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Parametrization of the elastic sector of the nucleon-nucleon scattering matrix. I.

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We present a method for parametrizing the 2×2 S matrix for coupled-channel nucleon-nucleon elastic scattering above the pion-production threshold, namely, we introduce two dummy nucleon-Roper-resonance channels and unitarize the consequent 4×4 S matrix. The elastic nonunitary 2×2 nucleon-nucleon S matrix is then recovered as a submatrix of the 4×4 matrix. We parametrize the unitary 4×4 S matrix by generalizing the prescription of Stapp, Ypsilantis, and Metropolis for 2×2 matrices. The six phases parametrizing the 2×2 submatrix are well behaved as the interaction vanishes, and automatically satisfy a unitarity condition which we derive, viz., $\det(1-S_{aa}^{\dagger}S_{aa})\geq 0$, where S_{aa} is the 2×2 submatrix. Of the six phases, three represent the absorption and parametrize a real, symmetric 2×2 matrix N, analogous to the absorption parameter η in the uncoupled case. A method is given for recovering the phase parameters from the S matrix or other parametrizations.

[NUCLEAR REACTIONS Scattering theory. Parametrized coupled-channel nucleon-nucleon elastic scattering above the pion-production threshold.]

I. INTRODUCTION

In this paper we present a method for parametrizing the scattering of two like Dirac (spin-$\frac{1}{2}$) particles when inelasticity is also present. We will only apply the method to nucleon-nucleon (NN) scattering above the pion production threshold,

$$NN\rightarrow NN \tag{1a}$$

$$\rightarrow NN\pi \tag{1b}$$

but we believe that, as far as the elastic-scattering sector [Eq. (1a)] is concerned, the method is perfectly general and can be applied to any of two like Dirac particles.

Our parametrization stems from the observation that the factor η used to parametrize inelasticity in uncoupled NN states,

$$S=e^{2i\delta}\rightarrow \eta e^{2i\delta},\quad |\eta|\leqslant 1\,, \tag{2}$$

occurs in a natural way if one hypothesizes a second baryon-baryon channel (call it N-Roper[1]) and then parametrizes the coupled NN-NR channels according to the Stapp-Ypsilantis-Metropolis (SYM) prescription[2]

$$S_{2\times 2}=\begin{pmatrix}\cos 2\epsilon e^{2i\delta} & i\sin 2\epsilon e^{i(\delta+\delta')}\\ i\sin 2\epsilon e^{i(\delta+\delta')} & \cos 2\epsilon e^{2i\delta'}\end{pmatrix}. \tag{3}$$

The upper-left matrix element, $\cos 2\epsilon\exp 2i\delta$, parametrizes the elastic-scattering sector, and $\cos 2\epsilon\equiv\eta$ is automatically required to have absolute value $\leqslant 1$ [Eq. (2)], as is required to conserve probability.

It is in parametrizing nonunitary elastic scattering in the *coupled* NN angular momentum states (e.g., $^3S_1-{}^3D_1$) where complications arise. MacGregor, Arndt, and Wright[3] and Hoshizaki[4] have devised models which they use in their phase-shift analyses. In this paper we present our own prescription which we obtain by generalizing Eq. (3) to accommodate the coupled angular momentum states as well as the coupled NN-NR channels. There results a 4×4 unitary S matrix which in the $NN\rightarrow NN$ sector appears as the SYM form modified by a 2×2 matrix N;

$$S_{2\times 2}=e^{i\delta}e^{2i\epsilon\sigma^1}e^{i\delta}\rightarrow e^{i\delta}e^{i\epsilon\sigma^1}Ne^{i\epsilon\sigma^1}e^{i\delta}\,, \tag{4a}$$

where

$$\delta=\begin{pmatrix}\delta_\alpha & 0\\ 0 & \delta_\beta\end{pmatrix}, \tag{4b}$$

σ^1 is the first Pauli spin matrix, and N is a symmetric 2×2 matrix specified by three parameters and subject to certain constraints, including $\det(1-N^2)\geqslant 0$. The effect of absorption on elastic scattering is all contained in N, which is a generalization of η; also the total reaction (inelastic) cross section depends only on N. Thus the roles of inelastic and elastic scattering are neatly separated.

