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General properties of energy independent nuclear optical model potentials

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An energy independent optical potential for nucleon-nucleus scattering is formally derived. A simple relation between energy dependent and energy independent potentials is established showing that the latter has the same thresholds as the former. A generalized dispersion relation for energy independent potentials is found and compared to the conventional dispersion relation of the generalized energy dependent optical potentials.

[REACTION THEORY General formulation of the energy independent optical potential.]

I. INTRODUCTION

In a recent paper[1] we formulated a new microscopic theory of the optical potential for elastic nucleon-nucleus scattering which, in contrast to the Green's function formalism[2] and also Feshbach's formalism[3] leads to a formally energy-independent optical potential. This energy-independent potential is complex and nonlocal, and can be written in an exact diagrammatic perturbation expansion based on the Rayleigh-Schrödinger (folded diagram) type of perturbation theory. Because of the energy-independent nature of the potential, however, it is not easily seen to what extent this potential incorporates the well known general properties of the energy dependent potentials such as threshold effects and dispersion relation. In this paper we therefore investigate the specific properties of the energy independent potential in more detail, in particular we study its threshold effects and also derive a generalized dispersion relation for it. To keep the equations as simple as possible and to concentrate only on the essential physical aspects we avoid all the complications in this paper which arise when one describes scattering processes in a second quantized form.[1,2,4,5] For this reason we work in the $\vec{r}$ or $\vec{k}$ representation with the projectile nucleon assumed to be distinguishable from the nucleons in the target. As usual the target nucleus is described by an antisymmetric wave function. The more general, fully antisymmetric formulation of the $(A+1)$-body problem (A = number of nucleons in the target) can be easily recovered from our formulation by using the methods described in Refs. 1,2,4,5).

In Sec. II of this paper we derive the energy independent optical potential starting from the full $(A+1)$-body problem. In Sec. III we discuss the threshold effects of the potential and formulate. also a type of dispersion relation which we compare with the well known results of the energy dependent theories. Section IV contains the conclusions.

II. DERIVATION OF THE ENERGY INDEPENDENT OPTICAL POTENTIAL

We consider a nucleus consisting of A nucleons and study the scattering of a nucleon by the nucleus. The total Hamiltonian of the system can be written as

$$H = T_o + H_A + V \quad , \tag{1}$$

where H_A is the Hamiltonian of the target system, T_o is the kinetic energy operator of the incident nucleon, and V represents the sum of interactions between the projectile nucleon and the nucleons in the target. The scattering problem we must study is represented by the wave equation

$$H \mid \Psi_{A+1}^{(+)}(E)\rangle = E \mid \Psi_{A+1}^{(+)}(E)\rangle \quad . \tag{2}$$

The function $\mid \Psi_{A+1}^{(+)}(E)\rangle$ is the exact antisymmetric $A+1$ body wave function and obeys the standard asymptotic boundary conditions [superscript $(+)$ indicates an incident plane wave in the elastic channel and radially outgoing waves in reaction channels]. We set our energy scale so that the target in its ground state $\mid \Psi_A{}^0\rangle$ is at zero energy. Then, the energy E in Eq. (2) is identical with the asymptotic kinetic energy of the incoming nucleon. In the case of elastic scattering, we are only interested in that part of $\mid \Psi_{A+1}^{(+)}(E)\rangle$ where A nucleons form the true ground state $\mid \Psi_A{}^0\rangle$ of the A-body system and where one nucleon is in a scattering state. Therefore we project out from $\mid \Psi_{A+1}^{(+)}(E)\rangle$ this particular part and define a model-space problem by

$$(T_o + V_{\text{opt}})P \mid \Psi_{A+1}^{(+)}(E)\rangle = EP \mid \Psi_{A+1}^{(+)}(E)\rangle \quad , \tag{3}$$

where the projection operator P is defined by

$$P = \int d\vec{k} \mid \vec{k},\Psi_A{}^0\rangle\langle \vec{k},\Psi_A{}^0 \mid ; \qquad \langle \vec{r} \mid \vec{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} \quad . \tag{4}$$

