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Inversion procedure for a pion-nucleon T matrix

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The solution of the inverse scattering problem for a separable model of the pion-nucleon T matrix is obtained. The model is based on the Feshbach-Villars Hamiltonian formulation of the Klein-Gordon equation, and includes the left hand cut in the complex energy plane associated with crossing, as well as the direct and crossed nucleon poles. Numerical results are given for the four P-wave channels.

[NUCLEAR REACTIONS Pion-nucleon elastic scattering; inversion procedure for separable T matrix; contains nucleon poles and crossing cut.]

I. INTRODUCTION

Over the years, many approaches have been developed in an attempt to determine the off-shell pion-nucleon T matrix. The various approaches have been outlined in Ref. 1 (hereafter referred to as F), where extensive references to the literature can be found. The proceedings[2] of the latest meson-nuclear physics conference is also useful for reviews and references on the πN T matrix.

In F the Feshbach-Villars[3] formulation of the Klein-Gordon equation was used to develop time dependent and time independent scattering theories for relativistic, spin-zero particles. It was shown that the time independent theory provides a natural framework for constructing separable potential models for the πN interaction which lead to scattering amplitudes with the analytic structure in the complex energy plane implied by quantum field theory, i.e., right and left hand cuts associated with unitarity and crossing, respectively, as well as direct and crossed nucleon poles. The potential models constructed by other workers (see Refs. 7-11 of F) do not include the nucleon poles and the crossing cut.

A separable potential model of the πN, P-wave, elastic scattering amplitude was constructed which, with a particular choice of the form factors, reproduces the Chew-Low (CL) T matrix.[4,5] A procedure for including nucleon recoil corrections was developed, which gives rise to a T matrix whose singularity structure agrees with a form of the Low equation obtained by Miller.[6]

Here we shall develop a procedure for obtaining the form factors in the separable potential model from the πN phase shifts. The method extends the well known solutions[7] of the inverse scattering problem for separable interactions so as to account for the left hand crossing cut and the nucleon poles.

In Sec. II the inversion procedure is developed. Its practicality is demonstrated in Sec. III, where the form factors for the four P-wave channels are obtained. Section IV gives a brief discussion of possible further work along the lines presented here. Throughout we work in units such that $\hbar = c = 1$.

II. INVERSION PROCEDURE

From the results of Sec. IV of F it follows that the πN T matrix in the c.m. frame can be written in the form

$$T^{\sigma\sigma'}(\vec{p},\vec{p}\,';z) = -\sum_{\alpha} G_{\alpha}^{\sigma}(p)\frac{P_{\alpha}(\hat{p},\hat{p}')}{d_{\alpha}(z)}G_{\alpha}^{\sigma'}(p')\,, \qquad (1)$$

where $\vec{p}$ and $\vec{p}\,'$ are the c.m. momenta of the π or N, and z is a complex energy parameter which for physical energies is the total c.m. energy minus the nucleon mass. The index α labels the four P-wave channels distinguished by the total isospin T and total angular momentum J according to

$$\alpha = 1,2,3,4;\quad 2T, 2J = 11, 13, 31, 33, \qquad (2)$$

and the indices σ and σ' are $+1$ for positive energies and -1 for negative energies. It is seen that there are two form factors $G_{\alpha}^{(\pm)}(p)$: one associated with the positive energy states, and the other with the negative energy states. The P_{α} are the projection operators for the four channels. The denominator function is given by

$$d_{\alpha}(z) = z\lambda_{\alpha}^{-1}\left[1 + z\lambda_{\alpha}\sum_{\sigma}\int_0^{\infty}\frac{dp\,p^2\sigma G_{\alpha}^{\sigma\,2}(p)}{\Omega_p^{\,2}(z-\sigma\Omega_p)}\right], \qquad (3)$$

where

$$\lambda_{\alpha} = \tfrac{2}{3}\left(\frac{f}{\mu}\right)^2 \times \begin{cases} -4, & \alpha = 1 \\ -1, & \alpha = 2 \\ -1, & \alpha = 3 \\ \ \ 2, & \alpha = 4\,, \end{cases} \qquad (4)$$

$$\Omega_p = \omega_p + p^2/2m\ , \tag{5}$$

and

$$\omega_p = (p^2+\mu^2)^{1/2}\ . \tag{6}$$

Here f is the renormalized πN coupling constant, μ the pion mass, and m the nucleon mass.