II. PARAMETRIZING THE SCATTERING MATRIX

Although we could have implemented the method outlined in the Introduction simply by writing down a 4×4 S matrix for coupled-channel, coupled angular momentum states, and then parametrizing this matrix, we found it more instructive to write down a model Lagrangian for coupled nucleon, Roper, and scalar meson fields, and to determine the S matrix through straightforward calculation. We were thus able to work with a Hermitian Hamiltonian so unitarity of the S matrix was assured. The Lagrangian was chosen to obey the usual invariance conditions for strong interactions (invariance under space-reflection, time-reversal, Lorentz transformation, and isospin-space rotation). We obtained momentum-space expressions for the S and M matrices for coupled NN and NR channels. We then expanded the M matrix into partial-wave matrix elements and imposed the unitarity condition. There resulted, for each value of total angular momentum j, a unitary 8×8 S matrix consisting of two 2×2 blocks along the diagonal for the singlet and triplet-uncoupled states, and a 4×4 block on the diagonal for triplet-coupled states. Each block was independently unitary. This result was entirely as expected and certainly not dependent on our particular choice of Lagrangian; however the specific calculation had the advantage of definiteness and provided formulas linking the partial-wave parameters to the scattering observables.

We ordered the matrix elements of the 4×4 matrix S into blocks of 2×2 matrices as follows:

$$S=\begin{pmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb}\end{pmatrix}, \tag{5a}$$

where $a(b)$ is the $NN(NR)$ channel index. Each 2×2 matrix was in turn ordered as in the following example:

$$S_{ab}=\begin{pmatrix} \langle a,j-1|S|b,j-1\rangle, & \langle a,j-1|S|b,j+1\rangle \\ \langle a,j+1|S|b,j-1\rangle, & \langle a,j+1|S|b,j+1\rangle \end{pmatrix}, \tag{5b}$$

where $j\pm 1$ refers to the orbital angular momentum. The total angular momentum index j and the spin index $s(=1)$ are suppressed in this notation.

To find the optimal parametrization for S, we first studied the SYM parametrization[2] for 2×2 matrices. The latter may be expressed as in Eq. (4);

$$S_{2\times 2}=e^{i\delta}e^{2i\epsilon\sigma^1}e^{i\delta}\,.$$

The 2×2 S matrix thus has the general form

$$S_{2\times 2}=\mathcal{O}\mathcal{O}^T\,, \tag{6}$$

where

$$\mathcal{O}=\prod_{i=1}^{3}e^{i\theta_i\Gamma_i}\,,$$

the Γ_i are the three symmetric 2×2 matrices $(1,\sigma^1,\sigma^3)$ and the θ_i are (linear combinations of) the phase parameters. (T denotes transpose.) Besides being manifestly symmetric [Eq. (6)] as required by time-reversal invariance, the SYM S matrix has the nice property that its phase parameters are proportional to the interaction Hamiltonian as the latter vanishes;

$$\lim_{V\to 0}S=(1+i\delta)(1+2\epsilon\sigma^1)(1+i\delta)\,, \tag{7a}$$

$$\lim_{V\to 0}(S-1)/2i=\begin{pmatrix}\delta_\alpha & \epsilon \\ \epsilon & \delta_\beta\end{pmatrix}. \tag{7b}$$

(This proportionality is not a property of the other 2×2 parametrization, that of Blatt and Biedenharn.[5])

In selecting a parametrization for the 4×4 matrix we therefore chose to generalize the SYM 2×2 prescription.

We set

$$S_{4\times 4}=\mathcal{O}\mathcal{O}^T\,, \tag{8a}$$

initially taking the most general representation of $\mathcal{O}$,

$$\mathcal{O}=\prod_{i=1}^{16}e^{i\theta_i\Gamma_i}\,. \tag{8b}$$

The Γ_i are 16 Hermitian matrices which span the 4×4 matrix space and the θ_i are 16 real constants. By definition [Eq. (8)] S is unitary and symmetric, and as in the 2×2 case, the θ_i are proportional to the interaction Hamiltonian as the latter vanishes. We chose as our set of 16 Hermitian matrices the unit matrix, Dirac's σ_j and ρ_j matrices,[6] and products of the σ_j and ρ_j matrices. Thus

$$\Gamma_i=1,\sigma_j,\rho_k,\sigma_j\rho_k,\quad j,k=1,2,3,$$

where

$$\sigma_j=\begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix},\quad j=1,2,3,$$

$$\rho_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\rho_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$

and

$$\rho_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

All the above are 4×4 matrices written in 2×2 block form. The σ_j and ρ_k matrices commute;

$$\sigma_j\rho_k=\rho_k\sigma_j,\quad j,k=1,2,3.$$

Thus, the Γ_i have the property that

$$\Gamma_i^{\,2}=1,\quad i=1,2,\ldots 16$$

whence, for real θ_i,

$$e^{i\theta_i\Gamma_i}=\cos\theta_i+i\Gamma_i\sin\theta_i\,.$$

Of these 16 matrices, 10 are symmetric and 6 antisymmetric. We put the exponentials of the six antisymmetric matrices on the right end of the product $\mathcal{O}$ [Eq. (8b)] and when $\mathcal{O}\mathcal{O}^T$ is evaluated, they cancel. Thus Eq. (8) reduces to

