as a series in the ψ's and their permuted terms. We thus provide a mapping between the Faddeev and Derrick-Blatt approaches and several examples of this are developed in Sec. III. This decomposition greatly simplifies the algebra involved in calculating matrix elements, allows us to identify wave function components which are small, greatly facilitates plotting the various parts of the wave function, and is the primary motivation for this work. Expressions for magnetic moments,[8] and charge densities,[9,10] for example, can be written entirely in terms of the u's and v's and simple geometrical factors. We will illustrate this in Sec. III by considering the isoscalar charge density (mass density) and the Coulomb energy of ^{3}He calculated using a partial wave projected Coulomb potential.[11] Numerical results are also presented. Although our technique clearly is applicable to both configuration space[9] and momentum space[1] Faddeev calculations, our approach emphasizes the former. Previous Faddeev calculations[12] have utilized the Derrick-Blatt classification scheme, but not in the manner we present next.

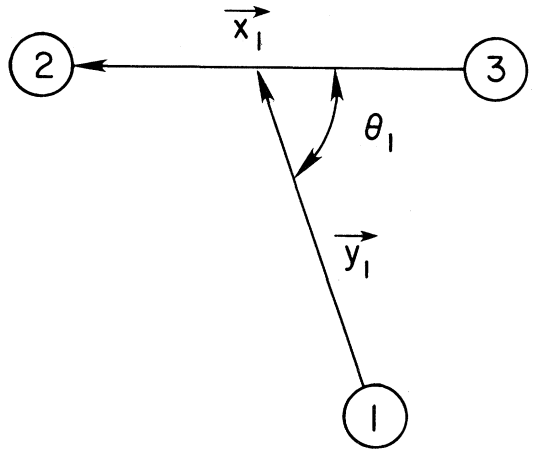

FIG. 1. Jacobi coordinates for the three-body problem.

II. ANGULAR MOMENTUM DECOMPOSITION

In order to be specific, we must choose the 5 angular factors which specify the orbital angular momentum's orientation with respect to the intrinsic spin. One choice corresponds to the Euler angle treatment of Derrick and Blatt[3]; others[4–6] have followed a different course, which we adopt.

The S waves are trivial since the angular factors are constant in space; the orbital wave function is denoted $G \equiv 1$. There is a single P-wave orbital function $\vec{x} \times \vec{y}$ which is unique except for an overall multiplicative factor which can depend on x, y, and μ. We write this in the form

$$H = 4\pi[Y_1(\hat{x}) \times Y_1(\hat{y})]_1 xy \ , \quad (3)$$

where all z components of tensors will be suppressed. Our choice of the three D-state components is

$$I = \sqrt{4\pi} Y_2(\hat{x}) , \tag{4a}$$

$$J = \sqrt{4\pi} Y_2(\hat{y}) , \tag{4b}$$

$$K = 4\pi [Y_1(\hat{x}) \times Y_1(\hat{y})]_2 (\tfrac{10}{3})^{1/2} . \tag{4c}$$

Any (complete) linear combination of these would also suffice.[13]

Different sets of orbital basis functions are possible, which differ only by changing the factors in front of H through K. Our choice for H was motivated by symmetry considerations ($\vec{x} \times \vec{y}$ is antisymmetric, while $\hat{x} \times \hat{y}$ is not), while simplicity motivated the choice of I, J, and K. It will be seen later that $x^2 I$, $y^2 J$, and xyK are also natural choices of these functions.

We will divide the problem into two parts: (a) relate the partial wave components of ψ illustrated in Eq. (1) to G, H, I, J, and K; (b) relate the permuted angular functions G_2 through K_2 and G_3 through K_3 (i.e., written with permuted variables $\hat{x}_2$, $\hat{x}_3$, etc.) in terms of G through K. We will not attempt the most general case for l_1 and l_2, but rather restrict ourselves to $(l_1,l_2) = (0,0)$, $(0,2)$, $(2,0)$, $(2,2)$, $(1,1)$, $(1,3)$, $(3,1)$, and $(3,3)$, which subsumes all previous Faddeev calculations of 18 channels or less. Two methods are effective. One may use the fact that the various functions $G - K$ are complete (but not orthogonal) and project, or one may specialize the coordinate system so that $\hat{y}$ lies along the z direction and all components of $\hat{x}$ are measured with respect to $\hat{y}$. Only $Y_{l_2 0}(\hat{y})$ is nonvanishing in this case.

