Eikonal corrections for spin-orbit potentials

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The first correction terms to the eikonal phases are evaluated in closed form for the case of scattering from a potential with a spin-orbit component. It is shown that the size of such effects is significant for the measurement of the polarization parameters in medium energy proton-nucleus scattering.

[NUCLEAR REACTIONS Spin-$\frac{1}{2}$ elastic scattering; eikonal corrections to polarization parameters.]

I. INTRODUCTION

The eikonal approximation[1] has proved to be a very efficient and economical approach to high energy potential scattering and has formed the basis of many calculations for the scattering of hadrons by nuclei. In a series of articles Wallace[2–4] has obtained systematic corrections to the eikonal phase in the case where the potential is spherically symmetric. Now in the elastic scattering of intermediate energy protons by spin-zero nuclei the strong spin-orbit force gives rise to interesting polarization effects. Such potentials were studied in the lowest order eikonal approximation by Glauber[1] but not by Wallace though he did suggest a possible treatment.[4] It is therefore the purpose of this paper to generalize Wallace's results in the case of scattering from a potential with a spin-orbit as well as a central component.

In Sec. II we review Wallace's approach, distinguishing carefully between the impact parameter dependence arising from the linear momentum and the angular momentum variation of the eikonal phases. This is crucial for the investigation of spin orbit potentials which are momentum dependent as well as nonspherically symmetric. An alternative operator to those of Wallace is introduced which keeps track of this problem. After establishing the eikonal formalism for spin-zero potential scattering in Sec. II, the required generalization is then simply obtained by noting that the Schrödinger equation does not mix states with total angular momentum $j = l + \frac{1}{2}$ with $j = l - \frac{1}{2}$. In this way the Wallace correction of an arbitrary order can, in principle, be obtained, but in Sec. IV we rederive the first order correction using the more laborious propagator expansion techniques[3] whereby we see more clearly the role of the momentum dependence of the potential. For medium energy proton-nucleus scattering the first order is the dominant correction, and in Sec. V we show that though it has negligible influence on the cross sections it is significant for the polarization and spin-rotation parameters. In our conclusions in Sec. VI we also compare our results with those of other formulations.

II. WALLACE CORRECTIONS FOR SPIN-ZERO SCATTERING

The amplitude for the scattering of a spin-zero particle by a spherically symmetric potential may be described by a partial wave decomposition

$$f(\theta)=\frac{1}{2ik}\sum_l (2l+1)(e^{2i\delta(l)}-1)P_l(\cos\theta) \ , \tag{2.1}$$

or by an impact parameter representation

$$f(q)=-ik\int_0^\infty b\, db J_0(qb)(e^{i\chi(b)}-1) \ , \tag{2.2}$$

where the scattering angle θ is related to the incident momentum k and momentum transfer q by

$$q=2k\sin(\theta/2) \ . \tag{2.3}$$

To derive a connection between these two descriptions Wallace[4,5] used the expansion of the Legendre polynomials in terms of the zeroth order Bessel function:

$$P_l(z)=\sum_{m=0}^{\infty}\frac{1}{(2m)!}\left[\frac{\partial}{\partial l}\right]^{2m} b_m\left[\frac{1}{4}\frac{\partial}{\partial l}(2l+l)\right] \times J_0\{(2l+1)\times[(1-z)/2]^{1/2}\} \ , \tag{2.4}$$

where the $b_m(x)$ are generalized Bernouilli polynomials with $b_0(x)=1$ and $b_1(x)=-x/6$. For a well behaved potential which gives rise to phase shifts which can be interpolated smoothly for real l, the eikonal phase may then be exactly related to the $\delta(l)$ through

$$e^{i\chi(b)}=(2l+1)^{-1}W(2l+1)e^{2i\delta(l)} \tag{2.5a}$$

by the operator

$$W=\sum_{m=0}^{\infty}\frac{1}{(2m)!}b_m\left[-\tfrac{1}{4}(2l+1)\frac{\partial}{\partial l}\right]\left[\frac{\partial}{\partial l}\right]^{2m} , \tag{2.5b}$$

where the impact parameter has its usual quantal definition

$$kb=l+\tfrac{1}{2} \ . \tag{2.6}$$

For high energy scattering where the phase shifts vary regularly with l this series converges rapidly and so to a good approximation

$$e^{i\chi(b)}\simeq\left[1+\frac{1}{48}\left[\frac{\partial}{\partial l}\right]^3(2l+1)\right]e^{2i\delta(l)} \ . \tag{2.7}$$

Since for a real potential the phase shifts are real by unitarity, it can be seen that the second term in Eq. (2.7) induces an imaginary part in $\chi(b)$ which leads to the designation of "unitarity" corrections for these higher order terms. Unitarity is diagonal in k so that the differentiations in Eq. (2.5) must be carried out with respect to l (or b) with k fixed.