Note that $\mid \vec{k},\Psi_A^0\rangle$ in Eq. (4) is an eigenstate of the operator $T_o + H_A$. This eigenstate, however, is not antisymmetrized with respect to the projectile and target nucleon coordinates. [One can, of course, introduce antisymmetrized wave functions $\mathfrak{a} \mid k,\Psi_A{}^0\rangle$ in Eq. (4) with $\mathfrak{a}$ being the antisymmetrization operator[3] but we avoid this complication here since the points of our discussion are independent from this formal consideration.] Now the essential problem to be solved consists in the derivation of the optical potential operator V_{opt}, since knowing V_{opt} we obtain readily from Eq. (3) a one body Schrödinger equation for the optical-model wave function

$$\rho_E(\vec{k}) = \langle \Psi_A{}^0,\vec{k} \mid \Psi_{A+1}^{(+)}(E)\rangle \quad . \tag{5}$$

Following Feshbach[3] we also introduce the projection operator Q which projects onto that part of $\mid \Psi_{A+1}^{(+)}\rangle$ not included in $P \mid \Psi_{A+1}^{(+)}\rangle$. Thus we have the relations[3]

$$P + Q = I \quad , \tag{6a}$$

$$QP = PQ = 0 \quad , \tag{6b}$$

where I is the identity operator. We emphasize that P and Q are not energy dependent.

In order to derive the energy independent potential we introduce, in addition to the projection operators P and Q, the Möller wave operator Ω which is defined by the equation

$$\mid \Psi_{A+1}^{(+)}\rangle = \Omega P \mid \Psi_{A+1}^{(+)}\rangle \quad . \tag{7}$$

The Möller wave operator Ω has the property that it reproduces the full many body wave function when acting on a projected wave function of the P subspace of the total Hilbert space. In Feshbach's theory of the optical potential, the Möller wave operator $\Omega_F^{(+)}$ is energy dependent, i.e.,

$$\Omega_F^{(+)} = P + \frac{1}{E^+ - QHQ} QVP \quad . \tag{8}$$

One can, however, also construct an energy independent wave operator. For example, the well-known Rayleigh-Schrödinger perturbation theory uses one. In the same spirit we shall now construct an energy independent Möller wave operator Ω for the optical model potential.

Following Lee and Suzuki,[6] we may write the operator Ω in the form

$$\Omega = P + Q\omega P \quad , \tag{9}$$

where the second term on the right hand side (rhs) of Eq. (9), $Q\omega P$, is an operator which transforms the P-space wave function $P \mid \Psi_{A+1}^{(+)}\rangle$ into the Q-

space wave function $Q\,|\Psi_{A+1}^{(+)}\rangle$. To obtain an equation for Ω, or $Q\omega P$, we insert the rhs of Eq. (7) into Eq. (2) with the result

$$H\Omega P\,|\Psi_{A+1}^{(+)}(E)\rangle = E\Omega P\,|\Psi_{A+1}^{(+)}(E)\rangle = \Omega PH\Omega P\,|\Psi_{A+1}^{(+)}(E)\rangle \ . \tag{10a}$$

The last equality sign in Eq. (10a) is obtained by replacing $EP\,|\Psi_{A+1}^{(+)}(E)\rangle$ in the first line of Eq. (10a) with Eq. (2), i.e., $EP\,|\Psi_{A+1}^{(+)}(E)\rangle = P(H\Omega)P\,|\Psi_{A+1}^{(+)}(E)\rangle$. It is *this* step which eliminates the energy dependence from our equations and which leads to an energy independent, but nonlinear operator equation[6] for the Möller wave operator Ω

$$H\Omega P = \Omega PH\Omega P \ . \tag{10b}$$

A similar equation has been derived by Bloch[7] for the case of a degenerate P space.