As an example of this procedure we illustrate the structure of the D-wave component which arises from the fifth channel in standard 5-channel calculations ($l_1 = l_2 = L = 2$). The factor $[Y_2(\hat{x}) \times Y_2(\hat{y})]_{2M}$ is obviously symmetric under the interchange of $\hat{x}$ and $\hat{y}$ and can be written as $a(I + J) + bK$, where a and b depend only on μ. Choosing $\hat{y}$ along the z direction, evaluating the single term in the coupling sum which remains for $M = 0$, 1, 2 successively, and comparing to I, J, and K in the same limit produces

$$4\pi [Y_2(\hat{x}) \times Y_2(\hat{y})]_2 = (\tfrac{10}{7})^{1/2} (I + J - 3\mu K/2) . \tag{5}$$

The complete decomposition is given in Table II. The structure is obvious. The $Y_l(\hat{x})$'s are irreducible tensors composed of l $\hat{x}$ vectors. When contracted with its corresponding element containing $\hat{y}$'s, there will in general be too many vectors left over for the irreducible tensor which remains; these must be contracted to form $\hat{x}^2 = 1$, $\hat{y}^2 = 1$, and $\hat{x} \cdot \hat{y} = \mu$. This is easily seen in Eq. (5), where two $\hat{x}$'s must be contracted for the J terms, two $\hat{y}$'s for the I terms, and one each of $\hat{x}$ and $\hat{y}$ for the K terms, which results in the μ.

The only remaining task is to evaluate the effect of permuting the coordinates. This is easily determined using the generalized addition theorem[7]:

$$G_1 = G_2 = G_3 , \tag{6}$$

$$H_1 = H_2 = H_3 , \tag{7}$$

$$I_2 = x^2 I/4x_2^2 + y^2 J/x_2^2 - xyK/2x_2^2 , \tag{8a}$$

$$I_3 = x^2 I/4x_3^2 + y^2 J/x_3^2 + xyK/2x_3^2 , \tag{8b}$$

$$J_2 = 9x^2 I/16y_2^2 + y^2 J/4y_2^2 + 3xyK/8y_2^2 , \tag{8c}$$

$$J_3 = 9x^2 I/16y_3^2 + y^2 J/4y_3^2 - 3xyK/8y_3^2 , \tag{8d}$$

TABLE II. Decomposition of $[Y_{l_1}(\hat{x}) \times Y_{l_2}(\hat{y})]_L$, denoted (l_1,l_2), in terms of G, H, I, J, K, and $\mu = \hat{x} \cdot \hat{y}$.

Faddeev channel	$L = 0$	$L = 1$	$L = 2$
(0,0)	$G(\equiv 1)$		
(0,2)			J
(2,0)			I
(2,2)	$\sqrt{5}(3\mu^2 - 1)/2$	$-\sqrt{5}\mu H/xy$	$(\frac{10}{7})^{1/2}(I + J - 3\mu K/2)$
(1,1)	$-\sqrt{3}\mu$	H/xy	$(\frac{3}{10})^{1/2} K$
(1,3)			$-\sqrt{5}\mu J + K/\sqrt{5}$
(3,1)			$-\sqrt{5}\mu I + K/\sqrt{5}$
(3,3)	$-\sqrt{7}(5\mu^3 - 3\mu)/2$	$-\sqrt{14}(1 - 5\mu^2)H/4xy$	$-\sqrt{35/3}\,(I + J - (25\mu^2 - 1)K/20)$

$$K_2 = -xyK/2x_2y_2 - y^2J/x_2y_2 + 3x^2I/4x_2y_2 , \quad (8e)$$

$$K_3 = -xyK/2x_3y_3 + y^2J/x_3y_3 - 3x^2I/4x_3y_3 . \quad (8f)$$

These results can be combined to form all the wave function components that arise for $l_1, l_2 \leq 3$.

