Alternative to Padé technique for solving scattering integral equations

Lauro Tomio and Sadhan K. Adhikari

Departamento de Física, Universidade Federal de Pernambuco, 50.000-Recife, Pe., Brazil

(Received 17 November 1980)

The aim of this work is to show how we can improve systematically the rate of convergence of a recently proposed method for iterative solution of scattering integral equations. The method relies on the introduction of an auxiliary equation containing an arbitrary function, whose kernel is much weaker than that of the original equation. The solution of the original equation is then expressed in terms of that of the auxiliary equation which is supposed to have a convergent iterative solution. In this work we introduce successive subtractions in the kernel in order to make it weaker. Such subtractions in the kernel are very simple to implement in practice and the present method appears to have some advantage over the other existing methods. We explain how to generalize the method for multichannel scattering equations. Using the present method we show how to modify a method by Fuda for three-particle scattering equations in order to have a method which uses the iterative solution only of nonsingular equations above the three-body breakup threshold. The method is used numerically to compute phase shifts, scattering lengths, and fully off-shell t matrix elements for neutron-deuteron scattering in the s-wave Amado model and for nucleon-nucleon scattering with the Reid $^1S_0$ potential. The iterative solution of the auxiliary equation is found to converge much faster than the conventional Padé technique in the case of the neutron-deuteron scattering and yields results with high precision.

I. INTRODUCTION

Iterative solutions of scattering integral equations are attractive because of their formal and numerical simplicity over methods using matrix inversions. A divergent iterative series can be summed by the use of Padé approximants.[1,2] The method of Padé approximants has been successfully used in the solution of few-body scattering integral equations.[3,4] An alternative method for iterative solutions of scattering equations was recently proposed.[5,6] Here we propose a generalization of the method of Refs. 5 and 6, which has certain advantages over the conventional Padé technique.

The method of Refs. 5 and 6 relies on solving an auxiliary equation whose kernel contains an arbitrary flexible function. The solution of the original equation is then related to that of the auxiliary equation. If the freedom in the choice of the arbitrary function is exploited the kernel of the auxiliary equation can be made sufficiently weak in order to have a convergent iterative solution. Another advantage of the method is that certain types of fixed point singularities can be removed from the kernel of the original equation so that the auxiliary equation has a less singular or nonsingular structure.[7]

Following Ref. 5, a single channel one variable partial wave Lippmann-Schwinger-type equation can be written as

$$t(p,r;E) = V(p,r)$$

$$+ \int dq\, q^2 f(q)\, V(p,q)\, G_0(q) t(q,r;E),$$

$$(1.1)$$

with $G_0(q) = (k^2 - q^2 + i\epsilon)^{-1}$, $k^2 = E$, in units $\hbar = 2\mu = 1$, where p, q, and r are momentum variables and μ is the reduced mass. Unless explicitly shown the integration limits in Eq. (1.1) and in the rest of the paper are from 0 to ∞. The function $f(q)$ is some weight function which in the case of s-wave Lippmann-Schwinger equations is a constant λ. As in Ref. 5 we introduce the auxiliary equation

$$\Gamma(p,r;E) = V(p,r) + \int dq\, q^2 A(p,q;E) \Gamma(q,r;E),$$

$$(1.2)$$

where

$$A(p,q;E) = [V(p,q) - V(p,k)\gamma(k,q)] f(q) G_0(q),$$

$$(1.3)$$

and where $\gamma(k,q)$ satisfies

$$\gamma(k,k) = 1.$$

$$(1.4)$$

The kernel A of Eq. (1.2) does not have the fixed point singularity of $G_0(q)$. We know from our previous experience that if the function γ is

conveniently chosen, the kernel A can be made very weak compared to that of Eq. (1.1).[5,6] Now $t(p,r;E)$ can be expressed in terms of $\Gamma(p,r;E)$ as[5,6]

$$t(p,r;E) = \left[\frac{\Gamma(p,k;E)}{\Gamma(k,k;E)} \, t(k) \, \frac{\Gamma(r,k;E)}{\Gamma(k,k;E)} \right]$$
$$+ \left[\Gamma(p,r;E) - \frac{\Gamma(p,k;E)\Gamma(k,r;E)}{\Gamma(k,k;E)} \right],$$

$$(1.5)$$

where

$$t(k) \equiv t(k,k;E)$$

$$= \Gamma(k,k;E)\left[1 - \int dq \, q^2 \gamma(k,q) f(q) G_0(q) \Gamma(q,k;E)\right]^{-1}.$$

$$(1.6)$$

In Eq. (1.2) we have introduced one subtraction in the kernel of Eq. (1.1) in order to make it weaker, and Eq. (1.2) with a proper choice of γ may have a convergent iterative solution. If the rate of convergence of the iterative solution of the auxiliary equation (1.2) is not satisfactory we show how we can improve the rate of convergence of the method by introducing further subtractions in the kernel. It is worthwhile to note that Eq. (1.5) has certain advantages as have been first pointed out by Kowalski and Noyes.[8] The quantity in the first set of square brackets is separable and is known in the literature as the Kowalski-Noyes approximation[8] for the t matrix, and it is exact for half-on-shell values of momentum variables. The term in the second set of square brackets is the residual term. This term is real and it is zero for half-on-shell values of momentum variables. A multichannel generalization of Eq. (1.5) was also given in Ref. 6.

The applicability of the present method is by no means limited to the case of scattering integral equations of the Lippmann-Schwinger type.[9] The present method can be applied to the case of general integral equations of the Fredholm type as has been demonstrated elsewhere.[9]

The method is applicable to the case of two- and three-body equations as has been shown in Refs. 5 and 6. In the case of Lippmann-Schwinger equations the fixed point singularity of the kernel of the original equation can be easily removed and we have a nonsingular auxiliary equation. The same is true below the breakup threshold for three-body equations. But above the breakup threshold it is difficult to write nonsingular three-particle scattering equations because of the appearance of complicated logarithmic singularities.[10] Fuda[11] proposed a method for solving three-particle scattering equations by solving

auxiliary nonsingular equations above and below the breakup threshold. We show that using ideas of Ref. 9 all the auxiliary equations of the method by Fuda can be solved by iteration. Hence it is possible to solve the three-particle scattering above the breakup threshold by solving auxiliary nonsingular equations by iteration.

The present method is simple to implement and for reasons discussed later is expected to yield accurate results. The generalization proposed in this paper maintains all the advantages of the previous papers.[5,6] The present method is equally good for on-shell, half-on-shell, and off-shell t matrix elements. In the case of three-body equations, of course, special care is needed to write nonsingular scattering equations above the three-body breakup threshold.[10] For example, one may use the technique of contour rotation[12,13] or the method by Fuda[11] in order to write nonsingular equations above the breakup threshold. It is known that iterative solution of scattering integral equations preserves the analytic structure of the solution.[14] Hence the correct analytic structure will be built in the approximate solution of the present method. Moreover, below the (three-body) breakup threshold each order of iterative solution of three-body equations preserves the constraints of unitarity.[7] Unlike in the method of Padé approximants[2] there are no spurious poles or singularities in the approximate solution of the present method.

We tested numerically the method in the simple case of s-wave elastic neutron-deuteron scattering with separable two-body interactions with Yamaguchi[15] form factors—commonly known as the Amado model[16]—for energies below the breakup threshold and also in the case of elastic nucleon-nucleon scattering with Reid $^1S_0$ interaction[17] and find that the method converges rapidly in both the cases. We find that the method converges significantly faster than the iterative solution of Refs. 5 and 6. We also find that in the case of neutron-deuteron Amado model the method converges much faster than the method of Padé approximants.[3] The present method is also slightly simpler to implement than the Padé technique because it is numerically somewhat more tedious to construct the Padé approximants using the iterative solution than to construct the solution using that of the auxiliary equation in the present method.

