Three-body calculations for reactions involving transfer of alpha cluster using the bound state approximation

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The three-body equations of Alt, Grassberger, and Sandhas have been solved for a system of three particles, viz., α, d, and ^{40}Ca, to obtain the on-shell transition amplitudes of various rearrangement processes, using what is called the bound state approximation. The input in this calculation, viz., the two-body t matrices, representing the interaction between the pairs of particles, is taken to be of a separable form conforming to the bound S states of the pairs. The absolute values of the differential cross section of the (^{6}Li,d) reaction on ^{40}Ca, leading to the ground state of the ^{44}Ti, obtained from this calculation are compared with the experimental results of Fulbright *et al.* to assess how far the three-body calculations have a bearing on α-transfer reactions.

[NUCLEAR REACTIONS Bound state approximation, α-transfer reaction, three-body calculation.]

I. INTRODUCTION

Transfer reactions of the kind $A + a = B + b$, where the particle a is composite, consisting of the outgoing particle b and transferred particle x, may be analyzed in the framework of three-body dynamics wherein the internal structures of the clusters A, b, and x are ignored. The interactions between the pairs of particles enter into the Alt, Grassberger, and Sandhas (AGS)[1] form of Faddeev equations through the two-body transition operators in three-body space, viz., $T_k(z)$. The bound state approximation (BSA) implies that in the spectral resolution of $T_k(z)$ only the contribution of the bound state of the interacting pair is retained and that due to the continuum states of the latter is ignored.[2] As a result, the matrix representing the operator $T_k(z)$ in the momentum representation becomes separable,[2,3] resulting in considerable simplification of an AGS equation when written in the angular momentum representation.[4,5] These equations are solvable provided one takes proper care of the singularities in the kernel.[5]

It may be pointed out that Greben and Levin[2] have recently tested the BSA within the framework of Mitra's three-body model[6] of nuclear stripping reactions using a different version of three-body equations. They observe that the agreement of the BSA with the "exact" " calculations is good only for incident energies close to the breakup threshold, which is to be expected since the neglected continuum terms in the BSA tend to become significant at ^{6}Li energy greater than the threshold value, thus impairing the agreement of BSA. Now it is known that a separable approximation called the unitary pole approximation (UPA)[7] for the two-body t matrix is valid even at positive energies,[8-10] so one would anticipate that the validity of BSA for transfer reactions may be extended to energies higher than the breakup threshold if one uses, in the separable (or bound state) approximation, a propagator $\tau_k^{Jk}(z - u_k^2)$ conforming to the unitarity requirement[7] rather than that having a form used by Greben and Levin [viz., Eq. (4), Sec. II]. This, of course, is naturally taken care of in the Mitra[6] model of stripping with separable pair potentials; the use of BSA within the Mitra three-body model merely amounts to using a propagator not conforming to the unitarity requirement. So one hopes that, in general, constructing a BSA using a propagator conforming to the unitarity criterion would make the approximation more effective for three-body calculations.

In the present paper we solve the AGS equations for a system of three particles, viz., α, d, and ^{40}Ca, to obtain the on-shell transition amplitudes pertaining to various rearrangement processes. In Sec. II the method of reduction of AGS equations is outlined and in Sec. III the relevant form factors g_k^{Jk} have been obtained using, as input, (i) the separation energy of the pairs of particles in the bound state and (ii) the S-wave phase shifts pertaining to the interaction. A procedure similar to that used by Bolsterli and Mackanzie[11] has been followed. In Sec. IV we give details of the computational methods used which are rather involved because of the presence of two kinds of singularities in the kernel of Eq. (5), viz., (i) the "pole" of the propagator τ_k^{Jk} at $E - u_k^2 = -\epsilon_{Bk}$, i.e., at the on-shell value Q_k of the momentum u_k, and (ii) logarithmic singularities in the region $0 \leq u_k \leq \sqrt{E}$ when $0 \leq q_i \leq \sqrt{E}$. To handle the former we have used a method based on the work of Sasakawa[12] and Kowalski,[13] and to deal with the latter we have used Doleschall's method of special quadratures.[5] Finally, to see if the three-body calculations have some bearing on the α-transfer reactions, the calculated absolute values of the cross sections of the (^{6}Li, d) reaction on ^{40}Ca leading to the ground state of ^{44}Ti are compared with those measured by Fulbright *et al.*,[14] for two different ^{6}Li energies (Sec. V). A general agreement is seen although the finer features of the angular distribution are not reproduced by the present calculation.