By acting with the projection operators P and Q on Eqs. (10) from the left we obtain the following two equations [hereafter, we use $|\Phi_E^{(+)}\rangle = P\,|\Psi_{A+1}^{(+)}(E)\rangle$ for brevity]

$$(PH\Omega P)\,|\Phi_E^{(+)}\rangle = E\,|\Phi_E^{(+)}\rangle \ , \tag{11a}$$

$$Q\Omega PH\Omega = QH\Omega \ . \tag{11b}$$

Equations (11a) and (11b) are fully equivalent to the $(A+1)$-Schrödinger equation of Eq. (2). The first equation, (11a), states the model state problem while Eq. (11b) is a defining equation for the effective interaction to be used in solving Eq. (11a). It is obvious that Eq. (11b) is nonlinear in Ω, and that it is independent of the energy E. Using Eq. (9) in Eqs. (11) and using the fact that $P(T_o + H_A)Q = 0$ we may rewrite Eqs. (11) in the form

$$[PT_0P + PVP + PV(Q\omega P)P]\,|\Phi_E^{(+)}\rangle = E\,|\Phi_E^{(+)}\rangle \ , \tag{12a}$$

$$QVP + QHQ(Q\omega P) = Q\omega P[PT_oP + PVP + PV(Q\omega P)P] \ . \tag{12b}$$

Equation (12b) can be solved for $Q\omega P$ which when inserted into Eq. (12a) leads to the energy independent optical model equation. The energy independent optical potential is then given by

$$\mathcal{V}_{\rm opt} = PVP + PVQ\omega P \ . \tag{13}$$

We want to emphasize that until now we have not used any boundary conditions for defining our model space problem. Therefore Eqs. (11) and (12) are equally applicable for bound and unbound problems. In the case of a bound state problem, one uses a P space corresponding to bound states in solving for $Q\omega P$ from Eq. (12b). In scattering problems, we use instead a P space composed of wave functions with appropriate scattering boundary conditions.

The above Eqs. (12a), (12b), and (13) can be solved explicitly and formally to give an energy independent optical potential. When $Q\omega P$ operates on *an eigenstate* of Eq. (12a), we obtain

$$(Q\omega P)\,|\Phi_E^{(+)}\rangle = \frac{1}{E^{(+)} - QHQ} QVP\,|\Phi_E^{(+)}\rangle \ . \tag{14}$$

Here, $E^{(+)} = E + i0^+$ ensures that $|\Phi_E^{(+)}\rangle$ obeys the right boundary conditions. It is important to notice that the whole set of solutions $|\Phi_E^{(+)}\rangle$ with variable E forms a complete basis of the P space.[8] This fact allows us to write the operator $Q\omega P$ in its spectral representation

$$Q\omega P = \sum\!\!\!\!\!\!\int dE' \frac{Q}{E'^{(+)} - QHQ} V\,|\Phi_{E'}\rangle\langle\widetilde{\Phi}_{E'}^{(+)}| \ , \tag{15}$$

where $\langle\widetilde{\Phi}_E^{(+)}|$ is the biorthogonal vector to $|\Phi_E^{(+)}\rangle$ normalized such that $\langle\widetilde{\Phi}_{E'}^{(+)}|\Phi_E^{(+)}\rangle = \delta(E' - E)$. At this point it is appropriate to emphasize the difference between Eq. (15) and Eq. (12b). Equation (15) is only valid, if and only if the wave functions $|\Phi_E^{(+)}\rangle$ are proper eigenstates of Eq. (12a) while Eq. (12b) is always valid. Equation (15) actually completes the proof that the operators $Q\omega P$ and $\Omega = P + Q\omega P$ exist as has been stated above. It also explicitly shows the energy independence of both operators. Insertion of the rhs of Eq. (15) into Eq. (13) gives us the energy independent optical potential

$$\mathcal{V}_{\rm opt} = PVP + \sum\!\!\!\!\!\!\int dE'\, PV \frac{Q}{E'^{(+)} - QHQ} V\,|\Phi_{E'}^{(+)}\rangle\langle\widetilde{\Phi}_{E'}^{(+)}| \ . \tag{16}$$