The plan of the paper is as follows. In Sec. II we give a brief description of the method. We show how to introduce successive subtractions in the kernel in order to make it weaker. In Sec. III we discuss the generality of the present approach and apply it to multichannel scattering

equations and to the three-body scattering above three-body breakup threshold. In Sec. IV we present numerical results for the s-wave neutron-deuteron scattering in the Amado model and for the nucleon-nucleon scattering for the Reid $^1S_0$ potential. Finally, in Sec. V we present a brief summary and concluding remarks.

II. THE METHOD

In Refs. 5 and 6 we introduced the auxiliary equation (1.2) and write the solution of Eq. (1.1) in terms of the solution of the auxiliary equation as in Eq. (1.5). In Eq. (1.2) we have made one subtraction in the kernel of Eq. (1.1) in order to make it weaker. With a proper choice of γ, Eq. (1.2) is supposed to have a convergent iterative solution. Here we show how to introduce successive subtractions in the kernel in order to improve the rate of convergence of the iterative scheme. Following Ref. 9 we introduce an auxiliary equation

$$\Gamma_1(p,r;E) = V(p,r) + \int dq\, q^2 B(p,q;E)\Gamma_1(q,r;E) ,$$

$$(2.1)$$

where

$$B(p,q;E) = A(p,q;E) - A(p,k_1;E)\gamma_1(k_1,q) , \quad k_1 \neq k.$$

$$(2.2)$$

The function $\gamma_1(k_1,q)$ is again an arbitrary flexible function such that

$$\gamma_1(k_1,k_1) = 1 , \qquad (2.3)$$

where k_1 is an arbitrary chosen point, if the kernel A [see Eq. (1.3)] does not have another fixed point singularity in the function $f(q)$.

Here we suppose that the kernel A has no other fixed point singularity, so we use the form B given by Eq. (2.2), and we note that in this case $B(p,k_1;E) = 0$, whereas A of Eq. (1.3), though nonsingular, is always nonzero. So we expect that B can be made much weaker than A and this will improve the rate of convergence of the auxiliary equation (2.1). Now it remains to relate the solution of Eq. (2.1) to that of Eq. (1.2). For this purpose we rewrite (1.2) as

$$\Gamma(p,r;E) = V(p,r) + \int dq\, q^2 B(p,q;E)\Gamma(q,r;E)$$

$$+ A(p,k_1;E) \int dq\, q^2 \gamma_1(k_1,q)\Gamma(q,r;E) .$$

$$(2.4)$$

Now using the method illustrated in Refs. 5 and 6 the solutions of Eqs. (2.1) and (2.4) are related

by

$$\Gamma(p,r;E) = \Gamma_1(p,r;E) + \mathcal{Y}_1(p,k_1;E)I(r;E) ,$$

$$(2.5)$$

where

$$\mathcal{Y}_1(p,r;E) = [\,\Gamma_1(p,r;E) - \Gamma_1(p,k;E)\gamma(k,r)\,]$$

$$\times f(r)\,G_0(r) \qquad (2.6)$$

and

$$I(r;E) = \int dq\, q^2 \gamma_1(k_1,q)\,\Gamma(q,r;E) . \qquad (2.7)$$

From Eqs. (2.5) and (2.6) we find that $I(r;E)$ satisfies

$$I(r;E) = I_1(r;E) + J_1(k_1;E)I(r;E) , \qquad (2.8)$$

where

$$I_1(r;E) = \int dq\, q^2 \gamma_1(k_1,q)\,\Gamma_1(q,r;E) \qquad (2.9)$$

and

$$J_1(r;E) = \int dq\, q^2 \gamma_1(k_1,q)\,\mathcal{Y}_1(q,r;E) . \qquad (2.10)$$

Now from Eqs. (2.5) and (2.8) we get

$$\Gamma(p,r;E) = \Gamma_1(p,r;E) + \frac{\mathcal{Y}_1(p,k_1;E)I_1(r;E)}{1 - J_1(k_1;E)} .$$

$$(2.11)$$

The method solves Eq. (2.1) by iteration and constructs the solution of Eq. (1.2) by Eqs. (2.6) and (2.9)–(2.11), and then constructs the solution of Eq. (1.1) by using Eq. (1.5). We have seen that the kernel A of Eq. (1.2) is already weak and may have a convergent Neumann series solution for two and three nucleon scattering equations. In the kernel B of Eq. (2.1) we have introduced another subtraction and we expect that Eq. (2.1) will have a more rapidly convergent iterative solution. After we construct the iterative solution for Γ_1 we need to evaluate only several integrals in order to find the t matrix elements. For example, in order to find the on-shell t matrix element $t(k)$ we need to evaluate only four integrals once the iterative solution of Γ_1 is known. If the rate of convergence of the iterative solution of Eq. (2.1) is not satisfactory it is easy to introduce further subtractions in the kernel in order to make it weaker.

In order to achieve this we repeat the same steps that give Eq. (2.11) from Eq. (2.1). We introduce another subtraction at the point k_2 and rewrite Eq. (2.1) as

$$\Gamma_1(p,r;E) = V(p,r) + \int dq\, q^2\, C(p,q;E)\Gamma_1(q,r;E)$$
$$+ B(p,k_2;E) \int dq\, q^2 \gamma_2(k_2,q)\Gamma_1(q,r;E) ,$$

$$(2.12)$$

where

$$C(p,q;E) = B(p,q;E) - B(p,k_2;E)\gamma_2(k_2,q) \quad (2.13)$$

with

$$\gamma_2(k_2,k_2) = 1 \quad (k_2 \neq k_1,k).$$

Now we introduce the auxiliary equation

$$\Gamma_2(p,r;E) = V(p,r) + \int dq\, q^2\, C(p,q;E)\,\Gamma_2(q,r;E) .$$

$$(2.14)$$

Relating Eq. (2.12) with Eq. (2.14) we have

$$\Gamma_1(p,r;E) = \Gamma_2(p,r;E) + \mathcal{Y}_2(p,k_2;k_1;E) I_2(r;E) ,$$

$$(2.15)$$

where

$$\mathcal{Y}_2(p,k_2;k_1;E) = [\Gamma_2(p,k_2;E) - \Gamma_2(p,k;E)\gamma(k,k_2)]\, f(k_2)\, G_0(k_2) - [\Gamma_2(p,k_1;E) - \Gamma_2(p,k;E)\gamma(k,k_1)]$$

$$\times f(k_1)\, G_0(k_1)\gamma_1(k_1,k_2) ,$$

and

$$I_2(r;E) = \int dq\, q^2 \gamma_2(k_2,q)\Gamma_1(q,r;E). \quad (2.17)$$

Using Eqs. (2.15) and (2.17) we get

$$I_2(r;E) = I_{22}(r;E) + J_2(k_2;E) I_2(r;E) , \quad (2.18)$$

where

$$I_{22}(r;E) = \int dq\, q^2 \gamma_2(k_2,q)\, \Gamma_2(q,r;E) , \quad (2.19)$$

and

$$J_2(k_2;E) = \int dq\, q^2 \gamma_2(k_2,q)\, \mathcal{Y}_2(q,k_2;k_1;E). \quad (2.20)$$

Now using Eqs. (2.15) and (2.18) we have

$$\Gamma_1(p,r;E) = \Gamma_2(p,r;E) + \frac{\mathcal{Y}_2(p,k_2;k_1;E) I_{22}(r;E)}{1 - J_2(k_2;E)} .$$

$$(2.21)$$

Using Eqs. (2.14)–(2.21) we can construct Γ_1 and hence the t matrix elements using Eqs. (2.6)–(2.11) and Eq. (1.5).