II. REDUCTION OF THREE-BODY EQUATIONS

The three-body AGS equations are

$$U_{ij}(z) = (1-\delta_{ij})(z - H_0) + \sum_{k=1}^{3} (1-\delta_{ik}) T_k(z) G_0(z) U_{kj}(z) , \quad (1)$$

where

$$T_k(z) = V_k + V_k G_k(z) V_k \quad (2)$$

and

$$G_k(z) = (z - H_k)^{-1} .$$

In BSA we have (Sec. III)

$$\langle \vec{r}_k \vec{u}_k | T_k(z) | \vec{r}_k' \vec{u}_k' \rangle = \delta(\vec{u}_k - \vec{u}_k') g_k(\vec{r}_k) \tau_k(z - u_k^2) g_k(\vec{r}_k') , \quad (3)$$

$\vec{u}_k$ being the momentum of the kth (noninteracting) particle, and $\vec{r}_k$ the relative momentum of the other two (interacting) particles, the momenta being expressed in $\sqrt{E}$ units.[3,5] The propagator $\tau_k(z - u_k^2)$ in Eq. (3) is given by

$$\tau_k(z - u_k^2) = (z - u_k^2 + \epsilon_{Bk})^{-1} , \quad (4)$$

$-\epsilon_{Bk}$ being the bound state energy of the interacting pair. This approximation, viz., Eq. (3), simplifies the problem from the computational viewpoint for, in the angular momentum basis,[4,5] viz., $|p_i q_i [(L_i S_i) J_i s_i] K_i l_i : JM\rangle \equiv |p_i q_i \alpha_i : JM\rangle \equiv |p_i q_i (L_i S_i) \beta_i : JM\rangle$, the AGS equation (1), reduces to the form[5]

$$T_{ij}(q_i q_j' \beta_i \beta_j : J) = K_{ij}(q_i q_j' \beta_i \beta_j : J) + \sum_k \sum_{\beta_k} \int u_k^2 du_k K_{ik}(q_i u_k \beta_i \beta_k : J) \tau_k^{Jk}(z - u_k^2) T_{kj}(u_k q_j' \beta_k \beta_j : J) , \quad (5)$$

where the Born term is given by

$$K_{ik}(q_i u_k \beta_i \beta_k : J) = (1 - \delta_{ik}) \int\int \frac{g_i^{Ji}(p_i) p_i^2 dp_i}{(z - p_i^2 - q_i^2)} \langle p_i q_i \alpha_i : JM | r_k u_k \alpha_k : JM \rangle r_k^2 dr_k g_k^{Jk}(r_k) , \quad (6)$$

and $g_i^{Ji}(p_i) = \sqrt{4\pi} g_i(\vec{p}_i)$. The overlap $\langle p_i q_i \alpha_i : JM | r_k u_k \alpha_k : JM \rangle$ may be evaluated by graphical methods of spin algebras[15] and, in such a simple case as ^{40}Ca, α, and the d system, wherein $K_i = K_j = K_k = 1$, it simplifies to[16]

$$\langle p_i q_i \alpha_i : J | r_k u_k \alpha_k : J \rangle = \frac{\delta_{l_i l_k} \{K_i l_i J\}}{2 B_{ik}^3} \int_{-1}^{+1} \frac{\delta(p_i - r_i) \delta(p_k - r_k)}{p_i^2 r_k^2} P_{l_k}(x) dx , \quad (7)$$