Let us emphasize again that $Q\omega P\,|\Phi_E^{(+)}\rangle$ as a whole is indeed energy dependent, yet $Q\omega P$ itself is *energy independent* as shown by Eq. (15). In fact, for any given P operator, Eq. (12b) can be used to solve for $Q\omega P$ and $\mathcal{V}_{\rm opt}$ of Eq. (13). Then Eq. (12a) will be used to solve the P-space problem. Different P space operator gives different effective interaction or optical potential. Yet the solution obtained from (12b) is energy independent. One naturally chooses an appropriate P space operator for a specific problem. For example, (a) for the low-lying states of a nucleus, one uses a shell-model P space, and (b) for the scattering of one nucleon off a nucleus, one can choose a P space composed of the wave functions of the one body mean field. The P space operator of Eqs. (15) and (16) corresponds to the choice (b) above. Here P is composed of the eigenfunctions of (12a), and it is with this particular choice that $Q\omega P$ and $\mathcal{V}_{\rm opt}$ can be expressed in the simple form of Eqs. (15) and (16). Some general properties of $\mathcal{V}_{\rm opt}$ will be discussed in Sec. III based on these equations.

III. THRESHOLD EFFECTS AND DISPERSION RELATION

To discuss the general properties of $\mathcal{V}_{\rm opt}$ we introduce a complete system of eigenfunctions of the Q-space Hamiltonian QHQ. In general the spectrum of QHQ consists of a discrete part and a continuum. We denote the states of the discrete part of the spectrum by $|q_i\rangle$ with eigenvalues ϵ_i:

$$QHQ\,|q_i\rangle=\epsilon_i\,|q_i\rangle\ . \tag{17}$$

The continuum states are denoted by $|\epsilon,\alpha\rangle$ with

$$QHQ\,|\epsilon,\alpha\rangle=\epsilon\,|\epsilon,\alpha\rangle\ , \tag{18}$$

where α labels the degenerate states corresponding to the eigenenergy ϵ. We now may insert the complete spectrum of QHQ from Eqs. (17) and (18) into Eq. (16) and obtain

$$\begin{aligned}V_{\rm opt}=PVP&+\sum_{\mu}\sum_{i}\frac{PV|q_i\rangle\langle q_i|V}{E'_{\mu}-\epsilon_i}|\Phi_{E'_{\mu}}\rangle\langle\tilde{\Phi}_{E'_{\mu}}|+\int_0^{\infty}dE'\sum_i\frac{PV|q_i\rangle\langle q_i|V}{E'^{(+)}-\epsilon_i}|\Phi_{E'}^{(+)}\rangle\langle\tilde{\Phi}_{E'}^{(+)}| \\ &+\sum_{\mu}\int d\alpha\int_0^{\infty}d\epsilon\frac{PV|\epsilon,\alpha\rangle\langle\alpha,\epsilon|V}{E'_{\mu}-\epsilon}|\Phi_{E'_{\mu}}\rangle\langle\tilde{\Phi}_{E'_{\mu}}| \\ &+\int_0^{\infty}dE'\int d\alpha\int_0^{\infty}d\epsilon\frac{PV|\epsilon,\alpha\rangle\langle\alpha,\epsilon|V}{E'^{(+)}-\epsilon}|\Phi_{E'}^{(+)}\rangle\langle\tilde{\Phi}_{E'}^{(+)}|\ .\end{aligned} \tag{19}$$

Here, the sums over μ and i represent the sums over the the bound state wave functions of the P space and of the Q space, respectively.