Although it is possible to improve the method to a desired degree of accuracy by introducing successive subtractions, in this paper we shall be limited to the consideration of two subtractions only; i.e., we shall consider only equations up to Eq. (2.11) in the following. With this introduction about the method we turn to a discussion about its applicability.

III. APPLICATIONS

In this section we consider examples where we can use the method of Sec. II. It is clear that the method is applicable to single channel Lippmann-

Schwinger-type equations. We can apply the present method to any type of Fredholm integral equations—single channel or multichannel, Lippmann-Schwinger type or not. First, we demonstrate the applicability of the method to the case of multichannel scattering equations and then to the case of a three-body method suggested by Fuda[11] in order to demonstrate the generality of the present method.

A. Multichannel scattering

The multichannel scattering equations in general have the same form as Eq. (1.1), but now the various variables have channel indices over and above the momentum labels. In explicit notation the multichannel generalization of Eq. (1.1) is[6]

$$t_{\beta\alpha}(p_\beta,r_\alpha;E) = V_{\beta\alpha}(p_\beta,r_\alpha)$$

$$+ \sum_\sigma \int dq_\sigma q_\sigma^2 f_\sigma(q_\sigma)\, V_{\beta\sigma}(p_\beta,q_\sigma)$$

$$\times G_\sigma(q_\sigma) t_{\sigma\alpha}(q_\sigma,r_\alpha;E) ,$$

$$(3.1)$$

where $G_\sigma(q_\sigma) = (k_\sigma^2 - q_\sigma^2 + i\epsilon)^{-1}$ carries the possible singularities of the σ channel, while $f_\sigma(q_\sigma)$ is a weight function, which is a constant for the s-wave Lippmann-Schwinger equation and k_σ is the on-shell momentum for channel σ.

Following Ref. 6 we can write the solution of (3.1) as

$$t_{\beta\alpha}(p_\beta,r_\alpha;E) = \Gamma_{\beta\alpha}(p_\beta,r_\alpha;E)$$

$$+ \sum_\sigma \Gamma_{\beta\sigma}(p_\beta,k_\sigma;E) I_{\sigma\alpha}(k_\sigma,r_\alpha;E) ,$$

$$(3.2)$$

where $I_{\sigma\alpha}$ is the solution of

$$I_{\beta\alpha}(k_\beta,r_\alpha;E) = d_{\beta\alpha}(k_\beta,r_\alpha;E)$$

$$+ \sum_\sigma d_{\beta\sigma}(k_\beta,k_\sigma;E) I_{\sigma\alpha}(k_\sigma,r_\alpha;E),$$

with

$$d_{\sigma\alpha}(k_\sigma,r_\alpha;E) = \int dq_\sigma q_\sigma^2 G_\sigma(q_\sigma) \gamma_\sigma(k_\sigma,q_\sigma)$$

$$\times \Gamma_{\sigma\alpha}(q_\sigma,r_\alpha;E). \tag{3.4}$$

In Eqs. (3.2) and (3.4), $\Gamma_{\beta\alpha}(p_\beta,r_\alpha;E)$ is the solution of the following auxiliary equation:

$$\Gamma_{\beta\alpha}(p_\beta,r_\alpha;E) = V_{\beta\alpha}(p_\beta,r_\alpha)$$

$$+ \sum_\sigma \int dq_\sigma q_\sigma^2 A_{\beta\sigma}(p_\beta,q_\sigma;E)$$

$$\times \Gamma_{\sigma\alpha}(q_\sigma,r_\alpha;E). \tag{3.5}$$

Here

$$A_{\beta\sigma}(p_\beta,q_\sigma;E) = [\, V_{\beta\sigma}(p_\beta,q_\sigma) - V_{\beta\sigma}(p_\beta,k_\sigma)\gamma_\sigma(k_\sigma,q_\sigma)]$$

$$\times f_\sigma(q_\sigma) G_\sigma(q_\sigma) \tag{3.6}$$

and $\gamma_\sigma(k_\sigma,q_\sigma)$ is a function which satisfies $\gamma_\sigma(k_\sigma,k_\sigma) = 1$.

Equations (3.2) to (3.6) are the fundamental equations of the multichannel formulation of Ref. 6. The final result, Eq. (3.2), can be written in the Kowalski-Noyes form[8] as has been shown in Ref. 6. In this subsection we introduce more subtractions in the kernel of the auxiliary equa-

tion (3.5) and writing the result in the Kowalski-Noyes form is not important from this point of view. In Eq. (3.5) we have introduced one subtraction in each channel. Of course, by taking some of the γ_σ's to be zero we can introduce one subtraction in some of the selected channels of Eq. (3.1). This is important because kernels of some of the channels may be very weak and may not need any subtraction so that we may take $\gamma_\sigma = 0$ for these channels. On the other hand, some of the channels can be very strong and may need more than one subtraction in order to have a rapidly convergent iterative solution of the auxiliary equation. A good example of this is the case of three-body equations with finite rank potentials. If we make an unitary pole expansion[18] for the two-body t matrix, the first term, called the unitary pole approximation (UPA) is the most important term. With such an expansion the three-body equations have the same structure as Eq. (3.1), where the channel containing the UPA gives a good account of the full solution and hence is more important and stronger compared to other channels. Hence a second subtraction in the kernel of the three-body equations in such channels could be useful in having an auxiliary equation with rapidly convergent iterative solution. Here we show how to introduce a second subtraction in Eq. (3.5).

For this purpose we rewrite Eq. (3.5) as

$$\Gamma_{\beta\alpha}(p_\beta,r_\alpha;E) = V_{\beta\alpha}(p_\beta,r_\alpha) + \sum_\sigma \int dq_\sigma q_\sigma^2 B_{\beta\sigma}(p_\beta,q_\sigma;E) \Gamma_{\sigma\alpha}(q_\sigma,r_\alpha;E)$$

$$+ \sum_\sigma A_{\beta\sigma}(p_\beta,k_{1\sigma};E) \int dq_\sigma q_\sigma^2 \gamma_{1\sigma}(k_{1\sigma},q_\sigma) \Gamma_{\sigma\alpha}(q_\sigma,r_\alpha;E), \quad k_{1\sigma} \neq k_\sigma \tag{3.7}$$

and introduce another auxiliary equation by

$$\Gamma_{1\beta\alpha}(p_\beta,r_\alpha;E) = V_{\beta\alpha}(p_\beta,r_\alpha) + \sum_\sigma \int dq_\sigma q_\sigma^2 B_{\beta\sigma}(p_\beta,q_\sigma;E) \Gamma_{1\sigma\alpha}(q_\sigma,r_\alpha;E), \tag{3.8}$$

where

$$B_{\beta\sigma}(p_\beta,q_\sigma;E) = A_{\beta\sigma}(p_\beta,q_\sigma;E) - A_{\beta\sigma}(p_\beta,k_{1\sigma};E)\gamma_{1\sigma}(k_{1\sigma},q_\sigma) \tag{3.9}$$

and $\gamma_{1\sigma}(k_{1\sigma},q_\sigma)$ is a function which satisfies $\gamma_{1\sigma}(k_{1\sigma},k_{1\sigma}) = 1$.