As a dynamical model for the phase shifts Wallace[4] uses the Wentzel-Kramers-Brillouin (WKB) approximation and its generalizations:

$$\delta^{\rm WKB}(l)=\int_{r_0}^{\Lambda}dr[k^2-(l+\tfrac{1}{2})^2/r^2-2mV(r)]^{1/2}-\int_{(l+1/2)/k}^{\Lambda}dr[k^2-(l+\tfrac{1}{2})^2/r^2]^{1/2} \ , \tag{2.8}$$

where r_0 is the turning point, assumed single, of a particle of mass m in a potential $V(r)$ and the limit $\Lambda\to\infty$ is understood. Wallace[4] is then able to expand this solution in a power series in the strength of the potential

$$\delta^{\rm WKB}(l)=\sum_{n=0}^{\infty}\delta_n(l) \ , \tag{2.9}$$

where

$$\delta_n(l=kb-\tfrac{1}{2})=-\frac{k(m/k^2)^{n+1}}{(n+1)!b^{2n}}\left[b^2\left[1+b\frac{d}{db}\right]\right]^n \times\int_0^\infty dz\ V^{n+1}[(b^2+z^2)^{1/2}] \tag{2.10}$$

and we shall lay aside questions of convergence of the series.

Different forms of the differential operator in Eq. (2.10) have been derived but all upon the assumption that the potential does not depend explicitly upon l. It can easily be shown by induction that Eq. (2.10) is completely equivalent to

$$\delta_n(l)=-\frac{(m)^{n+1}}{k(n+1)!}\left[\left[\frac{b}{k}\frac{\partial}{\partial b}-\frac{\partial}{\partial k}\right]\frac{1}{k}\right]^n \times\int_0^\infty dz\ V^{n+1}[(b^2+z^2)^{1/2}] \ , \tag{2.11}$$

where the differentiations $\partial/\partial b$ and $\partial/\partial k$ are carried out at fixed k and b, respectively. However,

$$\frac{b}{k}\left[\frac{\partial}{\partial b}\right]_k-\left[\frac{\partial}{\partial k}\right]_b=-\left[\frac{\partial}{\partial k}\right]_l \ , \tag{2.12}$$

so that, in terms of differentiations with respect to k at fixed l,

$$\delta_n(l) = -\frac{1}{2}\frac{(2m)^{n+1}}{(n+1)!}\left[\left[-\frac{\partial}{\partial k^2}\right]_l\right]^n \frac{1}{k} \times \int_0^\infty dz\ V^{n+1}\{[(l+\tfrac{1}{2})^2/k^2+z^2]^{1/2}\}\ . \tag{2.13}$$

It is not, of course, suprising that the WKB solution, which is diagonal in l, should be expressable in terms of an operator in which l is held constant. The form of Eq. (2.13) can be derived directly from Eq. (2.18) using the representation

$$\delta^{\rm WKB}(l) = \tfrac{1}{2}\int_{k^2}^\infty dk'^2 \int_0^\infty r\ dr([k'^2r^2-(l+\tfrac{1}{2})^2]^{-1/2}\theta(k'r-l-\tfrac{1}{2}) - \{[k'^2-2mV(r)]r^2-(l+\tfrac{1}{2})^2\}^{-1/2}\theta\{[k'^2-2mV(r)]r^2-(l+\tfrac{1}{2})^2\}). \tag{2.14}$$

With the help of the displacement operator in k^2 this becomes

$$\delta^{\rm WKB}(l) = \tfrac{1}{2}\int_{k^2}^\infty dk'^2\int_0^\infty r\ dr\left\{1-\exp\left[-2mV(r)\left[\frac{\partial}{\partial k'^2}\right]_l\right]\right\}[k'^2r^2-(l+\tfrac{1}{2})^2]^{-1/2}\theta(k'r-l-\tfrac{1}{2})\ . \tag{2.15}$$

Expansion of the exponential and integration with respect to k'^2 then yields the desired form of Eq. (2.13), which shows that this or Eq. (2.11) is valid even when V depends upon the orbital angular momentum.

The first two terms in the Wallace expansion are explicitly

$$\delta_0(l) = -\frac{m}{k}\int_0^\infty V(r)dz\ , \tag{2.16a}$$

$$\delta_1(l) = -\frac{m^2}{2k^3}\left[1+b\frac{\partial}{\partial b}-k\frac{\partial}{\partial k}\right]\int_0^\infty V^2(r)dz\ . \tag{2.16b}$$

It should be noted that Eq. (2.16a) is not the Born approximation to the phase shift $\delta(l)$, though if it were interpreted as half the eikonal phase $\frac{1}{2}\chi(b)$ it would be. Wallace[4] shows that if higher order corrections to the WKB approximation, such as those of Rosen and Yennie,[6] are included, the extra terms which are linear in V cancel with the unitarity correction of Eq. (2.5). A completely systematic approach therefore requires an evaluation of the higher order WKB term. However, for practical applications in medium energy hadron-nucleus scattering the first correction of Eq. (2.14) is the dominant effect and so the zeroth order and first Wallace correction to the eikonal phase are

$$\chi_0(b) = -\frac{m}{k}\int_{-\infty}^{+\infty} V(r)dz\ , \tag{2.17a}$$

$$\chi_1(b) = -\frac{m^2}{2k^3}\left[1+b\frac{\partial}{\partial b}-k\frac{\partial}{\partial k}\right]\int_{-\infty}^{+\infty} V^2(r)dz\ . \tag{2.17b}$$