In order to facilitate comparison with the CL theory, we define new form factors $v_\alpha(p)$ and $u_\alpha(p)$ according to

$$G_\alpha^{(+)}(p) = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{m+\omega_p}{2m\omega_p}\right)^{1/2} p v_\alpha(p)\ ,$$
$$G_\alpha^{(-)}(p) = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{m+\omega_p}{2m\omega_p}\right)^{1/2} p u_\alpha(p)\ , \tag{7}$$

which when put into (3) leads to

$$d_\alpha(z) = z\lambda_\alpha^{-1}\left\{1 - \frac{z\lambda_\alpha}{\pi}\int_\mu^\infty d\Omega_p \frac{p^3}{\Omega_p^2}\left[\frac{v_\alpha^2(p)}{\Omega_p - z} + \frac{u_\alpha^2(p)}{\Omega_p + z}\right]\right\}\ . \tag{8}$$

In the CL theory[4,5] all of the $v_\alpha(p)$ are the same cutoff function $v(p)$, and the $u_\alpha^2(p)$ are obtained from the nonlinear equations implied by crossing symmetry. A big advantage of the approach presented here is that is is not necessary to solve such nonlinear equations.

We see from (8) that $d_\alpha(z)$ is a real, analytic function of z with a right hand cut (RHC) beginning at $z=\mu$, a left hand cut (LHC) beginning at $z=-\mu$, and a simple zero at $z=0$. For z just above the RHC we shall write $z=\Omega+i\epsilon$, and for z just below the LHC we shall write $z=-\Omega-i\epsilon$, where, in both cases, $\Omega \geqslant \mu$ and ϵ is a positive infinitesimal. From (8) it follows that

$$k^3 v_\alpha^2(k) = -\mathrm{Im} d_\alpha(\Omega+i\epsilon)\ ,$$
$$k^3 u_\alpha^2(k) = -\mathrm{Im} d_\alpha(-\Omega-i\epsilon)\ , \tag{9}$$

where k is the momentum corresponding to the energy Ω. We can write

$$d_\alpha(\Omega+i\epsilon) = |d_\alpha(\Omega+i\epsilon)|\, e^{-i\theta_\alpha(\Omega)}\ ,$$
$$d_\Omega(-\Omega-i\epsilon) = |d_\alpha(-\Omega-i\epsilon)|\, e^{-i\phi_\alpha(\Omega)}\ . \tag{10}$$

The phases on the ohter sides of the cuts have opposite signs, since

$$d_\alpha(z^*) = d_\alpha^*(z)\ . \tag{11}$$

We shall show that $d_\alpha(z)$ can be determined from a knowledge of the phases $\theta_\alpha(\Omega)$ and $\phi_\alpha(\Omega)$. Knowing $d_\alpha(z)$, we can find the form factors from (9).

From (7), (9), and (10), we have

$$-\frac{G_\alpha^{(+)2}(k)}{d_\alpha(\Omega+i\epsilon)} = -\frac{1}{\pi}\frac{m+\omega_k}{m\omega_k} f_\alpha(\Omega)\ , \tag{12}$$

where

$$f_\alpha(\Omega) = e^{i\theta_\alpha(\Omega)}\sin\theta_\alpha(\Omega)/k\ , \tag{13}$$

and is the physical scattering amplitude. It is interesting to note that the factor that relates the on-shell T matrix and this amplitude agrees with the proper relativistic phase-space factor in the limit of nonrelativistic nucleons.[8]

We assume a CL-type crossing relation[4,5] to determine the ϕ_α. We have, with the help of (7), (12), and (10),

$$\frac{k^2 v_\alpha^2(k)}{d_\alpha(-\Omega-i\epsilon)} = \sum_\beta A_{\alpha\beta} f_\beta(\Omega) = \frac{k^2 v_\alpha^2(k)}{|d_\alpha(-\Omega-i\epsilon)|} e^{i\phi_\alpha(\Omega)}\ , \tag{14}$$

where the $A_{\alpha\beta}$ are given by Eq. (4.18) of F. We see from (13) and (14) that the phases $\theta_\alpha(\Omega)$ and $\phi_\alpha(\Omega)$ can be obtained from experiment.