Now it is not difficult to relate the solutions of Eqs. (3.7) and (3.8). Using methods of Ref. 9 we have

$$\Gamma_{\beta\alpha}(p_\beta,r_\alpha;E) = \Gamma_{1\beta\alpha}(p_\beta,r_\alpha;E) + \sum_\sigma [\Gamma_{1\beta\sigma}(p_\beta,k_{1\sigma};E) - \Gamma_{1\beta\sigma}(p_\beta,k_\sigma;E)\gamma_\sigma(k_\sigma;k_{1\sigma})] G_\sigma(k_{1\sigma})f_\sigma(k_{1\sigma}) I_{1\sigma\alpha}(k_{1\sigma},r_\alpha),$$

$$\tag{3.10}$$

where

$$I_{1\sigma\alpha}(k_{1\sigma},r_\alpha) = \int dq_\sigma q_\sigma^2 \gamma_{1\sigma}(k_{1\sigma},q_\sigma) \Gamma_{\sigma\alpha}(q_\sigma,r_\alpha;E) \tag{3.11}$$

satisfies the following set of linear equations:

$$I_{1\beta\alpha}(k_{1\beta},r_\alpha) = J_{\beta\alpha}(k_{1\beta},r_\alpha) + \sum_\sigma \left[J_{\beta\sigma}(k_{1\beta},k_{1\sigma}) - J_{\beta\sigma}(k_{1\beta},k_\sigma)\gamma_\sigma(k_\sigma,k_{1\sigma}) \right] G_\sigma(k_{1\sigma})f_\sigma(k_{1\sigma})I_{1\sigma\alpha}(k_{1\sigma},r_\alpha) ,$$
(3.12)

where

$$J_{\beta\alpha}(k_{1\beta},r_\alpha) = \int dq_\beta q_\beta^{\,2}\gamma_{1\beta}(k_{1\beta},q_\beta)\,\Gamma_{1\beta\alpha}(q_\beta,r_\alpha;E).$$
(3.13)

Equations (3.7) to (3.13) constitute the multichannel generalization of the method of Sec. II. As before the kernel B of Eq. (3.8) is expected to be weak and Eq. (3.8) is expected to have a rapidly convergent iterative solution. The solution for the t matrix is constructed by using the iterative solution of Eq. (3.8). With this discussion of the multichannel scattering equations we turn to a discussion of the Fuda method[11] for solving the three-body equations.

B. Fuda's method

Fuda[11] proposed a method for solving three-body equations with the help of auxiliary nonsingular equations. He considers three spinless bosons interacting via separable two-body transition operators

$$t(E) = |g\rangle\tau(E)\langle g|.$$
(3.14)

The half-off-shell partial wave three-body equations with this interaction are

$$X_L(q,k;E) = Z_L(q,k;E) + \int_0^\infty Z_L(q,q';E)q'^2 dq'\tau(E - \tfrac{3}{4}q'^2)X_L(q',k;E) ,$$
(3.15)

where

$$Z_L(q,q';E) = \int_{-1}^1 dx\, P_L(x)g(|\tfrac{1}{2}\vec{q}+\vec{q}'|)g(|\tfrac{1}{2}\vec{q}'+\vec{q}|)(E - q^2 - q'^2 - \vec{q}\cdot\vec{q}' + i\epsilon)^{-1} ,$$
(3.16)

where x is the cosine of the angle between $\vec{q}$ and $\vec{q}'$. Fuda separates the logarithmic singularities of Z_L by using the relation

$$Z_L(q,q';E) = W_L(q,q';E) + Y_L(q,q';E) ,$$
(3.17)

where

$$W_L(q,q';E) = \int_{-1}^1 \frac{dx\,P_L(x)}{E + i\epsilon - q^2 - q'^2 - \vec{q}\cdot\vec{q}'} \left\{ g(|\tfrac{1}{2}\vec{q}+\vec{q}'|)g(|\tfrac{1}{2}\vec{q}'+\vec{q}|) \right.$$
$$\left. - \Theta(c-q)g[(E-\tfrac{3}{4}q^2)^{1/2}]g[(E-\tfrac{3}{4}q'^2)^{1/2}]\Theta(c-q') \right\}$$
(3.18)

and

$$Y_L(q,q';E) = \Theta(c-q)g[(E-\tfrac{3}{4}q^2)^{1/2}]\frac{2}{qq'}Q_L\left(\frac{E+i\epsilon-q^2-q'^2}{qq'}\right)$$
$$\times \Theta(c-q')g[(E-\tfrac{3}{4}q'^2)^{1/2}],$$
(3.19)

with $c = (4E/3)^{1/2}$. The term Y_L contains the logarithmic singularities. Using Eqs. (3.15) and (3.17) we arrive at

$$X_L(q,k;E) = R_L(q,k;E)$$
$$+ \int_0^\infty dq'q'^2 R_L(q,q';E)$$
$$\times \tau(E-\tfrac{3}{4}q'^2)X_L(q',k;E) , \quad (3.20)$$

where R_L satisfies

$$R_L(q,q';E) = W_L(q,q';E)$$
$$+ \int_0^c Y_L(q,q'';E)q''^2 dq''$$
$$\times \tau(E-\tfrac{3}{4}q''^2)R_L(q'',q';E). \quad (3.21)$$

The logarithmic singularities of Eq. (3.21) can be treated by iterating the equation once to give

$$R_L(q,q';E) = B_L(q,q';E) + \int_0^c V_L(q,q'';E)q''^2 dq''R_L(q'',q';E),$$
(3.22)

where in operator form $B_L = W_L + Y_L\tau W_L$ and $V_L = Y_L\tau Y_L\tau$. Explicit representations of B_L and V_L are given in Ref. 11. Equation (3.22) is a nonsingular equation and may have a convergent iterative solution. It is interesting to note that the structure of Eq. (3.22) is not of the Lippmann-Schwinger type, but even in this case the present method can be applied. If the iterative solution of Eq. (3.22) does not converge, or if it converges slowly, we can introduce a subtraction in the kernel of Eq. (3.22) and rewrite it as

$$R_L(q,q';E) = B_L(q,q';E) + \int_0^c q''^2 dq'' A_L(q,q'';E) R_L(q'',q';E) + V_L(q,k;E) \int_0^c q''^2 dq'' \gamma(k,q'') R_L(q'',q';E),$$

$$(3.23)$$

where

$$A_L(q,q'';E) = [\, V_L(q,q'';E) - V_L(q,k;E)\gamma(k,q'')\,].$$

$$(3.24)$$

Now, following Ref. 9, we introduce the auxiliary equations

$$\Gamma_{1L}(q,q';E) = B_L(q,q';E)$$

$$(3.25)$$

$$+ \int_0^c A_L(q,q'';E) q''^2 dq'' \Gamma_{1L}(q'',q';E),$$

$$\Gamma_{2L}(q,q';E) = V_L(q,q';E)$$

$$+ \int_0^c A_L(q,q'';E) q''^2 dq'' \Gamma_{2L}(q'',q';E).$$

$$(3.26)$$

We can express the solution of Eq. (3.23) in terms of those of Eqs. (3.25) and (3.26) by

$$R_L(q,q';E) = \Gamma_{1L}(q,q';E) + \Gamma_{2L}(q,k;E) I(k,q'),$$

$$(3.27)$$

where

$$I(k,q') = \int_0^c \gamma(k,q'') q''^2 dq'' R_L(q'',q';E)$$

$$= \frac{\int_0^c \gamma(k,q) q^2 dq\, \Gamma_{1L}(q,q';E)}{1 - \int_0^c \gamma(k,q) q^2 dq\, \Gamma_{2L}(q,k;E)}. \quad (3.28)$$