With an ordinary potential the differentiation with respect to k in Eqs. (2.16b) and (2.17b) yields nothing. For a *momentum*- dependent potential,[7] such as the spin-orbit component discussed in the next section, a contribution remains though in such cases the validity of the WKB solution of Eq. (2.8) might have to be investigated. To compare the results of Eq. (2.17) with an exact resolution of the Schrödinger equation for a potential which only depends implicitly upon an energy parameter, the $\partial/\partial k$ must clearly be discarded. However, it has been argued[8] in connection with Coulomb distortions of the eikonal phase that if the optical potential is introduced into the Schrödinger equation minimally by letting $E\to E-V$ everywhere, including the implicit energy dependence of the optical potential itself,[9] then Eq. (2.17b) should be valid as it stands. In this event the $\partial/\partial k$ would also differentiate the energy dependence of the optical potential.

III. WALLACE CORRECTIONS FOR SPIN-$\frac{1}{2}$ SCATTERING

The amplitude for the scattering of a spin-$\frac{1}{2}$ particle may be written (see, for example, Joachain[10]),

$$\hat{f} = f + \vec{\sigma}\cdot\hat{n}g\ , \tag{3.1}$$

where $\hat{n}$ is the normal to the scattering plane,

$$\hat{n} = (\vec{k}_i \times \vec{k}_f)/|\vec{k}_i \times \vec{k}_f| \quad . \tag{3.2}$$

Assuming parity conservation, the states with $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ are uncoupled, and in terms of the corresponding phase shifts $\delta^+(l)$ and $\delta^-(l)$ the partial wave decompositions of the spin-non-flip and spin-flip amplitudes are

$$f(\theta) = \frac{1}{2ik}\sum_l [(l+1)(e^{2i\delta^+(l)} - 1) + l(e^{2i\delta^-(l)} - 1)]P_l(\cos\theta) \quad , \tag{3.3a}$$

$$g(\theta) = -\frac{1}{2k}\frac{\partial}{\partial\theta}\sum_l (e^{2i\delta^+(l)} - e^{2i\delta^-(l)})P_l(\cos\theta) \quad . \tag{3.3b}$$

The impact parameter representation for f goes through identically[11] as for the spinless case of Sec. II, whereas for the spin-flip amplitude we note that the derivative with respect to θ in Eq. (3.3b) commutes with the Wallace operator W of Eq. (2.5b). Consequently, if

$$f(q) \equiv ik\int_0^\infty b\, db\, \Gamma_f(b)J_0(qb) = -ik\int_0^\infty b\, db b\left[\frac{d\Gamma_f(b)}{db}\right]\frac{J_1(qb)}{qb} \quad , \tag{3.4a}$$

$$g(q) \equiv -ik\cos(\theta/2)\int_0^\infty b\, db\, \Gamma_g(b)J_1(qb) = -ik\sin\theta\int_0^\infty b\, db[kb\Gamma_g(b)]\frac{J_1(qb)}{qb} \quad , \tag{3.4b}$$

the profile functions are connected to the phase shifts by

$$\Gamma_f(b) = 1 - (2l+1)^{-1}W[(l+1)e^{2i\delta^+(l)} + le^{2i\delta^-(l)}] \tag{3.5a}$$

$$\Gamma_g(b) = \tfrac{1}{2}iW(e^{2i\delta^+(l)} - e^{2i\delta^-(l)}) \quad . \tag{3.5b}$$

In contrast to the definition of Glauber,[1] who was concerned primarily with near-forward scattering, the presence of the explicit $\cos(\theta/2)$ factor in Eq. (3.4b) is necessary to yield a simple unitarity relation.[12] It also ensures that the spin-flip amplitude vanishes in the backward direction.

Just as Wallace has shown in the spinless case, the dominant eikonal correction is dynamical rather than imposed by unitarity, so that we shall for simplicity only consider the leading (unit) term in the expansion of Eq. (2.5b) for W. The profile functions then become

$$\Gamma_f(b) \simeq 1 - e^{i\bar{\chi}(b)}\cos[\Delta\chi(b)] + i\Gamma_g(b)/2kb \quad , \tag{3.6a}$$

$$\Gamma_g(b) \simeq -e^{i\bar{\chi}(b)}\sin[\Delta\chi(b)] \quad , \tag{3.6b}$$

where $\bar{\chi}$ and $\Delta\chi$ are related to $\delta(l = kb - \frac{1}{2})$ by