where

$$p_i = r_i = \frac{|A_{ik} \vec{q}_i + \vec{u}_k|}{B_{ik}} ,$$

$$r_k = p_k = \frac{|\vec{q}_i + A_{ik} \vec{u}_k|}{B_{ik}} , \quad x = \hat{q}_i \cdot \hat{u}_k ,$$

$$B_{ik}^2 = (1 - A_{ik}^2) ,$$

and

$$A_{ik}^2 = \frac{m_i m_k}{(M - m_i)(M - m_k)} ,$$

M being the sum of masses of the three particles. From Eqs. (6) and (7) we have, for the Born term,

$$K_{ik}(q_i u_k \beta_i \beta_k : J) = \frac{\{K_i l_i J\} \delta_{l_i l_k} (1 - \delta_{ik})}{2 q_i u_k A_{ik} B_{ik}} \times \frac{1}{2} \int_{-1}^{+1} \frac{g_i^{Ji}(r_i) g_k^{Jk}(p_k) P_{l_k}(x) dx}{(D_{ik} - x)} , \quad (8)$$

where

$$D_{ik} = \frac{z B_{ik}^2 - q_i^2 - u_k^2}{2 A_{ik} q_i u_k} .$$

In the present case, $\beta_i \equiv l_i$ and $\beta_k \equiv l_k$, other angular momentum quantum numbers being redundant. Also the sums over $L_i S_i$ and $L_k S_k$, which appear in Doleschall's equations[5] become redundant in this case since $T_k(z)$ is one term separable and the form factors $g_k^{J_k}$ are related to a single bound state characterized by a definite (L_k,S_k) coupled to J_k. The amplitudes T_{ij} obtained by solving the coupled integral equation (5) may be related to the physical transition amplitudes $\langle\vec{Q}_i d_i J_i M_i|U_{ij}|\vec{Q}_j d_j J_j M_j\rangle$ by the relation

$$\langle\vec{Q}_i d_i J_i M_i|U_{ij}|\vec{Q}_j d_j J_j M_j\rangle=\left(\frac{Q_iQ_j}{4}\right)^{1/2}\left(\frac{N_iN_j}{4\pi}\right) \times\sum_{K_i l_i K_j l_j J} T_{ij}(Q_iQ_j\beta_i\beta_j:J)\sum_{k_i m_i k_j m_j M}(s_i d_i J_i M_i|K_i k_i)(s_j d_j J_j M_j|K_j k_j) \times(K_i k_i l_i m_i|JM)(K_j k_j l_j m_j|JM)Y_{l_i m_i}(\hat{Q}_i)Y^*_{l_j m_j}(\hat{Q}_j)\,, \tag{9}$$

where Q_i and Q_j are the on-shell values of the momenta q_i and q_j', respectively, so that $Q_i^2-\epsilon_{Bi}=Q_j^2-\epsilon_{Bj}=E$. Finally, the differential cross section for the transition $j\to i$ may be expressed in terms of the physical transition amplitudes by

$$\frac{d\sigma}{d\Omega}=(2\pi)^4\frac{\hbar^2}{2\mu_jQ_j^2}\left|\langle\vec{Q}_i d_i J_i M_i|U_{ij}|\vec{Q}_j d_j J_j M_j\rangle\right|_{\rm avg}^2. \tag{10}$$

III. TWO-BODY INPUT IN THE FRAMEWORK OF BSA

The spectral resolution of the two-body resolvent operator in three-body space is expressed as

$$G_k(z)=\sum_n\int\frac{d^3q_k|\vec{q}_k\phi_k^{(n)}\rangle\langle\vec{q}_k\phi_k^{(n)}|}{z-q_k^2+\epsilon_{Bk}^{(n)}} +\int\int\frac{d^3q_kd^3p_k|\vec{q}_k\psi_k^+\rangle\langle\vec{q}_k\psi_k|}{z-q_k^2-p_k^2}\,, \tag{11}$$