In order to determine $d_\alpha(z)$ from these phases, we shall write a dispersion relation involving $\ln d_\alpha(z)$. According to Eq. (4.6) of F, $d_\alpha(z)$ approaches a positive constant for large $|z|$. This constant can be taken to be one, since any positive, constant factor in $d_\alpha(z)$ can be absorbed into the form factors without affecting the T matrix [see Eq. (1)], so we can assume

$$d_\alpha(z) \underset{|z|\to\infty}{\sim} 1, \tag{15}$$

with the understanding that this one has the dimensions of μ^3 [see Eqs. (4) and (8)]. Since $d_\alpha(0)=0$, it is not convenient to work directly with $\ln d_\alpha(z)$; therefore we introduce a function

$$D_\alpha(z) = \frac{w+i\mu^{1/2}}{w-i\mu^{1/2}} d_\alpha(z)\ , \tag{16}$$

where

$$w = (z-\mu)^{1/2}\ ,$$
$$= |z-\mu|^{1/2} e^{i\zeta/2},\quad 0<\zeta<2\pi\ . \tag{17}$$

For z in the cut plane Im $(w)>0$, and so $w+i\mu^{1/2}$ cannot vanish; however, $w-i\mu^{1/2}$ can, and takes out the zero in d_α. From (16) and (17), we have

$$D_\alpha(0) = -4\mu/\lambda_\alpha\ , \tag{18}$$

and

$$D_\alpha(z) \underset{|z|\to\infty}{\sim} 1\ . \tag{19}$$

In writing a dispersion relation for $\ln D_\alpha(z)$, we shall work with the branch of the logarithm which has a cut when its argument is real and negative; therefore, we must check to see if $D_\alpha(z)$ can take on such values. From (4) and (18), we see that there is a difficulty for $\alpha=4$; however, we shall be able to circumvent it.

We can write

$$\ln D_\alpha(z) = \ln[(w + i\mu^{1/2})^2/\lambda_\alpha] + \ln g_\alpha(z)\,, \tag{20}$$

where

$$g_\alpha(z) = z^{-1}\lambda_\alpha d_\alpha(z)\,. \tag{21}$$

It follows from (8) that

$$\operatorname{Im} g_\alpha(z) = -\operatorname{Im}(z)\frac{\lambda_\alpha}{\pi}\int_0^\infty d\Omega_p \frac{p^3}{\Omega_p}\left[\frac{v_\alpha{}^2(p)}{|\Omega_p - z|^2} + \frac{u_\alpha{}^2(p)}{|\Omega_p + z|^2}\right], \tag{22}$$

so we see that $g_\alpha(z)$ is real only when z is real. Since in writing our dispersion relation we shall use a contour that excludes the real z axis for $|z| \geq \mu$, we need consider only $-\mu < z < \mu$. From (8) and (21), it follows that $g_\alpha(0) = 1$, thus $g_\alpha(z)$ cannot be real and negative, otherwise $g_\alpha(z)$ would have at least one zero on the range $-\mu < z < \mu$. This would contradict the assumption that the only zero in $d_\alpha(z)$ is at $z = 0$.

The only way $(w + i\mu^{1/2})^2$ can be real is if w is on the positive, imaginary axis, which from (17) means z is real and less than μ. Thus we see that for $\lambda_\alpha < 0$, $D_\alpha(z)$ cannot be real and negative, so for now we restrict our attention to $\alpha = 1, 2, 3$.

We can write

$$\ln D_\alpha(z) = \frac{1}{2\pi i}\oint_{C_1} dz' \frac{\ln D_\alpha(z')}{z' - z}\,, \tag{23}$$

where C_1 is the contour shown in Fig. 1. Using (10), (11), (16), (17), and (19), we have

$$\ln D_\alpha(z) = -\frac{1}{\pi}\int_\mu^\infty d\Omega\left[\frac{\theta_\alpha(\Omega)}{\Omega - z} + \frac{\phi_\alpha(\Omega)}{\Omega + z}\right] + F(z)\,, \tag{24}$$

where

$$F(z) = \frac{1}{\pi i}\int_\mu^\infty \frac{d\Omega}{\Omega - z}\ln\left[\frac{(\Omega - \mu)^{1/2} + i\mu^{1/2}}{(\Omega - \mu)^{1/2} - i\mu^{1/2}}\right]. \tag{25}$$