$$\bar{\chi}(b) = [\delta^+(l) + \delta^-(l)] \,, \tag{3.7a}$$

$$\Delta\chi(b) = [\delta^+(l) - \delta^-(l)] \quad . \tag{3.7b}$$

A potential with a spin-orbit component V_s as well as a central piece V_c

$$V(\vec{r}) = V_c(r) + V_s(r)\vec{\sigma}\cdot\vec{L} \quad , \tag{3.8}$$

does not mix the states $j = l \pm \frac{1}{2}$ and has diagonal elements

$$V^+(l,r) = V_c(r) + lV_s(r) \quad , \tag{3.9a}$$

$$V^-(l,r) = V_c(r) - (l+1)V_s(r) \quad . \tag{3.9b}$$

Though the effective potentials are l-dependent the WKB solution for $\delta^\pm(l)$ and its subsequent expansion in powers of V discussed in Sec. II should be equally valid. Therefore inserting Eq. (3.9) into Eq. (2.17a) the lowest order approximation for the eikonal phases is

$$\bar{\chi}_0(b) = \chi_c(b) - \tfrac{1}{2}\chi_s(b) \quad , \tag{3.10a}$$

$$\Delta\chi_0(b) = kb\chi_s(b) \quad , \tag{3.10b}$$

with

$$\chi_c(b) = -\frac{m}{k}\int_{-\infty}^{+\infty} V_c(r)dz \quad , \tag{3.11a}$$

$$\chi_s(b) = -\frac{m}{k}\int_{-\infty}^{+\infty} V_s(r)dz \quad . \tag{3.11b}$$

Apart from the $\cos(\theta/2)$ factor in Eq. (3.4b) these expressions differ from the analogous ones derived by Glauber[1] by the $i\Gamma_g/2kb$ in Eq. (3.6a) and the $\frac{1}{2}\chi_s$ term in Eq. (3.10a). The effect of these changes in the spin-non-flip amplitude is minimal because they cancel each other to first order in V_s and to second order they can be grouped to give

$$\Gamma_f(b) \approx 1 - e^{i\chi_c(b)}\cos[(k^2b^2 - \tfrac{1}{4})^{1/2}\chi_s(b)] \quad . \tag{3.12}$$

This differs from the Glauber result only by the replacement $(l+\frac{1}{2})^2 \to l(l+1)$ in the argument of the cosine.

The first Wallace correction follows from the substitution of Eq. (3.9) into Eq. (2.17b). Using the result that the differential operator does not act upon the combination kb, this leads to

$$\chi_1(b) = \chi_{cc}(b) - \chi_{cs}(b) + (k^2b^2 + \tfrac{1}{4})\chi_{ss}(b) \ , \tag{3.13a}$$

$$\Delta\chi_1(b) = kb[2\chi_{cs}(b) - \chi_{ss}(b)] \ , \tag{3.13b}$$

with phases defined by

$$\chi_{cc}(b) = -\frac{m^2}{2k^3}\left[1 + b\frac{\partial}{\partial b} - k\frac{\partial}{\partial k}\right] \times \int_{-\infty}^{+\infty}[V_c(r)]^2 dz \ , \tag{3.14}$$

and similarly for χ_{cs} and χ_{ss}.

The expressions in Eqs. (3.6), (3.10), and (3.13) can also be obtained from the spin-zero results of Sec. II using the commutation properties of the $\vec{\sigma}\cdot\vec{L}$ operator which give for arbitrary ϕ,

$$(\vec{\sigma}\cdot L + \tfrac{1}{2})^2 = (l+\tfrac{1}{2})^2 \ , \tag{3.15a}$$

$$\exp[i\phi(\vec{\sigma}\cdot\vec{L} + \tfrac{1}{2})/(l+\tfrac{1}{2})] = \cos\phi + i\sin\phi(\vec{\sigma}\cdot L + \tfrac{1}{2})/(l+\tfrac{1}{2}) \ . \tag{3.15b}$$

It is then convenient to rewrite the potential of Eq. (3.8) as

$$V(\vec{r}) = [V_c(r) - \tfrac{1}{2}V_s(r)] + V_s(r)(\vec{\sigma}\cdot\vec{L} + \tfrac{1}{2}) \ , \tag{3.16}$$

so that its square has a similar structure,

$$[V(\vec{r})]^2 = [V_c(r) - \tfrac{1}{2}V_s(r)]^2 + k^2b^2V_s^2(r) + 2[V_c(r) - \tfrac{1}{2}V_s(r)]V_s(r)(\vec{\sigma}\cdot L + \tfrac{1}{2}) \ . \tag{3.17}$$

Insertion of Eqs. (3.17) and (3.16) into Eqs. (2.17) and (2.2) followed by the application of $(\vec{\sigma}\cdot\vec{L} + \frac{1}{2})$ onto the plane wave states leads to the desired relations.

IV. EIKONAL EXPANSION FOR SPIN $\frac{1}{2}$ SCATTERING

Wallace has given two parallel derivations of the eikonal corrections in the spinless case, one involving the WKB approximation[4] as expounded in Secs. II and III, and the other based upon the expansion of the propagator around its eikonal limit.[3] This second method is rather tedious, even more so when the spin degrees of freedom are included, and we shall indicate only how the first Wallace correction may be derived in the case of spin-$\frac{1}{2}$ scattering.