$$S=\mathcal{O}_s\mathcal{O}_s^T\,, \tag{9a}$$

where

$$\mathcal{O}_s=\prod_{i=1}^{10} e^{i\theta_i\Gamma_i}\,, \tag{9b}$$

with the Γ_i the 10 symmetric Hermitian matrices. (We could, of course, have used just the 10 symmetric matrices at the outset.) The symmetric matrices are 1, σ_3, ρ_3, $\sigma_3\rho_3$, σ_1, and $\sigma_1\rho_3$, plus ρ_1, $\sigma_1\rho_1$, $\sigma_3\rho_1$, and $\sigma_2\rho_2$. The first six are block diagonal (do not bring about $NN\to NR$ transitions) and their exponentials can be ordered so as to recover the SYM form in the $NN\to NN$ (and $NR\to NR$) sectors; we so order, setting

$$\prod_{i=1}^{6} e^{i\theta_i\Gamma_i}=e^{i\Delta}e^{iE}\,, \tag{10a}$$

where

$$\Delta=\begin{bmatrix}\delta_\alpha & & & 0\\ & \delta_\beta & & \\ & & \delta'_\alpha & \\ 0 & & & \delta'_\beta\end{bmatrix} \tag{10b}$$

and

$$E=\begin{pmatrix}\epsilon\sigma^1 & 0\\ 0 & \epsilon'\sigma^1\end{pmatrix}_{4\times4}\,; \tag{10c}$$

δ_α, δ_β, δ'_α, δ'_β, ϵ, and ϵ' are linear combinations of the six θ_i.

The remaining 4 matrices are antiblock diagonal, and bring about $NN\to NR$ transitions. These are sketched in Fig. 1 for the case $j=1$. Thus $\prod_{i=7}^{10}\exp i\theta_i\Gamma_i$ provides 4 parameters to specify inelasticity. Yet the 2×2 $NN\to NN$ sector admits only 6 parameters (3 elastic, 3 inelastic) as it is nonunitary but symmetric. Therefore we drop one factor to eliminate redundancy.[7] We choose to omit $\exp i\theta_{10}\sigma_2\rho_2$ (this has the effect of making $\langle a,{}^3S_1|S|b,{}^3D_1\rangle=\langle a,{}^3D_1|S|b,{}^3S_1\rangle$). The remaining exponentials are ordered

$$e^{i\theta_7\rho_1}e^{i\theta_8\sigma_3\rho_1}e^{i\theta_9\sigma_1\rho_1}$$

more-or-less arbitrarily. Any permutation appears to be equally acceptable.

Thus our final parametrization of S is

$$S=e^{i\Delta}e^{iE}Ue^{iE}e^{i\Delta}\,, \tag{11a}$$

where Δ and E are defined in Eq. (10) and where

$$U=e^{i\Gamma\rho_1}e^{i\gamma\sigma_3\rho_1}e^{2i\phi\sigma_1\rho_1}e^{i\gamma\sigma_3\rho_1}e^{i\Gamma\rho_1}\,, \tag{11b}$$

with θ_7, θ_8, and θ_9 replaced by Γ, γ, and ϕ, respectively, for writing convenience. The explicit form of U is

$$\begin{aligned}U_{aa}=U_{bb}&=\cos2\Gamma\cos2\gamma\cos2\phi\\&\quad-\sigma^1\sin2\Gamma\sin2\phi\\&\quad-\sigma^3\sin2\Gamma\sin2\gamma\cos2\phi\end{aligned} \tag{12}$$

and

$$\begin{aligned}U_{ab}=U_{ba}&=i(\sin2\Gamma\cos2\gamma\cos2\phi\\&\quad+\sigma^1\cos2\Gamma\sin2\phi\\&\quad+\sigma^3\cos2\Gamma\sin2\gamma\cos2\phi)\,.\end{aligned}$$

The S matrix for the sector of interest, S_{aa}, may now be determined. Since $\exp i\Delta\,\exp iE$ is block diagonal,

$$S_{aa}=e^{i\delta}e^{i\epsilon\sigma^1}U_{aa}e^{i\epsilon\sigma^1}e^{i\delta}\,, \tag{13a}$$

where, from Eq. (12),

$$U_{aa}\equiv N=\begin{pmatrix}\cos2(\Gamma+\gamma)\cos2\phi & -\sin2\Gamma\sin2\phi\\ -\sin2\Gamma\sin2\phi & \cos2(\Gamma-\gamma)\cos2\phi\end{pmatrix}. \tag{13b}$$