III. RESULTS AND DISCUSSION

As an illustration of the preceding analysis we give three examples from a 5-channel Faddeev calculation (in j-j coupling): the functions u, $u_{P''}$, and (v_3, v_4). The first and second Faddeev channels (amplitudes) are s wave, spin singlet and triplet, respectively. Channels 3 and 4 are spin triplet, and couple a d wave in the interacting pair with an s wave spectator and *vice versa,* respectively. The fifth channel couples a d-wave interacting pair with a D-wave spectator to generate S-, P-, and D-state functions. Thus channels 1, 2, and 5 contribute to the dominant S state u; channels 3, 4, and 5 to the d-state functions v_3 and v_4; only channel 5 contributes to $u_{P''}$. Denoting by $\psi_n(i)$ the Faddeev amplitude of channel n as a function of variables $(\vec{x}_i, \vec{y}_i)$ we find

$$u(x,y,\mu) = \psi^S(1) + \psi^S(2) + \psi^S(3) , \quad (9a)$$

$$u_P''(x,y,\mu) = \psi^P(1) + \psi^P(2) + \psi^P(3) , \quad (9b)$$

where

$$\psi^S(x,y,\mu) = \frac{1}{\sqrt{2}}[\psi_1(1) - \psi_2(1) - \psi_5(1)P_2(\mu)] , \quad (10a)$$

$$\psi^P(x,y,\mu) = -\frac{\sqrt{3}}{2}\psi_5(1)\mu/xy . \quad (10b)$$

The additional numerical factors arise from L-S to j-j recoupling, and from rearranging the Faddeev channel spin-isospin functions in terms of those listed in Table I.

Using the classifications in Table I, the 5-channel D-state components can be written in the form $\Psi_D = v_4 \times \phi_3 - v_3 \times \phi_4$, where

$$v_4 = \frac{\sqrt{3}}{2}(\zeta_3 - \zeta_2) , \quad (11a)$$

$$v_3 = \tfrac{1}{2}(\zeta_2 + \zeta_3 - 2\zeta_1) , \quad (11b)$$

and

$$\zeta_1 = \overline{\psi}_3(1)I + \overline{\psi}_4(1)J + \overline{\psi}_5(1)K , \quad (12a)$$

$$\zeta_2 = x^2I[\overline{\psi}_3(2)/4 + 9\overline{\psi}_4(2)/16 - 9\hat{x}_2\cdot\hat{y}_2\overline{\psi}_5(2)/8] + y^2J[\overline{\psi}_3(2) + \overline{\psi}_4(2)/4 + 3\hat{x}_2\cdot\hat{y}_2\overline{\psi}_5(2)/2] + xyK[-\overline{\psi}_3(2)/2 + 3\overline{\psi}_4(2)/8 + 3\hat{x}_2\cdot\hat{y}_2\overline{\psi}_5(2)/4] , \quad (12b)$$

$$\zeta_3 = x^2I[\overline{\psi}_3(3)/4 + 9\overline{\psi}_4(3)/16 + 9\hat{x}_3\cdot\hat{y}_3\overline{\psi}_5(3)/8] + y^2J[\overline{\psi}_3(3) + \overline{\psi}_4(3)/4 - 3\hat{x}_3\cdot\hat{y}_3\overline{\psi}_5(3)/2] + xyK[\overline{\psi}_3(3)/2 - 3\overline{\psi}_4(3)/8 + 3\hat{x}_3\cdot\hat{y}_3\overline{\psi}_5(3)/4] . \quad (12c)$$

In the preceding expression any factors of I, J, and K must be coupled to ϕ_3 or ϕ_4 in the usual manner [e.g., $(I \times \phi_3)_{1/2}$]. In addition we have defined

$$\overline{\psi}_3(i) = [\psi_3(i) + \psi_5(i)/\sqrt{2}]/x_i^2 , \quad (13a)$$

$$\overline{\psi}_4(i) = [\psi_4(i) + \psi_5(i)/\sqrt{2}]/y_i^2 , \quad (13b)$$

$$\overline{\psi}_5(i) = \psi_5(i)/\sqrt{2}x_iy_i . \quad (13c)$$

Thus v_3 and v_4 can each be represented by the form

$$v = Iv^I + Jv^J + Kv^K \quad (14)$$

in terms of 3 scalar functions: v^I, v^J, and v^K, each different for v_3 and v_4.