Following closely the work of Wallace[3] the T matrix for the scattering from a state i to f is

$$T = -\frac{2\pi}{m}\hat{f}(\vec{q}) = \langle \vec{k}_f | V + V\mathcal{G}V | \vec{k}_i \rangle \ , \tag{4.1}$$

where $\vec{q} = \vec{k}_i - \vec{k}_f$ and the spin labels in the state have been suppressed. The inverse propagator has the representation

$$\mathcal{G}^{-1} = \frac{1}{2m}(k^2 - p^2) - V + i\epsilon \ , \tag{4.2}$$

where p is the particle momentum operator and $k^2/2m$ is its energy. Expanding $\vec{p}$ about the average momentum direction

$$\vec{k}_0 = \tfrac{1}{2}(\vec{k}_i + \vec{k}_f) \ , \tag{4.3}$$

but conserving energy leads to

$$(k^2 - p^2) = -(\vec{p} - k\vec{k}_0/k_0)^2 - 2k\vec{k}_0\cdot(\vec{p} - k\vec{k}_0/k_0)/k_0 \ . \tag{4.4}$$

In the spinless case the eikonal approximation of Glauber[1] follows if we drop the quadratic term and neglect the k/k_0 inside the other bracket. However, with the spin-orbit potential as in Eq. (3.8)

$$V(\vec{r}) = V_c(r) + V_s(r)\vec{\sigma}\cdot\vec{r}\times(-i\vec{\nabla}) \ , \tag{4.5}$$

the standard Glauber[1] result also requires that $-i\vec{\nabla}$ be replaced by $\vec{k}_0$ in the angular momentum operator of Eq. (4.5), thus eliminating the velocity dependence of the potential. Denote by $U(\vec{r})$ this approximate potential

$$U(\vec{r}) \equiv V_c(r) + V_s(r)\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0) \ . \tag{4.6}$$

The eikonal propagator is then

$$g^{-1} = \vec{v}\cdot(\vec{k}_0 - \vec{p}) - U \ , \tag{4.7}$$

where $\vec{v}$ is the velocity vector $k\vec{k}_0/mk_0$. The corresponding scattering wave function is

$$\psi^{\pm}_{\vec{k}_i}(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}}e^{i\chi_c^{+}(\vec{r})}e^{i\chi_s^{+}(\vec{r})\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0)} \ , \tag{4.8}$$

with the partial phase

$$\chi_c^{+}(\vec{r}) = -\frac{1}{v}\int_{-\infty}^{z} V_c(\vec{r}\,')dz' \ , \qquad (4.9)$$

and likewise for χ_s^{+}. For the outgoing states the integration for χ^{-} runs from z to $+\infty$, such that

$$\chi(b) = \chi^{+}(\vec{r}) + \chi^{-}(\vec{r}) \qquad (4.10)$$

for both the χ_c and χ_s of Eqs. (3.11).

The wave function of Eq. (4.8) leads directly to the scattering amplitude

$$T^0(q) = \langle \vec{k}_f \mid U \mid \psi^{+}_{\vec{k}_i} \rangle$$

$$= -iv\int d^2b \; e^{i\vec{q}\cdot\vec{b}}\Gamma^0(\vec{b}) \ , \qquad (4.11)$$

in which the profile function is

$$\Gamma^0(\vec{b}) = 1 - e^{i\chi_c(b)} e^{i\chi_s(b)\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0)}$$

$$= 1 - e^{i\chi_c(b)}\Big[\cos[bk_0\chi_s(b)] \qquad (4.12a)$$

$$+ i\frac{\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0)}{k_0 b}\sin[bk_0\chi_s(b)]\Big] \ . \qquad (4.12b)$$

Upon integration over the azimuthal angle of $\vec{b}$ we find the standard Glauber result discussed in Sec. III but with the $\cos(\theta/2)$ factor of Eq. (3.4b) due to the presence of the term $k_0 = k\cos(\theta/2)$ which arises from Eq. (4.6).

The first Wallace correction now consists of two parts, one coming from the approximation of the potential V by the U of Eq. (3.8), the other from the neglected terms in Eq. (4.4). Let us define

$$W \equiv V - U = V_s\vec{\sigma}\cdot[i\vec{\nabla}_p \times (\vec{p} - \vec{k}_0)] \quad (4.13)$$

and

$$N \equiv g^{-1} - W - \mathcal{G}^{-1}$$

$$= \frac{1}{2m}(p^2 - k^2) + \vec{v}\cdot(\vec{k}_0 - \vec{p}) \ . \qquad (4.14)$$

In the spirit of Wallace we seek to find the first correction $T^1(\vec{q})$ to Eq. (4.11) which is linear in either W or N but we shall omit terms of the type NW.