Several observations are now in order,which are also partly found for the energy-dependent optical potentials[3]:

(1) The numerators in expansion (19) are positive definite, i.e., for an arbitrary wave function $|\chi\rangle$ we have

$$\langle\chi|PV|q_i\rangle\langle q_i|VP|\chi\rangle=|\langle\chi|PV|q_i\rangle|^2\geq 0\ . \tag{20}$$

(2) The first three terms in expansion (19) contribute only to the real part of the optical potential. This is evident for the second and third term in Eq. (19), because the denominators do not vanish. The denominator in the first term vanishes if the "single particle state" $|E_{\mu'}\rangle$ is degenerate with a Q-space state $|\epsilon_i\rangle$. This might happen for the so called intruder state, which then has to be treated carefully. The problem of intruder states in bound state calculations has been extensively studied in Ref. 6.

(3) The only term which contributes to real absorption in the sense that incident flux goes to energetically open inelastic channels is the fourth term on the rhs of Eq. (19). The imaginary part of $V_{\rm opt}$ is given by the energy conserving contribution to this term

$$\begin{aligned}{\rm Im}\mathcal{V}_{\rm opt}=&-\pi\int\int dE'\,d\alpha PV|E',\alpha\rangle \\ &\times\langle\alpha,E'|V|\Phi_{E'}^{(+)}\rangle\langle\Phi_{E'}^{(+)}|\ .\end{aligned} \tag{21}$$

Furthermore, we find that the expectation value of ${\rm Im}\mathcal{V}_{\rm opt}$ taken with the optical model wave function $|\Phi_E^{(+)}\rangle$ is negative definite

$$\begin{aligned}\langle\Phi_E^{(+)}|{\rm Im}\mathcal{V}_{\rm opt}|\Phi_E^{(+)}\rangle&=-\pi\int d\alpha\langle\Phi_E^{(+)}|V|E,\alpha\rangle \\ &\quad\times\langle E,\alpha|V|\Phi_E^{(+)}\rangle \\ &=-\pi\int d\alpha|\langle\Phi_E^{(+)}|V|E,\alpha\rangle|^2\leq 0\ .\end{aligned} \tag{22}$$

This property we would have, of course, expected since $\mathcal{V}$ has to be an absorptive potential.

(4) Projecting Eq. (19) from left and right onto the elastic channel wave function $|\Phi_E^{(+)}\rangle$ we obtain the well known energy dependent optical potential

$$\mathcal{V}_{\text{opt}}(E) = P_E V P_E + \sum_i \frac{P_E V |q_i\rangle\langle q_i| V P_E}{E^{(+)} - \epsilon_i} + \int_0^\infty d\epsilon \int d\alpha \frac{P_E V |\epsilon,\alpha\rangle\langle\epsilon,\alpha| V P_E}{E^{(+)} - \epsilon} , \tag{23}$$

where $P_E = |\Phi_E^{(+)}\rangle\langle\tilde{\Phi}_E^{(+)}|$. From Eq. (23), it is clear that the scattering process of a nucleon with incident energy E is governed effectively by the *same* optical potential in the energy dependent and energy independent version of the optical model theory. The energy independent potential is just constructed in such a way that this condition holds. Therefore the energy independent potential includes the same threshold effects as the energy dependent potentials do.

Finally we formulate a type of dispersion relation From Eq. (19) we find the real part of the optical potential as

$$\text{Re}\mathcal{V}_{\text{opt}} = PVP + \int dE \sum_i \frac{PV|q_i\rangle\langle q_i|V|\Phi_E^{(+)}\rangle\langle\tilde{\Phi}_E^{(+)}|}{E - \epsilon_i} + \mathcal{P}\int dE \int d\epsilon \frac{\int d\alpha PV|\epsilon,\alpha\rangle\langle\epsilon,\alpha|V|\Phi_E^{(+)}\rangle\langle\tilde{\Phi}_E^{(+)}|}{E - \epsilon} \tag{24}$$