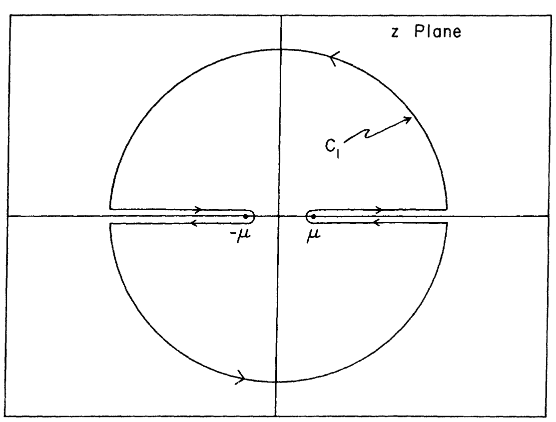

FIG. 1. The contour C_1 for the integrals in Eqs. (23) and (30). μ is the pion mass.

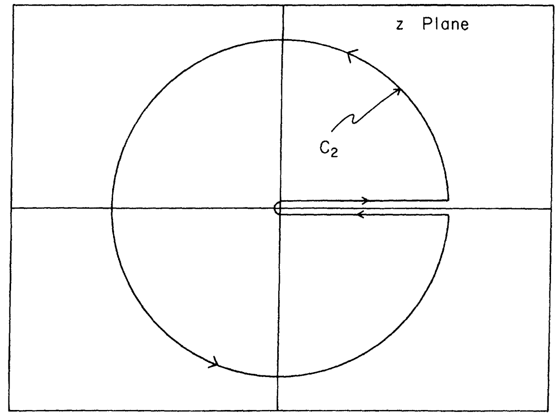

FIG. 2. The contour C_2 used in evaluating the integral in Eq. (25).

The integrand in (25) has a logarithmic cut for $0 < \Omega < \mu$ and a square root cut for $\Omega > \mu$. By considering the appropriate contour integral along C_2 in Fig. 2, we find

$$F(z) = \ln\left[\frac{(w + i\mu^{1/2})^2}{z - \mu}\right]. \tag{26}$$

From (16), (17), (25), and (26), it follows that

$$d_\alpha(z) = \frac{z}{z - \mu}\exp\left\{-\frac{1}{\pi}\int_\mu^\infty d\Omega\left[\frac{\theta_\alpha(\Omega)}{\Omega - z} + \frac{\phi_\alpha(\Omega)}{\Omega + z}\right]\right\}, \quad \alpha = 1, 2, 3. \tag{27}$$

For $\alpha = 4$, we consider $d_4(-z)$. The function $d_4(-z)$ behaves just like $d_\alpha(z)$ for $\alpha = 1, 2, 3$, but with the roles of $\theta_\alpha(\Omega)$ and $\phi_\alpha(\Omega)$ interchanged. From this it follows that

$$d_4(z) = \frac{z}{z + \mu}\exp\left\{-\frac{1}{\pi}\int_\mu^\infty d\Omega\left[\frac{\theta_4(\Omega)}{\Omega - z} + \frac{\phi_4(\Omega)}{\Omega + z}\right]\right\}. \tag{28}$$

In order to proceed we must establish the behavior of the phases for $\Omega \to \infty$ and for $\Omega \to \mu$. From (10) and (15) it follows that we can require that

$$\theta_\alpha(\Omega) \underset{\Omega\to\infty}{\sim} 0, \quad \phi_\alpha(\Omega) \underset{\Omega\to\infty}{\sim} 0\,. \tag{29}$$

We can learn something about the phases at $\Omega = \mu$ by deriving a modification of Levinson's[9] theorem. It is straightforward to show that

$$\frac{1}{2\pi i}\oint_{C_1} d\ln d_\alpha(z) = 1\,, \tag{30}$$

which, using (10), (11), and (29), leads to

$$\theta_\alpha(\mu) + \phi_\alpha(\mu) = \pi\,. \tag{31}$$

This shows that at least one of the two phases for each channel must be different from zero at threshold. An examination of (27) and (28) shows that we must have

$$\theta_\alpha(\mu)=\pi,\quad \phi_\alpha(\mu)=0,\quad \alpha=1,2,3,$$
$$\theta_4(\mu)=0,\quad \phi_4(\mu)=\pi\,. \tag{32}$$

These choices eliminate the spurious pole at $z=\mu$ in (27) and at $z=-\mu$ in (28). We shall see shortly how to do this explicitly.