Formal manipulation with the Lippmann-Schwinger equation readily yields

$$T^1 = \langle \psi^{-}_{\vec{k}_f} \mid W \mid \psi^{+}_{\vec{k}_i} \rangle + \langle \vec{k}_f \mid UgNgU \mid \vec{k}_i \rangle \ . \qquad (4.15)$$

The evaluation of this expression is considerably more difficult than in the spinless case due to the extra term in the potential which requires that care be taken with the commutation properties. There are extensive mutual cancelations and the final result may be written[13]

$$T^1(\vec{q}) = -iv\int d^2b \; e^{i\vec{q}\cdot\vec{b}} e^{i\chi_c(b)} e^{i\chi_s(b)\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0)}\gamma^1(\vec{b}) \qquad (4.16a)$$

with

$$\gamma^1(\vec{b}) = -\tfrac{1}{2}\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0)[\chi_s(b)]^2 + \frac{i}{2kv^2}\left[1 + b\frac{\partial}{\partial b} - k\frac{\partial}{\partial k}\right]$$

$$\times \int_{-\infty}^{+\infty} dz[V_c^2 + k^2b^2V_s^2 + 2V_cV_s\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0)] \ . \qquad (4.16b)$$

In the formula for γ^1 we have omitted terms of order V^3 but also those of order λV^2, where $\lambda = [1 - \cos(\theta/2)]$, relying on the ansatz of Wallace[3] that these latter terms will be canceled in the evaluation of the second Wallace correction.

The result in Eq. (4.16b) is very similar to order V^2 to that obtained by the WKB approach of Sec. III. Thus the first term, which is of order $1/kb$ lower than a typical term in the expansion of the exponentials, arises from the residual $\frac{1}{2}\chi_s$ in Eq. (3.10a) in the development of the exponential in Eq. (3.6b). The other terms are the *dominant* ones in the first Wallace correction of Eq. (3.13) and correspond, in the method of Sec. III, to evaluating the square of the potential from Eq. (4.6),

$$[U(\vec{r})]^2 = [V_c(r)^2 + k_0^2b^2V_s(r)^2]$$

$$+ 2V_c(r)V_s(r)\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0) \ , \qquad (4.17)$$

rather than the exact form of Eq. (2.17). In the present approach the small χ_{cs} and χ_{ss} pieces are missing because of the neglect of terms such as NW in the expansion.

It is interesting to note that the replacement of $\vec{p}$

by $\vec{k}_0$ in the spin-orbit potential of Eq. (4.5) is not the only choice that we could have made for the eikonal propagator. The Schrödinger equation may also be simply resolved if we let in the potential of Eq. (3.8)

$$\vec{p} \to (\vec{p}\cdot\hat{k}_0)\hat{k}_0 \tag{4.18}$$

and

$$g^{-1} = \vec{v}\cdot(\vec{k}_0 - \vec{p}) - V_c(r) - V_s(r)\vec{\sigma}\cdot(\vec{b}\times\hat{k}_0)(\vec{p}\cdot\hat{k}_0) \ . \tag{4.19}$$

This is equivalent to the standard eikonal equation but with an effective potential

$$V_{\rm eff} = (1 - b^2V_s^2/v^2)^{-1} \times \left[V_c - mb^2V_s - \frac{1}{v}V_s \times (V_c - vk_0)\vec{\sigma}\cdot(\vec{b}\times\hat{k}_0)\right] \tag{4.20a}$$

$$= (V_c - mb^2V_s^2) + (V_s - mV_sV_c/k_0^2)\vec{\sigma}\cdot(\vec{b}\times\vec{k}_0) + O(V^3) \ . \tag{4.20b}$$

The derivation of Eq. (4.12) goes through unchanged except that there are extra contributions to χ_s and χ_c of order V^2 (or higher). It is easy to see that these are nothing other than the $\partial/\partial k$ part of the correction term in Eq. (4.16b). Since these are canceled by the $\partial/\partial b$ differentiation, it seems that Glauber's original ansatz of Eq. (4.6) for the spin-orbit potential is the most efficacious for the lowest order eikonal phase.

V. APPLICATION TO PROTON-NUCLEUS ELASTIC SCATTERING

Proton-nucleus scattering at intermediate energy, being one of the domains where the eikonal formalism is widely used, we have tested the corrections derived in the previous sections to the particular case of 800 MeV p-^{12}C elastic scattering. This nucleus was chosen because Coulomb effects on polarization have been shown to be very small,[14] which is not true for heavy nuclei. Note that the Coulomb interaction can be easily introduced into the formalism of Secs. II and III (Ref. 8), but we refrained from so doing because we were primarily interested in the importance of the first Wallace correction to the nuclear amplitudes f and g. In the same spirit, comparisons with existing data are postponed to a subsequent work.