where $|\phi_k^{(n)}\rangle$ is one of the bound states of the interacting pair with energy $-\epsilon_{Bk}^{(n)}$, and $\vec{q}_k$ is the momentum of the noninteracting particle, the summation being made over all the bound states of the interacting pair. The second term $|\psi_k^+\rangle$ represents the continuum states of the interacting pair. Now, in BSA only the pole-dominant terms are to be retained, the contribution due to the continuum states of the interacting pair being neglected. Correspondingly, $T_k(z)$, the two-body t matrix in three-body space, defined by Eq. (2), is approximated as

$$T_k(z)\simeq\sum_n\int\frac{d^3q_kV_k|\vec{q}_k\phi_k^{(n)}\rangle\langle\vec{q}_k\phi_k^{(n)}|V_k}{(z-q_k^2+\epsilon_{Bk}^{(n)})}. \tag{12}$$

To restrict the size of the computational problem one may invoke a further approximation by considering only one bound state for each pair of particles. Thus for the initial and final partitions j and i of the compound system, the single bound states $|\phi_j\rangle$ and $|\phi_i\rangle$ that occur in an equation such as (12) to approximate the operators T_j and T_i, respectively, may be identified with the actual two-body bound states involved in the transfer reaction. For example, in the case of the (^{6}Li,d) reaction on ^{40}Ca leading to the ground state of ^{44}Ti, $|\phi_j\rangle$ may be identified as the ground state of ^{6}Li as an $(\alpha+d)$ system with $[L=0,\ S=1]$ $J=1$, while $|\phi_i\rangle$ as the ground state of ^{44}Ti is a (^{40}Ca$+\alpha$) system with $[L=0,\ S=0]\,J=0$. In addition, the bound state pertaining to the third partition may be identified with the ground state of ^{42}Sc as a (^{40}Ca$+d$) system with $[L=0,S=1]J=1$. Hence, dropping the summation as well as subscripts in Eq. (12), one gets

$$\langle\vec{r}_k,\vec{u}_k|T_k(z)|\vec{r}_k'\vec{u}_k'\rangle\simeq\frac{\langle\vec{r}_k|V_k|\phi_k\rangle\langle\phi_k|V_k|\vec{r}_k'\rangle}{z-u_k^2+\epsilon_{Bk}}\delta(\vec{u}_k-\vec{u}_k')\,, \tag{3'}$$

which, on comparison with the form of Eq. (3), gives

$$g(\vec{r}_k)\equiv\langle\vec{r}_k|g_k\rangle=\langle\vec{r}_k|V_k|\phi_k\rangle =-\langle\vec{r}_k|(\epsilon_{Bk}+h_0)|\phi_k\rangle=-(\gamma_k^2+\epsilon_{Bk})\phi_k(\vec{r}_k)\,, \tag{13}$$

showing the connection between the form factors $g_k(\vec{r}_k)$ and the two-body bound state wave function $\phi_k(\vec{r}_k)$, provided $\phi_k(\vec{r}_k)$ or $g_k(\vec{r}_k)$ is normalized. If $g_k(r_k)$ is not normalized then

$$\phi_k(\vec{r}_k)=-\frac{N_kg_k(\vec{r}_k)}{(\epsilon_{Bk}+r_k^2)}\,, \tag{13'}$$

where

$$N_k^{-2}=\int\frac{g_k^2(\vec{r}_k')d^3r_k'}{(\epsilon_{Bk}+r_k'^2)^2}.$$

Now, the additional requirement of "unitarity"[7] keeps the form of Eq. (3) intact with the following replacement for the propagator [Eq. (4)]:

$$\tau_k(z-u_k^2)=-\left[\frac{1}{\lambda}+\int\frac{g_k^2(r_k)d^3r_k}{z-u_k^2-r_k^2}\right]^{-1}, \tag{14}$$

where

$$\lambda^{-1} = \int \frac{g_k^2(\vec{r}_k) d^3 r_k}{(\epsilon_{Bk} + r_k^2)}.$$

Equations (3), (13′), and (14) may be regarded as the basis for our BSA.