Thus, defining

$$S_{aa}=\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix},$$

we have

$$\begin{aligned}S_{11}&=e^{2i\delta_\alpha}[(\cos2\epsilon\cos2\Gamma\cos2\gamma-\sin2\Gamma\sin2\gamma)\cos2\phi\\&\qquad-i\sin2\epsilon\sin2\Gamma\sin2\phi]\,,\\S_{12}&=e^{i(\delta_\alpha+\delta_\beta)}[i\sin2\epsilon\cos2\Gamma\cos2\gamma\cos2\phi\\&\qquad-\cos2\epsilon\sin2\Gamma\sin2\phi]\,,\\S_{21}&=S_{12}\,,\\S_{22}&=e^{2i\delta_\beta}[(\cos2\epsilon\cos2\Gamma\cos2\gamma+\sin2\Gamma\sin2\gamma)\cos2\phi\\&\qquad-i\sin2\epsilon\sin2\Gamma\sin2\phi]\,.\end{aligned} \tag{14}$$

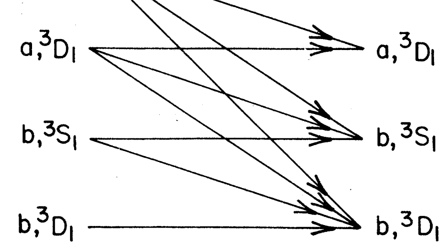

FIG. 1. Transitions between $^3S_1$ and $^3D_1$ states for coupled NN (a) and N-Roper (b) scattering. (Time-reversed transitions omitted for clarity.)

An alternate set of absorption parameters will sometimes be used: these are

$$\begin{aligned} \rho_\alpha &= \Gamma + \gamma \\ \rho_\beta &= \Gamma - \gamma\,. \end{aligned} \tag{15}$$

Note that S_{aa} reduces to the SYM representation when ρ_α, ρ_β, and ϕ vanish.

III. COMPARISON WITH OTHER PARAMETRIZATIONS

MacGregor, Arndt, and Wright (MAW) parametrize the coupled inelastic S matrix as follows;

$$S_{aa} = \begin{pmatrix} \cos\rho_- \cos 2\epsilon \exp 2i\delta_-\,, & i\sin 2\epsilon \exp i(\delta_- + \delta_+ + \phi) \\ i\sin 2\epsilon \exp i(\delta_- + \delta_+ + \phi), & \cos\rho_+ \cos 2\epsilon \exp 2i\delta_+ \end{pmatrix}. \tag{16}$$

Hoshizaki sets

$$S_{aa} = \begin{pmatrix} (r_-^2 - r_-r_+\sigma^2)^{1/2} \exp 2i\delta_-\,, & i(r_-r_+)^{1/2}\sigma \exp i(\delta_- + \delta_+ + \phi) \\ i(r_-r_+)^{1/2}\sigma \exp i(\delta_- + \delta_+ + \phi), & (r_+^2 - r_-r_+\sigma^2)^{1/2} \exp 2i\delta_+ \end{pmatrix}. \tag{17}$$

In fitting a certain nearly-unitary "experimental" S matrix with these and our own parametrizations, we discovered an important difference between the prescriptions; the experimental S matrix was

$$S_{aa} = \begin{pmatrix} 0.95 & 0.05 \exp 2i\delta_{12} \\ 0.05 \exp 2i\delta_{12} & 0.95 \end{pmatrix}, \tag{18}$$

where δ_{12} was varied from 0° to 180°. The phase parameters obtained in the three cases are plotted in Fig. 2. (We used the method given in the Appendix to determine our own phase parameters.[8]) One will note that the SYM and Hoshizaki parameter ϕ varies from 0° to 360°, even though the absolute magnitude of S_{12} is very small. This means that in applying this parametrization to NN scattering data analysis, it will be hard to search this ϕ to minimize χ^2. Our own ϕ remains small, probably because our θ_i are proportional to the interaction Hamiltonian in the limit as H^{int} vanishes.

We discovered another feature of our parametrization while attempting to convert sets of MAW parameters to our own. Certain sets of MAW parameters could not be converted. We found that this was because a certain unitarity condition was being violated. The condition is easily derived; to wit,

$$SS^\dagger = 1_{4\times4}\,,$$

so

$$S_{aa}S_{aa}^\dagger + S_{ab}S_{ba}^\dagger = 1_{2\times2}\,.$$

Thus

$$\det(1 - S_{aa}S_{aa}^\dagger) = \det(S_{ab}S_{ba}^\dagger) = \det S_{ab} \det S_{ba}^\dagger\,.$$

But

$$\det S_{ba}^\dagger = (\det S_{ab})^*$$

whence the unitarity condition[9]

$$\det(1 - S_{aa}S_{aa}^\dagger) = |\det S_{ab}|^2 \geq 0\,. \tag{19}$$

With our parametrization, Eq. (19) reduces to

$$\det(1 - N^2) \geq 0\,. \tag{20}$$

This condition is automatically satisfied for any choice of Γ, γ, and ϕ.