We want the amplitudes to have the correct residues at $z=0$. From (8), (27), and (28), it follows that

$$d_\alpha(z)=z\lambda_\alpha^{-1}\exp\left\{-\frac{z}{\pi}\int_\mu^\infty\frac{d\Omega}{\Omega}\left[\frac{\delta_\alpha(\Omega)}{\Omega-z}-\frac{\Delta_\alpha(\Omega)}{\Omega+z}\right]\right\},\quad \alpha=1,2,3,4, \tag{33}$$

where

$$\delta_\alpha(\Omega)=\theta_\alpha(\Omega)-\pi,\quad \Delta_\alpha(\Omega)=\phi_\alpha(\Omega),\quad \alpha=1,2,3,$$
$$\delta_4(\Omega)=\theta_4(\Omega),\quad \Delta_4(\Omega)=\phi_4(\Omega)-\pi\,. \tag{34}$$

The $\delta_\alpha(\Omega)$ we identify with the experimental phase shifts, since they all approach zero as $\Omega\to\mu$. From (9) and (33), we obtain

$$k^3v_\alpha^2(k)=\Omega\lambda_\alpha^{-1}\exp\left\{-\frac{\Omega}{\pi}\int_\mu^\infty\frac{d\Omega'}{\Omega'}\left[\frac{\delta_\alpha(\Omega')}{\Omega'-\Omega}-\frac{\Delta_\alpha(\Omega')}{\Omega'+\Omega}\right]\right\}\times\sin\delta_\alpha(\Omega)\,, \tag{35}$$

$$k^3u_\alpha^2(k)=-\Omega\lambda_\alpha^{-1}\exp\left\{-\frac{\Omega}{\pi}\int_\mu^\infty\frac{d\Omega'}{\Omega'}\left[\frac{\Delta_\alpha(\Omega')}{\Omega'-\Omega}-\frac{\delta_\alpha(\Omega')}{\Omega'+\Omega}\right]\right\}\times\sin\Delta_\alpha(\Omega)\,,$$

where the integrals are principal value integrals. In order for the $v_\alpha(k)$ and the $u_\alpha(k)$ to be real we must have [see Eq. (4)]

$$\delta_\alpha(\Omega)<0,\quad \Delta_\alpha(\Omega)>0,\quad \alpha=1,2,3,$$
$$\delta_4(\Omega)>0,\quad \Delta_4(\Omega)<0\,. \tag{36}$$

In (33) it appears as if $d_\alpha(z)$ diverges as $|z|\to\infty$; however, this is illusory [see (27) and (28)]. Because $\delta_\alpha(\Omega)\xrightarrow[\Omega\to\infty]{}-\pi$ for $\alpha=1,2,3$ and $\Delta_4(\Omega)\xrightarrow[\Omega\to\infty]{}-\pi$, the arguments of the exponentials develop logarithmic singularities for large $|z|$ which cancel the factor of z.

It is not difficult to show that (33) gives the solution of the inversion problem for the CL theory if the phases $\delta_\alpha(\Omega)$ and $\Delta_\alpha(\Omega)$ all become zero at high energies as well as at the elastic threshold. In the CL theory[4,5] the denominator function $d_\alpha(z)$ does diverge as z for large $|z|$. It should be kept in mind, however, that if this inversion procedure is used with experimental phases, an inconsistency will develop, as the form factors $v_\alpha(p)$ will depend on α. The CL theory has the same form factor in all four channels.

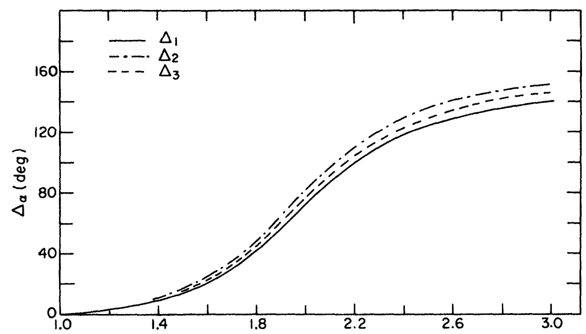

FIG. 3. The phases $\Delta_{1,2,3}$ calculated from Eqs. (13), (14), and (34). In this and the remaining figures Ω is the πN c.m. energy minus the nucleon mass, and μ is the pion mass.