Although Eq. (3.23) is not of the Lippmann-Schwinger type, the present method reduces it to Eqs. (3.25) and (3.26), which are to be solved by iteration, and then we construct the final solution through (3.27). Of course, as Eq. (3.23) is not of the Lippmann-Schwinger type, we have to solve two equations, Eqs. (3.25) and (3.26), by iteration. It is interesting to note that the kernels of these two auxiliary equations are the same, but the inhomogeneous terms are different. Once R_L is known we have to solve Eq. (3.20), which is of the Lippmann-Schwinger type , and the usual reduction technique of Refs. 5-7 is applicable. So by introducing an auxiliary equation we can solve Eq. (3.20) by iteration. Through the discussion of the present section we demonstrate two things. First, we develop a method to solve three-body scattering equations above the three-body breakup threshold, through the iterative solution of non-singular auxiliary equations. Second, the present method is applicable to general Fredholm equations and not only to equations of the Lippmann-Schwinger type. Before ending this section it is

worthwhile to comment that we can also remove the singularities of Eq. (3.15) by the method of contour rotation[12,13] and apply the reduction technique of Sec. II directly to this equation and solve it by the use of the iterative solution of nonsingular equations on the rotated contour.

IV. NUMERICAL RESULTS

Now we are left with the problem of choosing the function γ. In Refs. 5 and 6 we were mainly concerned with various ad hoc and intuitive attempts to choose γ. We shall consider in this paper the only choice, called choice D in Ref. 6, based on the idea of minimizing the square of the kernel of the auxiliary equations. In Eq. (1.2) a sufficient condition[19] for having a convergent iterative solution is

$$\int dq \left\{ \int dp [q^2 A(p,q;E)]^2 \right\} < 1. \quad (4.1)$$

Condition (4.1) is much stronger than the necessary condition for convergence.[18] We can minimize the left hand side of Eq. (4.1) by minimizing the integrand in the curly brackets of this expression. With a hope of getting a weaker and more general condition, we choose γ in order to minimize

$$\int dp\, \omega(p)[\, V(p,q) - V(p,k)\gamma(k,q)]^2 \quad (4.2)$$

with respect to small variations δ of γ, where $\omega(p)$ is a weight function. Hence we claim

$$\delta \int dp\, \omega(p)[\, V(p,q) - V(p,k)\gamma(k,q)]^2 = 0,$$

$$(4.3)$$

which yields the following functional form of γ:

$$\gamma(k,q) = \frac{\int \omega(p) dp\, V(p,k) V(p,q)}{\int \omega(p) dp\, V(p,k) V(p,k)}. \quad (4.4)$$

An interesting observation at this point is that for a one term separable potential the kernel A with choice (4.4) of γ is identically equal to zero and the zeroth iteration of the auxiliary equation (1.2) leads to the exact result for the t matrix. In such a case the Padé approximant gives exact results after two iterations,[2] and the present method is superior to the method of Padé approximants. In cases where the potential can be approximately represented by a single separable term, and this is the case in many of the physi-

cally interesting scattering equations, the present method is expected to work better than the method of Padé approximants.

Similarly, if we apply the same idea, which gives Eq. (4.4), to the kernel of Eq. (2.1), we get in a similar way the following functional form of γ_1:

$$\gamma_1(k_1,q) = \frac{\int dp\, \omega_1(p)A(p,q;E)A(p,k_1;E)}{\int dp\, \omega_1(p)A(p,k_1,E)A(p,k_1;E)},$$

(4.5)

where $\omega_1(p)$ is some weight function. In the case of multichannel equations the choice of γ_σ in Eq. (3.6) is given by the following generalization of Eq. (4.4) to the case of multichannel scattering equations:

$$\gamma_\sigma(k_\sigma,q) = \frac{\sum_\alpha \int dp\, \omega_\sigma(p)\, V_{\alpha\sigma}(p,k_\sigma)\, V_{\alpha\sigma}(p,q)}{\sum_\alpha \int dp\, \omega_\sigma(p)\, V_{\alpha\sigma}(p,k_\sigma)\, V_{\alpha\sigma}(p,k_\sigma)}.$$

(4.6)

Now it is not difficult to write the functional form for γ for other auxiliary equations. In this work we use a simple analytic form for $\omega(p)$, e.g., $\omega(p)=p^n$, where n is small integer, e.g., $1, 2, 3, \ldots$. Such a choice enhances or suppresses parts of the integrals in Eqs. (4.4)–(4.6) and hence generates a wide class of γ. It should be noted that an important restriction on the choice of $\omega(p)$ should be such that the integrals in Eqs. (4.2)–(4.6) are well defined and convergent.

We apply the present method to compute numerically the t matrices for nucleon-nucleon elastic scattering with the Reid $^1S_0$ potential[17] and for neutron-deuteron scattering in the Amado model.[16] The Lippmann-Schwinger equation that we solve in the case of the Reid $^1S_0$ potential has appeared in Ref. 5 and we do not write it explicitly here. The Amado model equations that we shall use have appeared in Ref. 6 and for the sake of completeness we quote them here.

The equations for neutron-deuteron elastic scattering in the Amado model are given by

$$X_{nn'}^{L,S}(p,p';E) = Z_{nn'}^{L,S}(p,p';E) + \sum_m \int_0^\infty q^2 dq\, Z_{n,m}^{L,S}(p,q)f_m(q)G_m(q)X_{mn'}^{L,S}(q,p';E)$$

(4.7)

with

$$Z_{nn'}^{L,S}(p,p';E) = \frac{8\pi^2}{3}\, J_{nn'}^S \int_{-1}^1 \frac{dx\, P_L(x)g_n(|\tfrac{1}{2}\vec{p}+\vec{p}'|)g_{n'}(|\tfrac{1}{2}\vec{p}'+\vec{p}|)}{p^2+p'^2+pp'x-E-i\epsilon},$$

(4.8)

where x is the cosine of the angle between $\vec{p}$ and $\vec{p}'$, and $g_n(q)$ are the Yamaguchi form factors given by

$$g_n(q) = N_n(q+\beta_n^2)^{-1},$$

(4.9)

where

$$N_n = \pi^{-1}[|\alpha_n|\beta_n(\alpha_n+\beta_n)^3]^{1/2}, \quad n=0,1$$

(4.10)

where α_0^2 is the deuteron binding energy and α_1^2 is the energy of the singlet state. Here

$$G_0(p) = (k^2-p^2+i\epsilon)^{-1},$$

(4.11)

$$G_1(p) = -\tfrac{3}{4}\{|\alpha_1| + [\alpha_0^2+\tfrac{3}{4}(p^2-k^2)]^{1/2}\}^{-2}$$

(4.12)

and

$$f_n(p) = -\frac{2}{\pi}\frac{\{\beta_n+[\alpha_0^2+\tfrac{3}{4}(p^2-k^2)]^{1/2}\}\{|\alpha_n|+[\alpha_0^2+\tfrac{3}{4}(p^2-k^2)]^{1/2}\}}{|\alpha_n|(\alpha_n+\beta_n)\{2\beta_n+\alpha_n+[\alpha_0^2+\tfrac{3}{4}(p^2-k^2)]^{1/2}\}}$$

(4.13)

and J's are the spin-isospin overlap factors. The nonzero spin-isospin overlap factors are given by $J_{00}^{1/2} = J_{11}^{1/2} = \tfrac{1}{4}$, $J_{01}^{1/2} = J_{10}^{1/2} = -\tfrac{3}{4}$, and $J_{00}^{3/2} = -\tfrac{1}{2}$. The on-shell three-body momentum k is defined by

$$E = \tfrac{3}{4}k^2 - \alpha_0^2.$$

(4.14)

Now the auxiliary equations for Γ's are given by

$$\Gamma_{nn'}^{LS}(p,p';E) = Z_{nn'}^{LS}(p,p';E) + \sum_m \int q^2 dq\, A_{nm}^{LS}(p,q;E) \times \Gamma_{mn'}^{LS}(q,p';E),$$

(4.15)

with

$$A_{nm}^{LS}(p,q;E) = [Z_{nm}^{LS}(p,q;E) - Z_{nm}^{LS}(p,k';E)\gamma_m(k',q)] \times f_m(q)G_m(q).$$

(4.16)

If we would like to introduce more subtractions at this stage, it would be straightforward to write the auxiliary equations and we need not show them explicitly here. This completes the definition of the basic equations in the Amado model.