The central and spin-orbit complex potentials were constructed from free nucleon-nucleon scattering amplitudes using the Kerman, McManus, and Thaler[15] formalism in first order. The resulting optical potentials were then reliably fitted with Fermi-type distributions. Explicitly for ^{12}C at 800 MeV,

$$V_c(r) = (12 - i66)F_c(r)\,MeV \ , \tag{5.1a}$$

$$V_s(r) = (11 + i14)\frac{1}{r}\frac{d}{dr}F_s(r)\,MeV\,\text{fm}^2 \ , \tag{5.1b}$$

with

$$F_\alpha(r) = \left[1 + \exp\left[\frac{r - R_\alpha}{a_\alpha}\right]\right]^{-1} , \tag{5.2}$$

where α stands for c or s. The best fit values for c_α and a_α are

$$R_c = 2.2 \text{ fm}, \ a_c = 0.49 \text{ fm} \ , \tag{5.3a}$$

$$R_s = 2.175 \text{ fm}, \ a_s = 0.53 \text{ fm} \ . \tag{5.3b}$$

With these potentials we have calculated the unpolarized differential cross section $d\sigma/d\Omega$, the polarization P, and the spin rotation parameter Q defined in terms of the amplitudes f and g by[17]

$$\frac{d\sigma}{d\Omega} = |f|^2 + |g|^2 \ , \tag{5.4a}$$

$$P = \frac{2\,\text{Re}(fg^*)}{|f|^2 + |g|^2} \ , \tag{5.4b}$$

$$Q = \frac{2\,\text{Im}(fg^*)}{|f|^2 + |g|^2} \ . \tag{5.4c}$$

The results are shown in Figs. 1 and 2, where we compare the lowest order eikonal approximation [Eqs. (3.10) and (3.11), dashed curve] to the results including the first Wallace correction in χ only [Eq. (3.13a), long dashed curve], or in χ and $\Delta\chi$ [Eqs. (3.13), solid curve]. Unpolarized differential cross sections (Fig. 2) are seen to be sensitive just to the χ_{cc} correction and the effects are sizable only in the vicinity of the minima or at large angles. In contrast, the polarization P and the spin rotation Q are significantly changed by the corrections χ_1 and $\Delta\chi_1$ taken separately, but including both of them leads to a curve very similar in shape to the original eikonal approximation. This is easily understood.

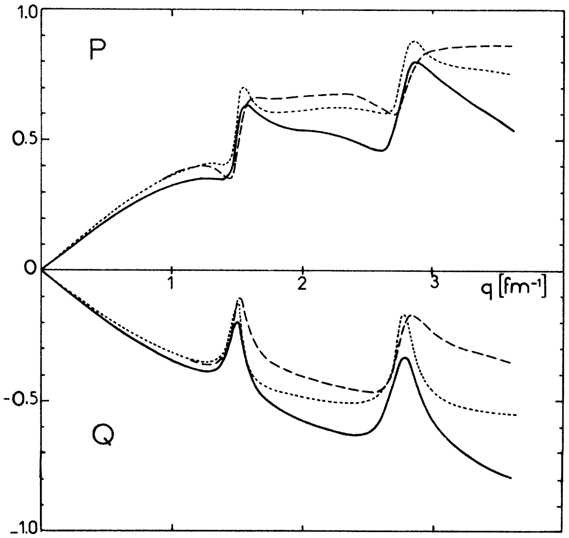

FIG. 1. Polarization parameter P and spin-rotation parameter Q for p-^{12}C elastic scattering at 800 MeV. The result of a calculation with the lowest order eikonal approximation is shown as the dashed curve, with the first Wallace correction in $\mathcal{X}(b)$ only as the long dashed curve and also including this modification in $\Delta\mathcal{X}(b)$ as the solid curve.

It is well known[1] that for a spin-orbit potential of the Thomas form,

$$V_s(r)=\xi\frac{1}{r}\frac{d}{dr}V_c(r) \ , \qquad (5.5)$$

which, in view of the parameter quoted in Eq. (5.3), is a good approximation to the present case, the lowest order eikonal phase satisfies the relation

$$\Delta\mathcal{X}_0(b)=k\xi\frac{\partial}{\partial b}\mathcal{X}_0(b) \ . \qquad (5.6)$$

On the other hand, the first Wallace corrections to the phases are related, to order ξ, by

$$\Delta\mathcal{X}_1(b)=k\xi\frac{\partial}{\partial b}\left[\mathcal{X}_1(b)+\frac{m^2}{k^3}\int_{-\infty}^{+\infty}V_c{}^2(r)dz\right] \ . \qquad (5.7)$$

If only the first term were present in Eq. (5.7), the total phases $\Delta\mathcal{X}$ and $\mathcal{X}$ would also satisfy the relation (5.6) and P and Q would be identical to their eikonal values except perhaps in the vicinity of the minima of the differential cross sections. This is not true in practice because the second term in Eq. (5.7) is small compared to the first only for such large values of the impact parameters b that the lowest order $\Delta\mathcal{X}_0$ is overwhelming. However, due to the