For the MAW parametrization, Eq. (19) indicates that

$$(\cos 2\epsilon)^4(\sin\rho_-)^2(\sin\rho_+)^2 \geq (\cos^2\epsilon \sin 2\epsilon)^2[(\cos\rho_-)^2 + (\cos\rho_+)^2 - 2\cos\rho_- \cos\rho_+ \cos 2\phi]. \tag{21}$$

Equation (21) is not satisfied by an arbitrary choice of ϵ, ρ_-, ρ_+, and ϕ. In particular[10] it is not satisfied when $\rho_- \neq 0$, $\phi = 0$, and $\rho_+ = 0$ (unless $\sin 4\epsilon = 0$). Thus with the MAW parametrization, it is not possible to represent a case of inelasticity in just the lower of two coupled channels, simply by setting ϕ and ρ_+ to zero.

The formula for the total reaction cross section σ_r due to the coupled states brings out another feature of the three parametrizations. In general,

$$\sigma_r = (\pi/4q^2)\Sigma_j(2j+1)\text{tr}(1 - S_{aa,j}^\dagger S_{aa,j})\,,$$

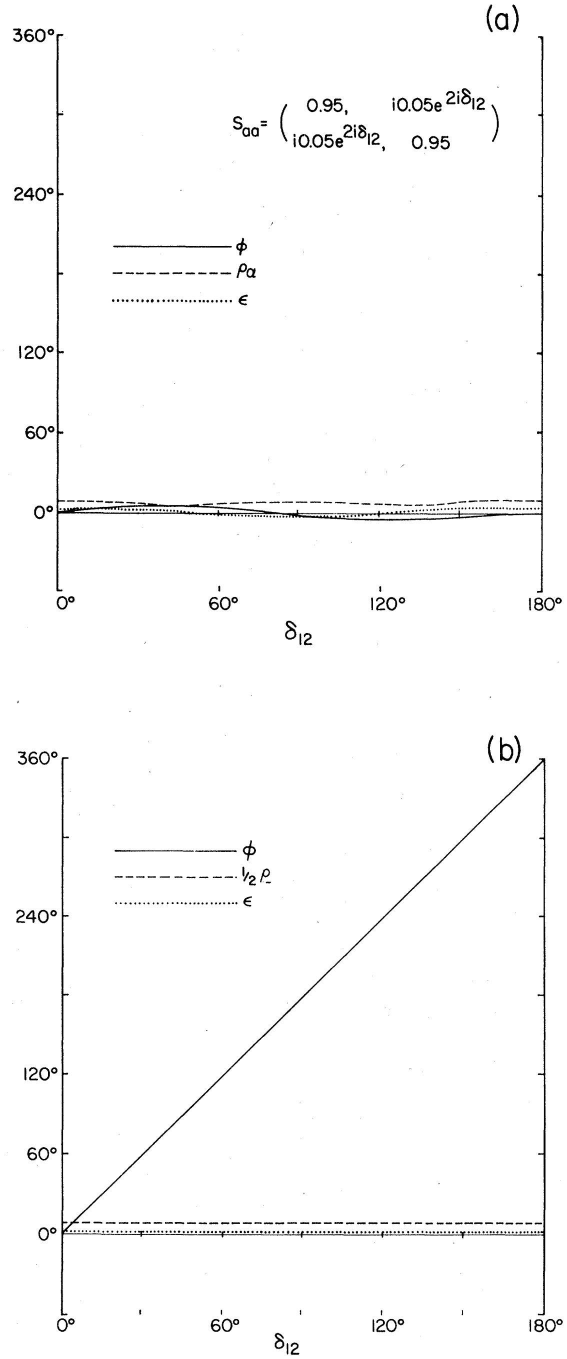

FIG. 2. (a) Plot of our phases ϕ, ρ_α, and ϵ which parametrize (Ref. 8) the S matrix shown in the inset [Eq. (18) in text]; δ_{12} varies from 0 to 180°. For this S matrix, $\rho_\beta=\rho_\alpha$, and $\delta_\beta=\delta_\alpha$; δ_α varies more-or-less sinusoidally with $2\delta_{12}$, but is never more than a line-width from zero. (b) Plot of MAW phases ϕ, ρ_-, and ϵ parametrizing this same S matrix; just ϕ varies with δ_{12}. $\rho_+=\rho_-$ and $\delta_+=\delta_-=0$. With regard to the Hoshizaki parameters, only ϕ varies with δ_{12} and is identical to the MAW ϕ; $r_+=r_-=0.9513$, $\rho=0.053$, and $\delta_+=\delta_-=0$.

where now S_{aa} is indexed by the total angular momentum j; q is the momentum of either nucleon in the center-of-mass system. For our parametrization,

$$\sigma_r=(\pi/4q^2)\Sigma_j(2j+1)\mathrm{tr}(1-N_j^2),$$

where N is also indexed. For Hoshizaki's parameters

$$\sigma_r=(\pi/4q^2)\Sigma_j(2j+1)(2-r_{j-}^2-r_{j+}^2).$$

However for MAW,

$$\sigma_r=(\pi/4q^2)\Sigma_j(2j+1)(\cos 2\epsilon)^2(\sin^2\rho_-+\sin^2\rho_+).$$

Thus in this last case, an "elastic-scattering" parameter appears in the formula for the absorption cross section.