In order to see how the method works in practice we now present numerical results for elastic

nucleon-nucleon scattering with the Reid soft-core $^1S_0$ potential and for elastic neutron-deuteron s-wave scattering in the Amado model. The numerical parameters for these two problems are defined in Refs. 5 and 6. We use $\hbar^2/m = 41.47$ MeV fm^2, where m is the nucleon mass. We convert momentum space integrals in the range 0 to ∞ to integrals in x in the range -1 to $+1$ by the transformation

$$q = c \, \frac{1-x}{1+x} \, , \tag{4.17}$$

and transform the x integrals to summation by using Gauss-Legendre quadratures.

First we present numerical results for the Reid $^1S_0$ potential. In Eq. (4.17) we took $c = 5$ fm^{-1} and approximated the x integral in the range -1 to $+1$ by 48 point Gauss-Legendre quadratures, which was enough for obtaining results with a high degree of accuracy. In order to make a numerical calculation we have to choose γ, γ_1, and k_1 of Eqs. (1.3) and (2.2). For ω and ω_1 in Eqs. (4.4) and (4.5) we choose simple weight factors such as p^n, where n is a small positive integer. We found some γ's of Eq. (4.4) which gave good convergence but in our calculation we use $\gamma = 1.0$ because in this case the convergence was equally good and it is simpler to use. It was previously found that such $\gamma = 1.0$ gives good convergence for a wide range of potentials in s-wave Lippmann-Schwinger equations.[5,20] Next we are left with choosing $\omega_1(p)$ and k_1 of Eq. (4.5). We found that a wide range of $\omega_1(p)$ and k_1 gave good convergence. This is because γ_1 of Eq. (4.5), by construction, is expected to give a good convergence rate of the auxiliary equation. After some experimentation we found that at zero energy $\omega_1(p) = p^3$ and $k_1 = 18.4$ fm^{-1} gave the best convergence. We

stress that other factors of ω_1, for example, $\omega_1(p) = p, p^2$, also gave good convergence for a certain range of k_1. The convergence in this case is particularly striking because the Born series of the original s-wave Lippmann-Schwinger equation diverges very strongly at zero energy—the limit μ of the ratio of successive terms in the Born series after a large number of iterations being -15.9. We repeated the calculation with the same γ, γ_1, and k_1 at other energies. The results for the scattering length and phase shifts in this case for various iterations are shown in Table I. The off-shell and half-on-shell t matrix elements at zero and other energies converges equally rapidly. In Figs. 1(a) and 1(b) we show the convergence of the off-shell t matrix elements $t(p, 0.77)$ and $t(0.77, p)$ at $E_{c.m.} = 72$ MeV for different values of iterations N. We use the same definition of t matrix elements as in Ref. 5. In all tables and figures in this paper $N = 0$ corresponds to no iterations and corresponds to keeping the "Born" term ($\Gamma_1 = V$) in the auxiliary equations. It is easy to check that the convergence of the present method is significantly faster than that of Ref. 20 where only one subtraction is used in a half-shell version of the present method.

Next we present numerical results for the Amado model below the three-body breakup threshold. In this case we took $c = 0.1$ fm^{-1} in Eq. (4.17) and approximated the x integral in the range -1 to $+1$ by a 32 point Gauss-Legendre quadrature. In the case of the spin quartet Amado model the equations are of the single channel type and hence the formulation of Sec. II can be directly applied. But in the case of the spin doublet Amado model the equations couple two channels and hence are of the multichannel type. We introduce two subtractions in each of these cases. In the spin

TABLE I. The on-shell phase shifts for the Reid $^1S_0$ potential for different N. Entries for zero energy are scattering lengths in fermis.

|	$E_{c.m.}$ (MeV)							
N	0	12	24	48	72	104	152	176
0	0.067 44	−0.061 74	−0.104 24	−0.175 50	−0.235 39	−0.303 25	−0.387 28	−0.423 46
1	−1.136 9	0.243 50	0.184 09	0.052 48	−0.062 82	−0.192 14	−0.348 79	−0.415 11
2	−12.162	0.781 71	0.613 41	0.375 79	0.202 36	0.023 26	−0.182 16	−0.266 74
3	−16.931	0.852 50	0.675 60	0.430 15	0.252 32	0.069 05	−0.141 27	−0.228 03
4	−17.145	0.859 64	0.683 28	0.438 41	0.260 90	0.077 84	−0.132 48	−0.219 33
5	−17.139	0.860 51	0.684 41	0.439 88	0.262 62	0.079 79	−0.130 30	−0.217 09
6	−17.143	0.860 65	0.684 60	0.440 15	0.262 95	0.080 19	−0.129 81	−0.216 57
7	−17.151	0.860 71	0.684 66	0.440 22	0.263 03	0.080 30	−0.129 68	−0.216 42
8	−17.147	0.860 70	0.684 66	0.440 22	0.263 05	0.080 33	−0.129 63	−0.216 37
9	−17.146	0.860 69	0.684 65	0.440 22	0.263 05	0.080 33	−0.129 63	−0.216 37
10	−17.147	0.860 69	0.684 65	0.440 23	0.263 05	0.080 33	−0.129 63	−0.216 36
11	−17.147	0.860 69	0.684 65	0.440 23	0.263 05	0.080 33	−0.129 63	−0.216 36
12	−17.147	0.860 69	0.684 65	0.440 23	0.263 05	0.080 33	−0.129 63	−0.216 36
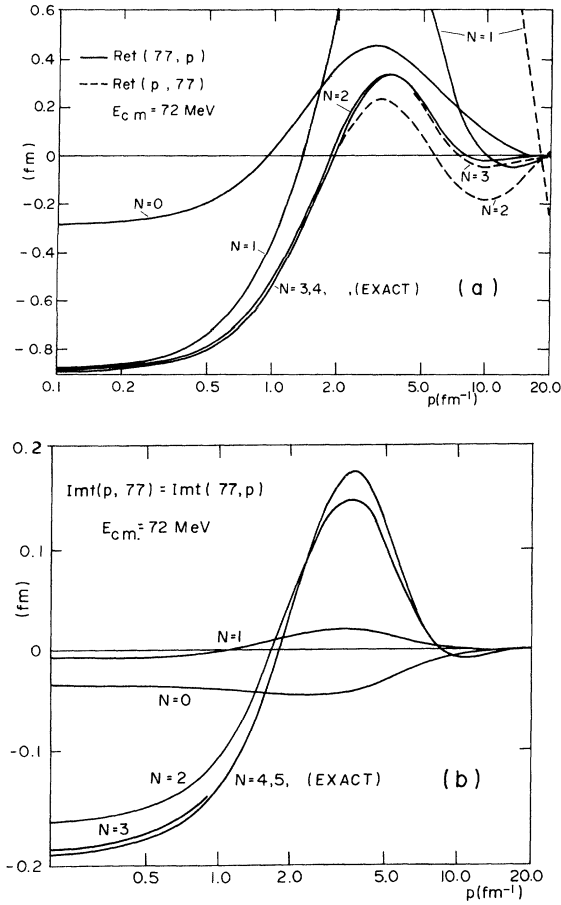
FIG. 1. (a) The real and (b) the imaginary parts of the off-shell t matrix elements for the Reid $^1S_0$ potential for different N.