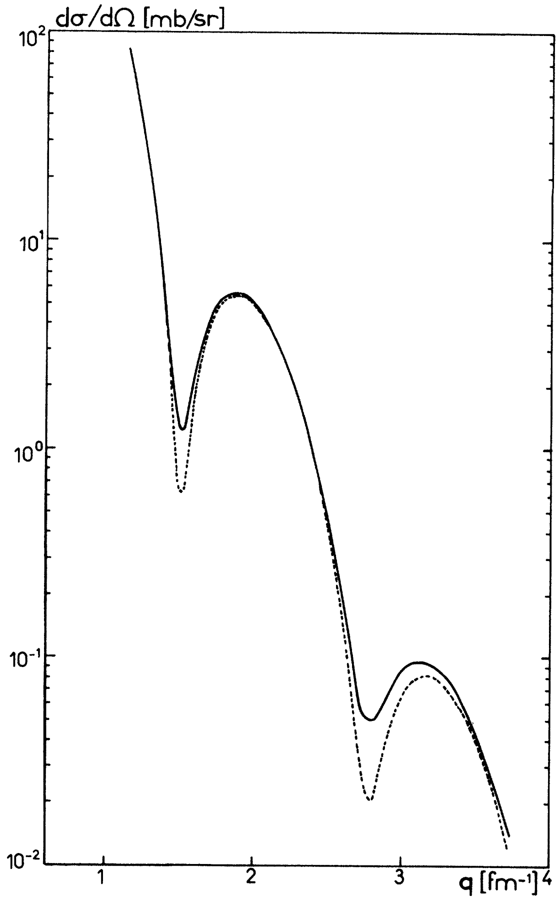

FIG. 2. Unpolarized differential cross section for p-^{12}C elastic scattering at 800 MeV. Here the long dashed curve would be undistinguishable from the solid one.

dominance of the imaginary part of V_c in Eq. (5.1a), $V_c{}^2$ and $\mathcal{X}_1$ are mainly real so that the difference between the relations (5.6) and (5.7) generates primarily an extra phase in the g amplitude with respect to the f. This results only in a slight change in P and Q with P^2+Q^2 almost invariant, and this is clearly seen in the graphs of Fig. 2. This property is independent of the sign of the real part of V_c.

VI. CONCLUSION AND DISCUSSION

Using two different techniques discussed by Wallace[3,4] we have been able to generalize his eikonal corrections to the case of spin-dependent potentials. The two methods give similar results except for minor terms, which are at least $1/kR$ smaller than the principal ones, arising form the truncation of the eikonal propagator expansion. In Sec. V we showed that the first correction terms of Eq. (3.13) could be significant in the description of the polarization parameters in medium energy elastic proton-nucleus scattering, and that in such a case it would be counterproductive to keep just the correction to the spin-independent amplitude. We contented ourselves with the elucidation of only the first correction though in the WKB approach of Sec. III spin is an "inessential complication" and we can work to as high an order in the Wallace expansion as in the spinless case.

The only other attempt to investigate eikonal corrections for spin-half (proton) scattering is that of Bleszynski and Osland.[16] Their method, which is based upon an expansion of the free propagator in powers of $1/k$ plus certain other *ad hoc* assumptions, was developed in terms of multiple scattering from the constituents of a nucleus. However, the single potential limit can easily be obtained by letting the constituent overlap in an independent particle model be total and taking the limit of a large number of nucleons, though this might weaken the justification for some of the approximations. Defining the spin-orbit potential as in Eq. (5.1b) through an auxiliary function $H(r)$

$$V_s(r) = \frac{1}{r}\frac{d}{dr}H(r) \ , \tag{6.1}$$

it follows from their work[18] that the first corrections to the eikonal phases should be

$$\mathcal{X}_1(b) = \chi_{cc}(b) + k^2b^2\chi_{ss} + R_1(b) \ , \tag{6.2a}$$

$$\Delta\mathcal{X}_1(b) = 2kb\,\chi_{cs}(b) + R_2(b) \ , \tag{6.2b}$$

where

$$R_1(b) = -\frac{m^2}{2k}\vec{\nabla}_b{}^2\int_{-\infty}^{+\infty} H^2(r)dz \ , \tag{6.3a}$$

$$R_2(b) = -\frac{m^2}{k^2}\frac{\partial}{\partial b}\int_{-\infty}^{+\infty} H(r)V_c(r)dz \ . \tag{6.3b}$$

Apart from the quantities R_1 and R_2, which cannot be expressed simply as functions of V_s, the relations in Eq. (6.2) coincide with those of the eikonal expansion derived in Sec. IV. To second order in the potential the R_1 term does not contribute to the scattering amplitude f in the forward direction. From an integration by parts it is seen to be equivalent to a term proportional to q^2H^2 which, in the Wallace[3] approach, is expected to be canceled by higher order corrections which we have not considered in the derivation of Sec. IV. On the other hand, the R_2 term does not vanish in the forward direction and will contribute to $\Delta\mathcal{X}_1$ though not significantly at large b. Since the long range parts are the least damped by the exponential in Eq. (3.6), the differences with our results should be minimized. It is, however, interesting to note that in the special case where $H(r) = \xi V_c(r)$ in Eq. (5.5) the R_2 contribution exactly cancels that of the second term in Eq. (5.7) so that Eq. (5.6) would be approximately valid for the total phases, and the Bleszynski and Osland[16] approach should lead to smaller noneikonal effects for the P and Q parameters than the ones presented in Fig. 2.

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