IV. CONCLUSION

We have presented a parametrization of the 2×2 S matrix (S_{aa}) for probability-nonconserving elastic nucleon-nucleon scattering, in which the (six) parameters (i) are well behaved for small excursions of S_{aa} about the unit matrix, and (ii) automatically obey a unitarity condition that we derive, namely $\det(1-S_{aa}S_{aa}^\dagger)\geq 0$. Neither of these features is a property of the other two parametrizations employed in the literature, those due to Hoshizaki and to MacGregor, Arndt, and Wright.

In our scheme, inelasticity is measured by a real, symmetric 2×2 matrix (N) imbedded in the standard Stapp-Ypsilantis-Metropolis form;

$$S_{aa}=e^{i\delta}e^{i\epsilon\sigma^1}Ne^{i\epsilon\sigma^1}e^{i\delta}.$$

Three real phases parametrize N and are nicely decoupled from the three ordinary phases. The partial-wave reaction cross section is proportional to $\mathrm{tr}(1-N^2)$.

An important application of our parametrization would be large-scale analysis of nucleon-nucleon scattering data. We anticipate no difficulty in computing S-matrix elements from our phase parameters [Eq. (14)] rather than from those of Hoshizaki [Eq. (17)] or MacGregor, Arndt, and Wright [Eq. (16)], even though our equations are somewhat longer, because most machine time is devoted to computing the M matrix at all the experimental scattering angles that go with each choice of S matrix. In fact, phase-shift analysis using our parameters should proceed faster because our ϕ does not execute large excursions when S_{aa} is varied slightly, as noted above.

The only drawback to our scheme appears to be in obtaining our phase parameters from a given S matrix. The formulas are complicated and there is a double solution (ρ_α, ϕ, ρ_β) for N. We shall discuss this further in a later publication.

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APPENDIX: EXTRACTING PHASE PARAMETERS FROM S_{aa}

While it is trivial to compute S_{aa} from our parameters using Eq. (14), it is not so straightforward to extract our parameters from S_{aa}. We have found the following method.

Assume that we are given S_{aa} numerically in the format

$$S_{aa}=\begin{pmatrix} R_{11}\exp 2i\delta_{11} & iR_{12}\exp 2i\delta_{12} \\ iR_{12}\exp 2i\delta_{12} & R_{22}\exp 2i\delta_{22}\end{pmatrix}. \quad \text{(A1)}$$

Then, taking guidance from Eq. (14), we introduce the new parameters θ_α, θ, and θ_β;

$$\delta_{11}=\delta_\alpha-\theta_\alpha\,,$$
$$2\delta_{12}=\delta_\alpha+\delta_\beta-\theta\,, \quad \text{(A2)}$$

and

$$\delta_{22}=\delta_\beta-\theta_\beta\,.$$

Comparing Eqs. (A1) and (A2) with Eq. (14) then reveals the three triangular relationships illustrated in Fig. 3. Two more equations may be obtained by taking the determinant of S_{aa}; from Eq. (A1)

$$\det S_{aa}=e^{2i(\delta_\alpha+\delta_\beta)}[R_{11}R_{22}e^{-2i(\theta_\alpha+\theta_\beta)}+R_{12}{}^2e^{-2i\theta}]\,,$$

while from Eqs. (13) and (14)

$$\det S_{aa}=e^{2i(\delta_\alpha+\delta_\beta)}[(\cos 2\gamma\cos 2\phi)^2-(\sin 2\Gamma)^2]\,.$$

Dividing out $\exp 2i(\delta_\alpha+\delta_\beta)$ and equating the real and imaginary parts of the remainder yields

$$\det N\equiv\Delta=(\cos 2\gamma\cos 2\phi)^2-(\sin 2\Gamma)^2 \quad \text{(A3a)}$$

and

$$R_{11}R_{22}\sin 2(\theta_\alpha+\theta_\beta)+R_{12}{}^2\sin 2\theta=0\,. \quad \text{(A3b)}$$