quartet case the two subtractions are introduced in the only existing channel and in the spin doublet case one subtraction is introduced in each of the two channels. Next we have to choose γ, γ_1, and k_1 (k is always the on-shell point). Convergence was much easier to obtain in this case than in the case of the Reid soft core $^1S_0$ potential. This is because the original multiple scattering series diverges slowly here. Equation (4.7) for the spin quartet state diverges at zero incident neutron energy, the limit of the ratio of successive terms in the multiple scattering series μ being -2.68. In the spin doublet case the corresponding value of μ is 2.45. We may recall that in the case of the Reid $^1S_0$ potential the value of μ was -15.9. We use the same functional form for γ and γ_1 and we find after some experimentation that the best convergence was obtained for (a) the spin $\frac{3}{2}$ case with $\omega(p)=p^3$, $\omega_1(p)=p^3$, and $k_1=0.3$ fm^{-1}; and (b) the spin $\frac{1}{2}$ case with $\omega(p)=p^3$, $\omega_1(p)=p^3$, and k_1 $=0.1102$ fm^{-1} (this is one of the mesh points). Very good convergence was obtained with other sets of parameters also. We recall that in the spin quartet case both subtractions are introduced in the same channel—there is only one channel in this case. But in the spin $\frac{1}{2}$ case the first subtraction is introduced in the deuteron channel and the second subtraction is introduced in the channel containing the spin singlet virtual nucleon-nucleon state. Table II shows the result for scattering length and phase shifts corresponding to incident neutron energy $E_{\rm lab}=0$, 2.45, and 3.27 MeV. We compare the zero energy result with that obtained by Brady and Sloan[3] with the use of

TABLE II. The on-shell phase shifts for the spin doublet and spin quartet neutron–deuteron scattering in the s-wave Amado model for different N. Entries for zero energy are scattering lengths in fermis.

		$E_{\rm lab}$ (MeV)						
		Spin doublet				Spin quartet		
		0				0		
N	Padé	Present calc.	2.45	3.27	Padé	Present calc.	2.45	3.27
0		5.522 57	1.916 94	1.708 27		6.283 40	1.997 24	1.878 47
1		−7.213 78	2.967 34	2.820 06		6.321 99	2.003 50	1.889 16
2	7.08	−2.509 14	2.919 61	2.841 45	6.642	6.317 83	2.002 93	1.887 55
3		−1.096 34	2.852 21	2.786 28		6.317 93	2.002 90	1.887 38
4	2.55	−1.038 04	2.840 34	2.767 86	6.321	6.317 92	2.002 90	1.887 36
5		−1.037 73	2.839 42	2.766 49		6.317 92	2.002 90	1.887 36
6	−0.68	−1.035 74	2.839 34	2.766 79	6.317	6.317 92	2.002 90	1.887 36
7		−1.035 44	2.839 35	2.766 88		6.317 92	2.002 90	1.887 36
8	−1.01	−1.035 41	2.839 35	2.766 88	6.317	6.317 92	2.002 90	1.887 36
9		−1.035 40	2.839 35	2.766 88		6.317 92	2.002 90	1.887 36
10	−1.04	−1.035 40	2.839 35	2.766 88		6.317 92	2.002 90	1.887 36

Padé approximants. We find that the rate of convergence of the present method is significantly faster than that obtained by the use of Padé approximants. The convergence is much faster than that obtained by Whiting and Fuda[20] who use essentially a half-shell version of the present method with one subtraction only and with an *ad hoc* choice of γ. The fully off-shell t matrix elements also converges very rapidly. In Fig. 2 we show some fully off-shell t matrix elements at $E_{lab} = 0$ MeV for the spin quartet state, calculated by the use of formulas presented in Secs. I and II.

From Tables I and II, and Figs. 1 and 2, we see that in both cases the convergence of the iterative solution is very good. By increasing the number of integration points we have verified that the results of Tables I and II are correct to the number of places quoted, and the numerical error in the converged result is less than 0.001% for the Reid $^1S_0$ potential and is less than 0.0001% in the s-wave Amado model (both spin quartet and spin doublet cases). (All the calculations were carried out in double precision in a Digital computer.) The only numerical work in this method is the evaluation of a small number of integrals whereas other methods usually need calculation of determinants or inverse of matrices. So there is less chance of making a numerical error in this method. Hence the present method is expected to be simpler and more accurate compared to methods using calculations of determinants and inversion of matrices,[21] especially of large dimension,[13] as needed in other methods including the method of Padé approximants.
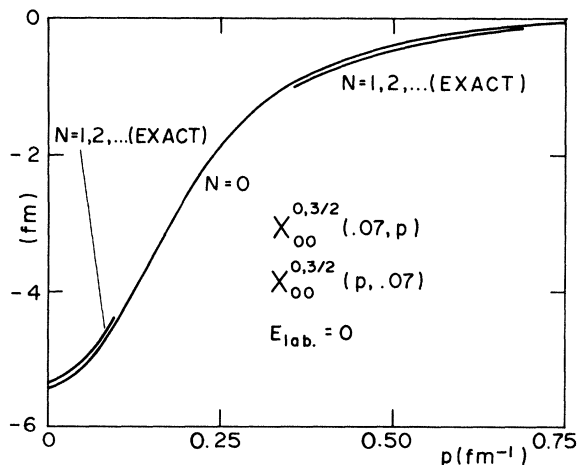
V. SUMMARY AND CONCLUSION

Here we show how to increase systematically the convergence rate of a recently proposed method for iterative solution of scattering integral equations. The method uses the solution of an auxiliary equation whose kernel can successively be made weaker compared to the kernel of the original equation. We also show how to generalize the method to the case of multichannel scattering equations. We show that the applicability of the method is not limited to the case of Lippmann-Schwinger type equations and it can be easily generalized to the case of general integral equations of the Fredholm type.[19] Then we incorporate the ideas of the present method in a method for three-body equations by Fuda[11] and show how we can solve three-body equations above the three-body breakup threshold by the use of an iterative solution of nonsingular auxiliary equations. We illustrate the method numerically in the case of nucleon-nucleon elastic scattering with Reid soft-core $^1S_0$ potential[17] and in the case of neutron-deuteron elastic scattering in the Amado model.[16] We employ an iterative solution of the auxiliary equation and show that it has a rapidly convergent Neumann series solution. As the numerical work involved in the present method is only the evaluation of a small number of integrals it is expected to yield results with high precision with a relatively small amount of numerical work. Usual methods for scattering integral equations[19,21] use the calculation of inverse and of determinants of matrices, sometimes of large dimensions. This is especially true for methods such as degenerate kernel methods, method of moments, direct solution by matrix inversions, and solution by the use of Padé approximants. Usually there will be loss of accuracy in such methods dealing with determinants and inversion of large matrices. In order to support our conjecture we note a comment by Brady and Sloan[3] who find that even to reproduce the second column in Table II they needed double precision, but the rate of convergence and precision obtained by them is much inferior to those of the present calculation as is clear from a glance at Table II. We show in our numerical calculation, in Table II, that the present method gives results with much higher precision as compared to the Padé technique using the same number of iterations and should be considered as an alternative to the Padé technique for solving scattering integral equations.