The determination of the form factor $g_k(\vec{r}_k)$ from the bound state using Eq. (13′) is not possible since the two-body bound state wave function $\phi_k(\vec{r}_k)$ is not delineated except for its tail. We use an alternative to construct the form factor $g_k(\vec{r}_k)$ from the two-body binding (or separation) energy and the S-wave phase shifts of elastic scattering of the interacting pair. The separation energies for the bound states of three pairs (in their ground states) are accurately known. Also, for the α-d system, the S-wave phase shift as a function of energy is available.[17] For the other two pairs of particles the S-wave phase shifts have been estimated from the real parts of the optical potentials[18,19] using the Wentzel-Kramers-Brillouin-Jeffreys (WKBJ) method.[20] Instead of trying arbitrary forms for the form factors with flexible parameters and adjusting the latter to fit the two-body data (binding energy and phase shifts), we have followed a method[11] wherein the form factor is calculated numerically using the equation

$$g_k(\vec{r}_k) = r_k^{-3/2}(r_k^2 + \epsilon_{Bk})^{1/2} \sin^{1/2}\delta_0(r_k) e^{(1/2)\mathcal{P}\tilde{\delta}_0(r_k)}, \quad (15)$$

where

$$\mathcal{P}\tilde{\delta}_0(r_k) = \frac{2}{\pi} \mathcal{P} \int \frac{\delta_0(p')p'dp'}{r_k^2 - p'^2}. \quad (15')$$

The numerical values of g_k are then fitted to a suitable analytic form, e.g., a sum of Yamagauchi type terms

$$g_k(\vec{r}_k) = \sum_{i=1}^{3} \frac{A_i}{(r_k^2 + \beta_i^2)} + \frac{D \sin(r_k a)}{r_k(r_k^2 + \beta^2)} \quad (16)$$

(see Table I).

IV. COMPUTATIONAL DETAILS

The kernel $K_{ik}(q_i u_k \beta_i \beta_k : J)\tau_k^{Jk}(E - u_k^2 + io)$ of the coupled integral equation (5) contains essentially two kinds of singularities, viz., (i) the pole of the propagator τ_k^{Jk} at $E - u_k^2 = -\epsilon_{Bk}$, i.e., at $u_k = Q_k$, and (ii) logarithmic singularities in the region $0 \leq u_k \leq \sqrt{E}$ when $0 \leq q_i \leq \sqrt{E}$.[21] To deal with the pole we follow a method based on the one outlined by Sasakawa[12] and Kowalski.[13] Thus instead of Eq. (5), the following coupled integral equations are solved:

$$\Gamma_{ij}(q_i Q_j l_i l_j : J) = K_{ij}(q_i Q_j l_i l_j : J) + \sum_{k=1}^{3} \sum_{l_k} du_k \Lambda_{ik}(q_i u_k l_i l_k : J) \times \Gamma_{kj}(u_k Q_j l_k l_j J). \quad (17)$$

Here, the new kernel Λ_{ik} is obtained from the kernel of Eq. (5) by subtracting out the pole term at $u_k = Q_k$, i.e.,

$$\Lambda_{ik}(q_i u_k l_i l_k : J) = \left[K_{ik}(q_i u_k l_i l_k : J)\tau_k^{Jk}(E - u_k^2) - \frac{K_{ik}(q_i Q_k l_i l_k : J) N_k^2 (Q_k^2 + 1)}{(E - u_k^2 + \epsilon_{Bk} + io)(u_k^2 + 1)} \right] u_k^2. \quad (18)$$

The new kernel obviously vanishes at $u_k = Q_k$ since

$$\tau_k(E - u_k^2 + io) \xrightarrow[u_k \to Q_k]{} \frac{N_k^2}{E - u_k^2 + \epsilon_{Bk} + io}.$$