We now solve Eq. (A3b) for $(\theta_\alpha+\theta_\beta)$ by noting from Eq. (A2) that

$$\theta=\theta_\alpha+\theta_\beta+\delta\,, \quad \text{(A4a)}$$

defining

$$\delta=\delta_{11}+\delta_{22}-2\delta_{12}\,. \quad \text{(A4b)}$$

We get

$$\tan 2(\theta_\alpha+\theta_\beta)=-R_{12}{}^2(\sin 2\delta)/(R_{11}R_{22}+R_{12}{}^2\cos 2\delta)\,. \quad \text{(A5)}$$

The quantity $(\theta_\alpha-\theta_\beta)$ may next be determined using the fact that the two triangles illustrated in Fig. 3 have the same height;

$$R_{11}\sin 2\theta_\alpha=R_{22}\sin 2\theta_\beta\,. \quad \text{(A6)}$$

Equation (A6) can be put in the form

$$\tan(\theta_\alpha-\theta_\beta)=[(R_{22}-R_{11})/(R_{22}+R_{11})]\times\tan(\theta_\alpha+\theta_\beta)\,. \quad \text{(A7)}$$

We now determine θ_α and θ_β individually and then write down δ_α and δ_β from Eq. (A2).

$$\delta_\alpha=\delta_{11}+\theta_\alpha\,.$$
$$\delta_\beta=\delta_{22}+\theta_\beta\,.$$

The remaining "elastic" parameter, ϵ, may be

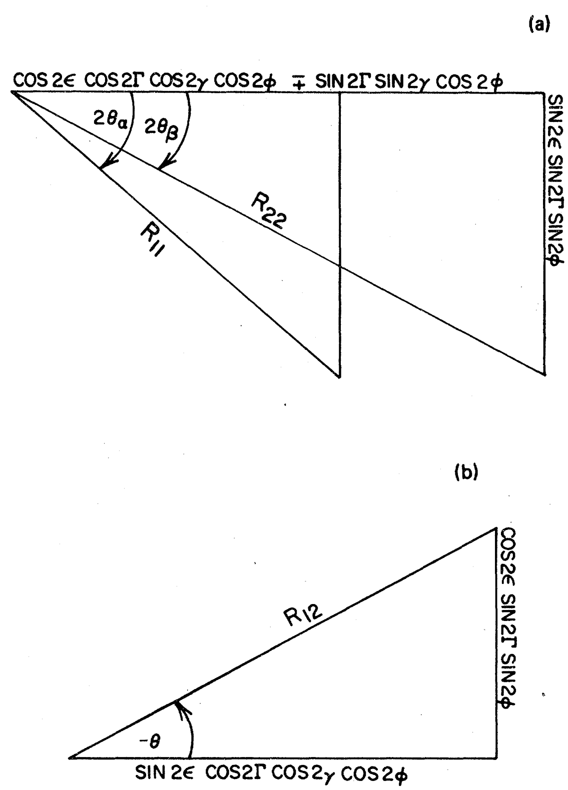

FIG. 3. Relationship between R_{11}, R_{22}, R_{12}, θ_α, θ_β, θ, and ϵ, Γ, γ, and ϕ. In part (a), α (β) corresponds to the minus (plus) sign.

found by taking the ratio

$$2R_{12}(\cos\theta)/(R_{11}\cos 2\theta_\alpha + R_{22}\cos 2\theta_\beta)\,, \tag{A8}$$

which from Fig. 3 is seen to

$$= \tan 2\epsilon\,.$$

Next we solve for the parameters Γ, γ, and ϕ of the absorption matrix $U_{aa} \equiv N$. First we determine the experimental values of N. From Eq. (12),

$$N = \begin{pmatrix} (\cos 2\Gamma \cos 2\gamma - \sin 2\Gamma \sin 2\gamma)\cos 2\phi & -\sin 2\Gamma \sin 2\phi \\ -\sin 2\Gamma \sin 2\phi & (\cos 2\Gamma \cos 2\gamma + \sin 2\Gamma \sin 2\gamma)\cos 2\phi \end{pmatrix}. \tag{A9}$$

Thus

$$N_{22} + N_{11} = 2\cos 2\Gamma \cos 2\gamma \cos 2\phi \tag{A10a}$$

which, from Fig. 3(a)

$$= (R_{22}\cos 2\theta_\beta + R_{11}\cos 2\theta_\alpha)/\cos 2\epsilon\,. \tag{A10b}$$

Similarly

$$N_{22} - N_{11} = 2\sin 2\Gamma \sin 2\gamma \cos 2\phi \tag{A11a}$$

which, from Fig. 3(a)

$$= R_{22}\cos 2\theta_\beta - R_{11}\cos 2\theta_\alpha\,. \tag{A11b}$$

N_{11} and N_{22} may now be determined.