The present method has several advantages over the commonly used Padé technique. Firstly, there are serious questions on the uniformity of convergence in the method of Padé approximants

FIG. 2. Off-shell spin quartet t matrix elements at incident neutron energy $E_{lab} = 0$ for various N.

because of the appearance of spurious poles in the denominator.[2] No such problem arises in the present method. Secondly, it is numerically more tedious to construct the Padé approximants using the iterative solution than to construct the solution using the present method. Thirdly, we have noted in the discussion after Eq. (4.4) that for potentials which are almost separable the present method is expected to converge faster than the Padé technique. For example, for a one term separable potential the present method converges to the exact value without any iteration, whereas the Padé technique gives the exact value after two iterations.[2] Finally, the rate of convergence of the method of Padé approximants depends in a complicated way on the structure of the equations. Tjon and Nieland[22] have noted that, for higher partial waves of the Lippmann-Schwinger equation, the solutions using Padé approximants converges rather poorly, possibly because of the threshold factors in these equations. The kernel of the higher partial waves are weaker compared to the s-wave kernel and the present method has been demonstrated[20] to converge much faster for higher partial waves. In this work we demonstrated that even for the s wave the present method converges faster than the method of Padé approximants; hence for higher partial waves the present method will converge even faster compared to the method of Padé approximants.

The applicability of the present technique can be easily extended to the case of three-body problems with two-body local interactions. This is because we can make finite rank approximations for such potentials and after a partial wave decomposition the three-body equation reduces to multichannel Lippmann-Schwinger–type equations in one variable. Then we can introduce subtractions in some of these channels and solve the full equation by iteration. From this experience we conjecture that we shall need to introduce subtraction in only one or two channels of the equation,

and the auxiliary equation after a few such subtractions will have a sufficiently weak kernel in order to have a rapidly convergent iterative solution. The present method could be easier and simpler than the perturbation techniques of Alt, Grassberger, and Sandhas[23] and of Sloan,[23] who solve part of the three-body equations by inversion and treat the rest by iteration, whereas in the present method we solve the full equation by iteration.

The present method can be easily used in the context of Karlsson-Zeiger equations[10] in order to write a nonsingular representation of three-body equations, which are interesting from a numerical and formal point of view. Such representation may lead to schemes for making unitary approximations above the three-body breakup threshold and to simple iterative methods for numerical solutions. It is interesting to note that the calculation of the present paper yields unitary results for each order of iteration below the three-body breakup threshold. The method, although applicable above the three-body breakup threshold with the use of contour deformation technique, will not yield unitary results for each order of iteration. We conclude that the present method should be considered as an important alternative for solving scattering integral equations and will lead to various formal and numerical approximation schemes in the future, which use the iterative solution of auxiliary equations.

[1]H. Padé, Ann. Ec. Norm. <u>9</u>, Suppl., 1 (1892).
[2]*The Padé Approximant in Theoretical Physics*, edited by G. A. Baker, Jr. and J. L. Gammel (Academic, New York, 1970).
[3]T. J. Brady and I. H. Sloan, Phys. Lett. <u>40B</u>, 55 (1972).
[4]J. A. Tjon, Phys. Lett. <u>63B</u>, 391 (1976); W. M. Kloet and J. A. Tjon, Ann. Phys. (N.Y.) <u>79</u>, 407 (1973); J. A. Tjon, Phys. Rev. D <u>1</u>, 2109 (1970).
[5]L. Tomio and S. K. Adhikari, Phys. Rev. C <u>22</u>, 28 (1980).
[6]L. Tomio and S. K. Adhikari, Phys. Rev. C <u>22</u>, 2359 (1980); S. K. Adhikari, *ibid.* <u>19</u>, 1729 (1979).
[7]T. Sasakawa, Prog. Theor. Phys. Suppl. <u>27</u>, 1 (1963);

K. L. Kowalski, Nucl. Phys. <u>A190</u>, 645 (1972); Phys. Rev. C <u>10</u>, 366 (1974); F. Coester, *ibid.* <u>3</u>, 525 (1971).
[8]K. L. Kowalski, Phys. Rev. Lett. <u>15</u>, 798 (1965); H. P. Noyes, *ibid.* <u>15</u>, 538 (1965).
[9]S. K. Adhikari, J. Comp. Phys. (to be published).
[10]B. R. Karlsson, Phys. Rev. D <u>17</u>, 642 (1978); R. T. Cahill, Nucl. Phys. <u>A194</u>, 599 (1972); K. L. Kowalski, Phys. Rev. D <u>5</u>, 395 (1972); T. Sasakawa, Nucl. Phys. <u>A203</u>, 496 (1973); D. Eyre and T. A. Osborn, *ibid.* <u>A299</u>, 301 (1978); B. R. Karlsson, Phys. Lett. <u>69B</u>, 13 (1977); S. K. Adhikari, Phys. Rev. D <u>21</u>, 2251 (1980); B. R. Karlsson and E. M. Zeiger, *ibid.* <u>11</u>,

939 (1975).

[11]M. G. Fuda, Phys. Rev. Lett. 32, 620 (1974).

[12]J. H. Hetherington and L. Schick, Phys. Rev. 137, B935 (1965).

[13]R. Aaron, R. D. Amado, and Y. Yam, Phys. Rev. 136, B650 (1964); 140, B1291 (1965); R. T. Cahill and I. H. Sloan, Nucl. Phys. A165, 161 (1971).

[14]S. K. Adhikari and R. D. Amado, Phys. Rev. D 9, 1467 (1974); S. K. Adhikari, ibid. 18, 4264 (1978).

[15]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954); Y. Yamaguchi and Y. Yamaguchi, ibid. 95, 1635 (1954).

[16]R. D. Amado, Phys. Rev. 132, 485 (1963); C. Lovelace, ibid. 135, B1225 (1964).

[17]R. V. Reid, Jr., Ann. Phys. (N.Y.) 50, 411 (1968).

[18]E. Harms, Phys. Rev. C 1, 1667 (1970); M. G. Fuda, Nucl. Phys. A116, 83 (1968); S. Weinberg, Phys. Rev. 131, 440 (1963).

[19]F. W. Byron, Jr. and R. W. Fuller, Mathematics of Classical and Quantum Physics (Addison-Wesley, Reading, Massachusetts, 1970), Vol. II, Chaps. 8 and 9.

[20]D. Blasczak and M. G. Fuda, Phys. Rev. C 8, 1665 (1973); J. Whiting and M. G. Fuda, ibid. 12, 320 (1975).

[21]E. Harms, Phys. Lett. 41B, 26 (1972); Nucl. Phys. A222, 125 (1974); T. A. Osborn, Nucl. Phys. A211, 211 (1973); I. H. Sloan and S. K. Adhikari, ibid. A235, 352 (1974); Yu. V. Vorobyev, Method of Moments in Applied Mathematics (Gordon and Breach, New York, 1965).

[22]J. A. Tjon and H. M. Nieland, See Ref. 2.

[23]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967); I. H. Sloan, ibid. A182, 549 (1972); S. C. Pieper, ibid. A193, 529 (1972); Phys. Rev. C 6, 1157 (1972); G. Bencze and P. Doleschall, Phys. Lett. 44B, 235 (1973).