III. SOME RESULTS

Here we present some preliminary numerical results to demonstrate the feasability of the inversion procedure presented in the previous section. We use the experimental phase shifts of Zidell, Arndt, and Roper,[10] which are for pion lab kinetic energies up to 350 MeV, and are shown in Figs. 3(d) and 3(f)-3(h) of Ref. 10. The Δ_α calculated from (13), (14), and (34) are shown in Figs. 3 and 4. It is seen that they are almost identical for $\alpha=1,2,3$, and, in fact, are very similar to the p_{33} phase shift [see Fig. 3 (h) of Ref. 10]. This is not surprising, since the resonant p_{33} amplitude dominates the sum in (14).

The only phase that violates the conditions given in (36) is the p_{11} phase shift which changes sign close to inelastic threshold.[10] In calculating the form factors the following forms have been used for δ_1 above $\Omega=1.6\mu$, and for the others above $\Omega=3.0\mu$:

$$\delta_\alpha(\Omega)=-\pi+\frac{a_\alpha}{\Omega+b_\alpha},\quad \alpha=1,2,3$$
$$\delta_4(\Omega)=\frac{1680}{(\Omega-4.10)^2+10.5}\,. \tag{37}$$

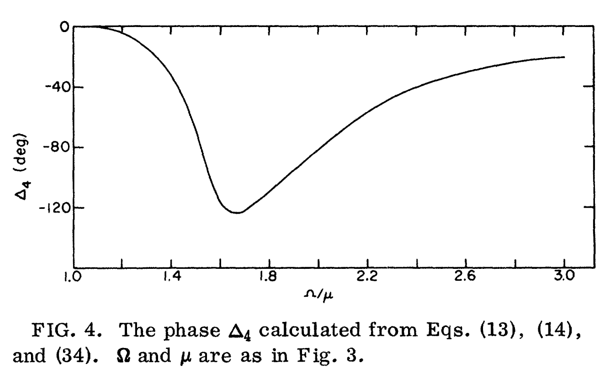

FIG. 4. The phase Δ_4 calculated from Eqs. (13), (14), and (34). Ω and μ are as in Fig. 3.

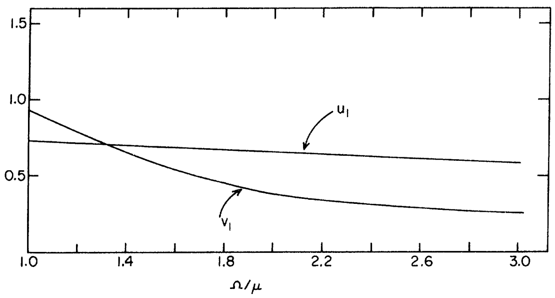

FIG. 5. Form factors for the P_{11} channel. Ω and μ are as in Fig. 3.

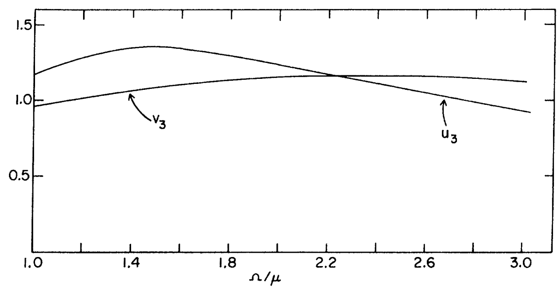

FIG. 7. Form factors for the P_{31} channel. Ω and μ are as in Fig. 3.

The parameters a_α and b_α were determined by matching the phase shifts and their first derivatives at the energies given above. The parameters in δ_4 were chosen so as to give a rough fit to the p_{33} phase shifts in Ref. 11 for $3\mu \leq \Omega \leq \leq 6\mu$.

In calculating the form factors from (35), it is also necessary to know the λ_α. For our purposes it is most convenient to define them by the relation

$$f_\alpha(\Omega) \underset{\Omega\to 0}{\longrightarrow} \frac{k_0^2\lambda_\alpha}{\Omega}, \qquad (38)$$

where k_0 is the imaginary momentum corresponding to $\Omega=0$ ($k_0 \approx i\mu^{1/2}$). From (7), (12), and (13), it follows that

$$k^2 v_\alpha^2(k) = d_\alpha(\Omega + i\epsilon) f_\alpha(\Omega), \qquad (39)$$

which when combined with (8) and (38) leads to

$$v_\alpha(k_0) = 1. \qquad (40)$$

In calculating the λ_α from (4), a value of $f^2=0.086$ has been used, which is consistent with the value $f^2=0.08$ used by Ernst and Johnson[12] in their work on the CL theory, since their cutoff function is normalized to one at $k=0$, not at $\Omega=0$.