N_{12} is readily evaluated from Eq. (A9) since

$$N_{12} = -\sin 2\Gamma \sin 2\phi \tag{A12a}$$

which, from Fig. 3(b)

$$= R_{12}(\sin\theta)/\cos 2\epsilon\,. \tag{A12b}$$

Thus the matrix elements of N are determined.

We now may solve for Γ, γ, and ϕ. We begin with Γ. Squaring Eq. (A10a) gives

$$(\mathrm{tr}N)^2 = (2\cos 2\Gamma \cos 2\gamma \cos 2\phi)^2\,,$$

while from Eq. (A3a)

$$(\cos 2\gamma \cos 2\phi)^2 = \Delta + (\sin 2\Gamma)^2\,.$$

Thus

$$(\mathrm{tr}N)^2 = (2\cos 2\Gamma)^2[\Delta + (\sin 2\Gamma)^2]\,.$$

This can be put in the form

$$(1+\cos 4\Gamma)^2 - 2(\Delta+1)(1+\cos 4\Gamma) + (\mathrm{Tr}N)^2 = 0\,.$$

The solution is

$$\cos 4\Gamma = \det N \pm [\det(1-N^2)]^{1/2}\,, \tag{A13}$$

where we have used the identity

$$(\Delta+1)^2 - (\mathrm{Tr}N)^2 = \det(1-N^2)\,. \tag{A14}$$

One sees that Γ may assume four values, two through the choice of plus or minus root in Eq. (A13), and a further doubling since 4Γ may fall in either the upper or lower half of the unit circle.

We next solve for ϕ. From Eq. (A12a) one has

$$\sin 2\phi = -N_{12}/\sin 2\Gamma\,. \tag{A15}$$

We will require, with no loss in generality, that 2ϕ fall in the right half of the unit circle.

Finally we determine γ. Dividing Eq. (A11a) by Eq. (A10a) yields

$$\tan 2\gamma = (N_{22} - N_{11})/(N_{22} + N_{11})\tan 2\Gamma\,. \tag{A16}$$

Γ, ϕ, and γ are now determined, except for the ambiguity as to whether 2γ falls in the right or left half of the unit circle. We resolve this by invoking Eq. (A10a);

$$\mathrm{Tr}N = 2\cos 2\Gamma \cos 2\phi \cos 2\gamma\,.$$

$\cos 2\Gamma \cos 2\phi$ is always positive, so for $\mathrm{Tr}N > 0$ ($\mathrm{Tr}N < 0$), 2γ must fall in the right (left) half of the unit circle.

That there are four sets of absorption parameters [(Γ, ϕ, γ) or equivalently, $(\rho_\alpha, \phi, \rho_\beta)$], which parametrize a given matrix N should come as no surprise since there are four sets which parameterize N when $N_{12} = 0$, namely

$$(\rho_\alpha = \pm \tfrac{1}{2}\arccos N_{11}, \quad \phi = 0, \quad \rho_\beta = \pm \tfrac{1}{2}\arccos N_{22})\,. \tag{A17}$$

Even when N_{12} is unequal to zero, there remains one simple relation between pairs of solutions; if $(\rho_\alpha, \phi, \rho_\beta)$ is a solution, so is $(-\rho_\alpha, -\phi, -\rho_\beta)$. However, in the other pairing of solutions, corresponding to taking plus and minus roots in Eq. (A13), there is no simple relation in general.

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[1]The Roper resonance is a baryon resonance with the quantum numbers of the nucleon but a mass of ~ 1450 MeV and a width of ~ 200 MeV.

[2]H. P. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. 105, 302 (1957).

[3]M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. 169, 1149 (1968).

[4]N. Hoshizaki, Progr. Theoret. Phys. Suppl. 42, 1 (1968).

[5]J. M. Blatt and L. C. Biedenharn, Phys. Rev. 86, 399 (1952).

[6]P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Oxford, London, 1958), p. 257.

[7]Whereas a symmetric unitary 4×4 matrix provides one parameter too many for the 2×2 sector, a symmetric unitary 3×3 matrix provides one too few. We have shown this rigorously [R. A. Bryan (unpublished)] but it can be surmised from the fact that a single inelastic channel, call it c, provides only two inelastic transitions, e.g., $\langle a, {}^3S_1 | S | c\rangle$ and $\langle a, {}^3D_1 | S | c\rangle$, hence two parameters, whereas three are needed.

[8]In our scheme there are in general two solutions $(\rho_\alpha, \phi, \rho_\beta)$, nontrivially related, for a given absorption matrix N. We plot in Fig. 2 the solution corresponding to taking the minus sign in Eq. (A13). If the plus sign is taken, the values of ρ_α and ϕ are interchanged (for this particular matrix).

[9]Dr. B. J. VerWest has proved this unitarity condition for an arbitrary number of two-body absorption channels.

[10]I would like to thank Dr. VerWest for pointing this out to me.