The kernel given by Eq. (18) still has logarithmic singularities which are taken care of by Doleschall's method of special quadratures.[5,22] Accordingly, the range of the u_k integration, as also that of continuous parameter q_i, is divided into three intervals: (i) 0 to $\sqrt{E}$, (ii) $\sqrt{E}$ to $[(E + 2\epsilon_{Bk})]^{1/2}$, and (iii) $[(E + 2\epsilon_{Bk})]^{1/2}$ to ∞. Within these intervals 6, 3, and 6 mesh points, respectively, are chosen, being essentially the Gauss-Legendre points mapped on to the respective intervals. In the first interval, the mesh points are determined by mapping the Gauss-Legendre points t onto this interval by the substitution

$$\frac{u_k}{\sqrt{E}} \equiv y_k = \left[1 - \left(\frac{1-t}{2}\right)^{1+a}\right]^{1/(1+b)}$$

so as to prevent any one of the mesh points u_k from coming close to the value of $A_{ik}\sqrt{E}$, and to eliminate the branch point behavior of the quantities Γ_{ik}. In this calculation the values of a and b are chosen to be 1.0 and 0.5, respectively.

TABLE I. Numerical values of the parameters of the form factors [Eq. (16)] in the case of the three pairs of clusters.

Particle pair	A_1 ($fm^{-3/2}$)	A_2 ($fm^{-3/2}$)	A_3 ($fm^{-3/2}$)	β_1^2 (fm^{-2})	β_2^2 (fm^{-2})	β_3^2 (fm^{-2})	D ($fm^{-5/2}$)	β^2 (fm^{-2})	a (fm)	ϵ_{Bk} (fm^{-2})
α-d	1.45×10^{-3}	2.26	−5.17	7.7×10^{-5}	0.677	0.92	−0.066	1.24	7.42	0.0943
^{40}Ca-d	2.56×10^{-2}	18.73	−6.2	1.84×10^{-3}	3.35	0.126	59.8	1.99	1.67	0.9526
^{40}Ca-α	9.1×10^{-3}	8.0	7.0	7.0×10^{-5}	0.065	0.49	−32.5	2.77	7.85	0.9109

For the second and third intervals, the Gauss-Legendre points t, which are 3 and 6 in number, respectively, in the two cases, are mapped onto the respective intervals by the transformations

$$u_k=[(t+1)\epsilon_{Bk}+E]^{1/2}$$

and

$$u_k=\frac{2}{1-t}[(E+2\epsilon_{Bk})]^{1/2},$$

thus requiring the on-shell point $u_k=Q_k$ to be one of the mesh points.[23] The logarithmic singularity which occurs in the interval $0\le u_k\le\sqrt{E}$ when $0\le q_i\le\sqrt{E}$ is taken care of by subtracting, from the kernel Λ_{ik}, the singular function

$$\frac{\delta_{l_i l_k}(1-\delta_{ik})\{K_i l_i J_i\}}{2B_{ik}A_{ik}}\frac{u_k}{q_i}Q_{l_k}(D_{ik})g_i^{J_i}(\tilde{r}_i)g_k^{J_k}(\tilde{p}_k)\tau_k^{J_k}(E-u_k^2+io), \tag{18'}$$

and adding the same term subsequently. (This is to be done only when both q_i and u_k lie between 0 and $\sqrt{E}$.) Here

$$Q_l(D)\equiv\tfrac{1}{2}\int_{-1}^{+1}\frac{P_l(x)dx}{D-x}$$

is the Legendre function of the second kind which contains logarithmic singularity. The quantities $\tilde{r}_i$ and $\tilde{p}_k$, which are to be finite functions of q_i and u_k, must bo so chosen that, for $x_i=D_{ik}$, r_i, and p_k join smoothly with $\tilde{r}_i$ and $\tilde{p}_k$, respectively. The following choice has been made:

$$\tilde{r}_i=(E-q_i^2)^{1/2}\text{ and }\tilde{p}_k=(E-u_k^2)^{1/2}.$$

The subtracted part of the kernel thus becomes free from logarithmic singularity and the integral over u_k of such a function (multiplied by the unknown function Γ_{kj}) may be expressed in terms of ordinary quadratures. A special quadrature must, however, be developed for expressing that integral which includes the singular function $Q_{l_k}(D_{ik})$. To express an integral of the form