with $\mathcal{P}$ meaning Cauchy principal value. The imaginary part $\text{Im}\mathcal{V}_{\text{opt}}$ has already been given in Eq (21). By comparison of Eqs. (21) and (24) it is evident that $\text{Im}\mathcal{V}_{\text{opt}}$ involves a single integration over the energy E' while the last term of $\text{Re}\mathcal{V}_{\text{opt}}$ in Eq. (24) involves a double integration over energies E and ϵ. Therefore a "simple" replacement of the kernel of the principal value integral by $\text{Im}\mathcal{V}_{\text{opt}}$ is not possible as it is in the energy dependent optical potentials.[3] However, one can add a zero term to Eq. (24)

$$\text{Re}\mathcal{V}_{\text{opt}} = PVP + \sum dE' \frac{PV|q_i\rangle\langle q_i|V|\Phi_E^{(+)}\rangle\langle\tilde{\Phi}_E^{(+)}|}{E' - \epsilon_i} + \sum_\mu \int_0^\infty d\epsilon \int d\alpha \frac{PV|\alpha,\epsilon\rangle\langle\alpha,\epsilon|V}{E_\mu - \epsilon} |\Phi_{E_\mu}\rangle\langle\tilde{\Phi}_{E_\mu}|$$
$$+ \mathcal{P}\int_0^\infty dE' \int d\epsilon \frac{d\alpha PV[|\alpha,\epsilon\rangle\langle\alpha,\epsilon| - |\alpha,E'\rangle\langle\alpha,E'|]V|\Phi_{E'}^{(+)}\rangle\langle\tilde{\Phi}_{E'}^{(+)}|}{E' - \epsilon} . \tag{25}$$

It can be noticed that the subtractive term in the numerator of the last term of Eq. (25) integrated over E' is just the imaginary part of Eq. (21). This term actually does not contribute to the principal value double integral because it is a "zero term", but it has the nice feature that it makes the numerator vanish for $\epsilon = E'$ so that the kernel in the double integral is smoothed and therefore well behaved. It is interesting to multiply Eq. (25) from left and right with the projection operator $P_E = |\Phi_E^{(+)}\rangle\langle\tilde{\Phi}_E^{(+)}|$ since then we recover the dispersion relation for the energy dependent potentials

$$\text{Re}(P_E \mathcal{V}_{\text{opt}} P_E) = P_E V P_E + \sum_i \frac{P_E V|a_i\rangle\langle q_i|VP_E}{E - \epsilon_i} - \frac{1}{\pi}\mathcal{P}\int \frac{\text{Im}[\mathcal{V}_{\text{opt}}(\epsilon)]}{E - \epsilon} d\epsilon , \tag{26}$$

where we have dropped the zero term in the principal value integral. Because of the fact that Eq. (25) implicitly contains the dispersion relation in Eq. (26) we may view Eq. (25) as a generalized relation for the energy independent optical potential.

IV. CONCLUSIONS

A derivation of the energy independent optical model potential for nucleon nucleus scattering has been made using the methods of Ref. 6. The projectile nucleon has been treated as distinguishable from the nucleons in the target. This assumption is made in order to keep the formulations as transparent as possible although a fully antisymmetric treatment can also be introduced in a straightforward way. An energy independent potential is derived whose structure shows the relation between energy dependent and energy independent potentials very clearly. We find that both types of potentials possess the same thresholds. In addition, we obtain a generalized dispersion relation for the energy independent potential. This dispersion relation reduces to the conventional dispersion relation for energy dependent potentials when projected onto the elastic channel described by $|\Phi_E^{(+)}\rangle$. A further remarkable feature of our theory is that the potential obtained can be used in nuclear structure and nuclear reaction calculations. Finally we want to mention the following point. The present formulation of the energy independent optical potential provides an alternative view to the diagrammatic theory of $\mathcal{V}_{opt}$ which we proposed earlier.[1] Both are very useful to understand the general properties of $\mathcal{V}_{opt}$, and microscopic calculations can also be performed in both frameworks.

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