A further example of the utility of this procedure is given by standard configuration space Faddeev calculations which include the Coulomb interaction[11] in the $^1S_0$ partial wave *only.* This partial wave potential for nucleons 2 and 3 has the schematic form

$$\hat{V}_c(x) = (4\pi)^{-2}V_c(x)\hat{e}_2\hat{e}_3\int\int d\Omega_x d\Omega_y$$

when acting on the total (Schrödinger) wave function Ψ, where $\hat{e}$ is the proton projection operator and $V_c(x) = \alpha/x$. (A similar angular projection acts to the left also.) Clearly the total P- and D-wave components H, I, J, and K vanish when integrated over the angles of $\hat{x}$ and $\hat{y}$, and only S waves contribute. Furthermore, the antisymmetry of v_2 when the coordinates 2 and 3 are interchanged or $\mu \rightarrow -\mu$ causes that contribution to vanish, since $(4\pi)^{-2}\int\int d\Omega_x d\Omega_y = \frac{1}{2}\int_{-1}^{1} d\mu$. Ignoring isospin $\frac{3}{2}$ pieces generated by $\hat{e}_2\hat{e}_3$ acting on the ϕ's we find

$$\hat{V}_c(x)\Psi = V_c\int_{-1}^{1} d\mu(u - v_1)(\phi_0 + \phi_2)/6 \equiv V_c\overline{(u - v_1)}(\phi_0 + \phi_2)/3 . \quad (15)$$

In perturbation theory, therefore, the energy shift is

$\int\int V_c(\overline{u-v_1})^2$, where the integrals are over the trinucleon coordinates $\vec{x}$ and $\vec{y}$. This simple result demonstrates why the eigenvalue difference in s-wave projected Coulomb Faddeev calculations containing D states is $\approx 10\%$ smaller than what one obtains using the complete (all partial waves) Coulomb potential: the $8-10\%$ D-wave contribution is missing.

As a further illustration we calculate the mass or isoscalar charge density ρ_s using the wave function we have broken into components. The various $L = 0,1,2$ partial waves do not interfere. We find, as before,

$$\rho_s = \int\int \Psi^2 , \tag{16a}$$

where

$$\Psi^2 = (u^2 + v_1^2 + v_2^2) + \tfrac{3}{2}(1-\mu^2)x^2y^2[u_P^2 + (v_1^P)^2 + (v_2^P)^2 + (v_3^P)^2 + (v_4^P)^2] + (v_3^I)^2 + (v_4^I)^2 + (v_3^J)^2 + (v_4^J)^2 + (3+\mu^2)[(v_3^K)^2 + (v_4^K)^2] + (3\mu^2-1)(v_3^I v_3^J + v_4^I v_4^J) + 4\mu[v_3^K(v_3^I + v_3^J) + v_4^K(v_4^I + v_4^J)] . \tag{16b}$$

Only the D-wave components lack a convenient sum of squares form, and this prohibits us from writing probabilities for these components in the usual way. As we stressed earlier, a wide variety of forms are possible, by taking linear combinations of the I, J, and K bipolar harmonics to generate new ones. An alternate set with more attractive symmetry properties is K, L, and M, where

$$K = 4\pi(\tfrac{10}{3})^{1/2}[Y_1(\hat{x}) \times Y_1(\hat{y})]_2 , \tag{17a}$$

$$L = [Y_2(\hat{x}) + Y_2(\hat{y})]\sqrt{4\pi} = I + J , \tag{17b}$$

$$M = [Y_2(\hat{x}) - Y_2(\hat{y})]\sqrt{4\pi} = I - J . \tag{17c}$$

With this set, symmetry alone ($\hat{x} \leftrightarrow \hat{y}$) dictates that M is orthogonal to K and L upon integration, whereas the overlap of the K and L components is $4\hat{x}\cdot\hat{y}$. Thus, in order to generate a completely orthogonal set of basis functions we must replace the vectors $\hat{x}$ and $\hat{y}$ in Eq. (17) by vectors $\hat{R}_1$ and $\hat{R}_2$ such at $\hat{R}_1 \cdot \hat{R}_2 \equiv 0$. This was the procedure of Ref. 4, although the choice of $\hat{R}_1$ and $\hat{R}_2$ is arbitrary, in general. For any such choice, the D-state contribution to ρ_s is generated by setting $\mu = \hat{x}\cdot\hat{y}$ to zero in Eq. (16b) and is given by