All of the forms factors shown in Figs. 5-8, except for u_4, are slowly varying in the range shown, and are close to one in value. Near $\Omega=1.4\mu$, u_4 is an order of magnitude larger than the other form factors. In Sec. IV of F an argument is given which anticipates this result. The form factor v_4 is very similar to the one obtained by Ernst and Johnson.[12] Of course, the form factors depend on the extrapolation chosen for the high energy phase shifts. It has been found that the sensitivity to the extrapolation is not very dramatic in the range shown in the figures. This can be traced back to the fact that we have forced the denominator function $d_\alpha(z)$ given in (33) to behave in a certain way near $z=0$. It is concluded that the inversion procedure given in the previous section is practical and leads to reasonable form factors.

IV. DISCUSSION

The separable potential model in F and this work can be extended and improved upon in a number of ways. Inelasticity effects should be taken into account, the p_{11} phase shift should be allowed to change sign, the treatment of crossing and its associated singularities should be improved, and the model should be made fully relativistic.

The work of Londergan *et al.*[13] shows how to construct rank one separable potentials that take account of inelasticity effects. The form they derive is obtained by eliminating the coupling to the inelastic channels by means of Feshbach's[14]

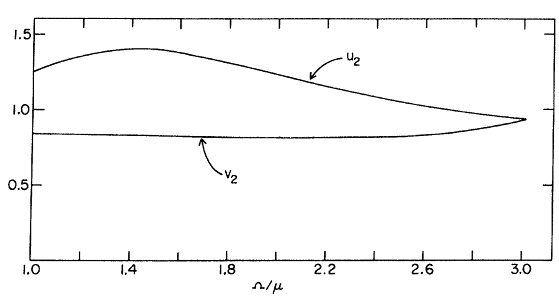

FIG. 6. Form factors for the P_{13} channel. Ω and μ are as in Fig. 3.

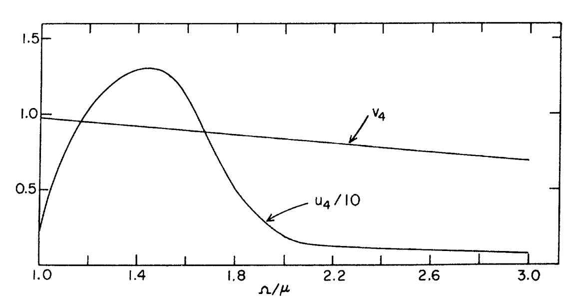

FIG. 8. Form factors for the P_{33} channel. Ω and μ are as in Fig. 3.

technique. It should be possible to adapt this approach to the author's formalism,[1] since his equations are similar in structure to those that occur in nonrelativistic potential scattering. This procedure can be checked by seeing if the resulting modification of the author's T matrix can reproduce the CL theory with inelasticity effects included.[12]

It will be interesting to see if including inelasticity effects will give a p_{11} phase shift that changes sign. Ernst and Johnson[15] state that the behavior of this phase shift is beyond the model of Ref. 13. However, recent results[16] based on an updated version[17] of the CL theory indicate that the sign change is associated with the inelasticity. Since the author's model is closer to the results of quantum field theory than the model of Ref. 13, it is quite possible that inclusion of the inelasticity will do the trick. If this does not work it is clear that a higher rank separable potential can be found that will produce the sign change.

Strictly speaking, the crossing relations used in Eq. (14) are valid only in the static limit. When recoil is taken into account all angular momentum states get coupled together. This should be taken into account in determining the form factors $v_\alpha(p)$ and $u_\alpha(p)$. This will not be difficult in the author's approach, since the crossing relations are used only to determine the phases $\Delta_\alpha(\Omega)$; it is not necessary to "solve" those relations.

It should be possible to obtain the author's T matrix as a limit of a covariant formulation. In their work on relativistic T matrices, Celenza *et al.*[18] have obtained T matrix equations which bear a strong resemblance to Eq. (3.30) of F in that they involve both positive and negative energy meson states. At present an attempt is under way to use their work as a basis for a covariant generalization of the present work.

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