$$3[(v_3^K)^2 + (v_3^M)^2 + (v_4^K)^2 + (v_4^M)^2] + (v_3^L)^2 + (v_4^L)^2 , \tag{18a}$$

where

$$v^L = \frac{v^I + v^J}{2} \tag{18b}$$

and

$$v^M = \frac{v^I - v^J}{2} \tag{18c}$$

are obtained by writing Eq. (14) in terms of K, L,

TABLE III. Ground state properties of ^{3}He and ^{3}H are listed at the top: binding energy, rms charge radii, and percentage of various wave function components. The P-state percentages are broken down further in the middle. The bottom rows list overlaps of the various D-state components, according to two different schemes.

E_B (MeV)	$\langle r^2\rangle_{\rm He}^{1/2}$ (fm)	$\langle r^2\rangle_{\rm H}^{1/2}$ (fm)	P_S (%)	$P_{S'}$ (%)	P_P (%)	P_D (%)
7.023	1.895	1.698	88.91	1.67	0.083	9.34
	$P(^2P'')$ (%)	$P(^2P')$ (%)	$P(^4P)$ (%)	P_P (%)		
	0.0083	0.0373	0.0373	0.0828		
	$P(K^2)$ (%)	$P(L^2)$ (%)	$P(M^2)$ (%)	$P(2KL)$ (%)	P_D (%)	
(K,L,M)	138.20	126.82	4.89	−260.57	9.34	
(K',L',M')	4.89	0.21	4.24	0	9.34	

and M.

As an illustration we resort to expediency and choose K, L, and M as defined with the vectors $\hat{x}$ and $\hat{y}$ in Eq. (17), and replace these vectors by $\vec{x} \to \vec{R}_1 = \hat{x} + \hat{y}$ and $\vec{y} \to \vec{R}_2 = \hat{x} - \hat{y}$ to generate K', L', and M'. The resulting wave function probabilities P and ground state properties for the Reid Soft Core 5-channel potential model are shown in Table III from our recent large basis calculation. Symmetry considerations are sufficient to show that $\int\int v_1^2 = \int\int v_2^2$ and $\int\int v_3^2 = \int\int v_4^2$ for both P and D waves. As remarked earlier, we expect $P(^2P'')$ to be small. Note that $P_D = P(^4D)$ differs[14] from the results of Ref. 9, that the uncertainty in E_B is a few keV and a few attometers for the radii, while the other numbers are stable to within 1 in the last digit. Our choice of K and L obviously has resulted in two "vectors" almost antiparallel, while K' and L' are orthogonal, with L' being very small. Clearly expediency is a good guide for the choice of D-state basis functions. Many choices are available.

In summary, we have developed spherical harmonic identities which allow us to rewrite the usual channel decomposition of Faddeev calculations in terms of a Blatt-Derrick type classification scheme. The latter scheme is nonunique, because a complete set of bipolar harmonics for the D-state components is nonunique. This technique was illustrated by the S-, P-, and D-wave function rearrangement of a 5-channel Faddeev model, and by an analysis of the contributions to the Coulomb energy using a partial wave ($^1S_0$) projected Coulomb potential. We have calculated D-wave component probabilities according to two different schemes, one orthogonal and the other partially so.

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[14]The formula used to calculate P_D in Refs. 9 and 11 was in error in a term involving channel 5. This also caused the *projected* Coulomb energies calculated in perturbation theory to be incorrect. We find 9.34%/9.41% for P_D without/with the Coulomb interaction, and $\Delta E_p/\Delta \bar{E}_P$ to be 558/535 keV, in agreement with the eigenvalue differences $\Delta E_F/\Delta \bar{E}_F$. No other significant changes occurred, and the 3-channel calculations were unaffected.