$$\int_0^{\sqrt{E}}Q_{l_k}(D_{ik})F(u_k)du_k, \tag{19}$$

one transforms the y_k points further, by the transformation

$$y_k=z\left(d-\frac{5-4d}{2}z^2+\frac{2d-3}{4}z^4\right), \tag{20}$$

to prevent the huddling of the quadrature points near the end of the interval. In this calculation $d=0.5$. For a given set of values of $q_i(=E^{1/2}x_i)$ and l_k, the set of "special weights" $[h_l^s(x_i l_k)]$ is determined by evaluating the integrals

$$I_n(x_i,l_k)\equiv\int_0^1 Q_{l_k}[x_i y_k(z)]P_m(z)dz$$

for different values of n ranging from 1 to 6, analytically, and equating them to the representative sums, viz.,

$$\sum_{l=1}^{6}h_l^s(x_i,l_k)P_m(z_l)\equiv\sum_{l=1}^{6}P_{nl}h_l^s,$$

where $P_{nl}\equiv P_m(z_l)$ and $m=2n-2$ if l_k is odd, and $2n-1$ if l_k is even.[24] The matrix equation

$$I_n=\sum_{l=1}^{6}P_{nl}h_l^s \tag{21}$$

then yields the set of special weights h_l^s. Such a set of special weights may be computed for different sets of values of x_i and l_k.

Finally, the set of coupled equations (17) may be converted into a set of linear algebraic equations by using appropriate quadratures in each of three intervals of the u_k integration, and these may be solved for the amplitudes $\Gamma_{kj}(u_kQ_jl_kl_j:J)$ by matrix inversion. In terms of these amplitudes, the on-energy shell solutions of Eq. (5) are given by

$$T_{ij}(Q_iQ_jl_il_j:J)=\Gamma_{ij}(Q_iQ_jl_il_j:J)+\sum_{k=1}^{3}\Gamma_{ik}(Q_iQ_kl_il_k:J)I_{kj}. \tag{22}$$

Here $l_i=l_j=l_k$ and the matrix I_{kj} is given by

$$I_{kj}-d_{kj}=\sum_{p=1}^{3}d_{kp}I_{pj}, \tag{23}$$

where

$$d_{kp}=d_{kp}^{(R)}+id_{kp}^{(Im)}$$

$$d_{kp}^{(R)}=(Q^2+1)N_k^2\mathcal{P}\int_0^\infty\frac{u_k^2du_k\Gamma_{kp}(u_kQ_pl_kl_p:J)}{(u_k^2+1)(Q_k^2-u_k^2)},$$

and

$$d_{kp}^{(Im)}=-\frac{\pi}{2}N_k^2Q_k\Gamma_{kp}(Q_kQ_pl_kl_p:J).$$

Consequently, $I_{kj}=I_{kj}^{(R)}+iI_{kj}^{(Im)}$, where $I_{kj}^{(R)}$ and $I_{kj}^{(Im)}$ may be determined from Eq. (23) by inversion. Finally, we have from Eq. (22)

$$T_{ij}(Q_iQ_jl_il_j:J)=T_{ij}^{(R)}(l_i)+iT_{ij}^{(Im)}(l_i), \tag{24}$$

where

$$T_{ij}^{(R)}(l_i) = \Gamma_{ij}(Q_i Q_j l_i l_j : J) + \sum_{k=1}^{3} \Gamma_{ik}(Q_i Q_k l_i l_k : J) I_{kj}^{(R)} \tag{25}$$

and

$$T_{ij}^{(Im)}(l_i) = \sum_{k=1}^{3} \Gamma_{ik}(Q_i Q_k l_i l_k : J) I_{kj}^{(Im)}. \tag{26}$$

Now

$$|\langle \vec{Q}_i d_i J_i | U_{ij} | \vec{Q}_j d_j J_j \rangle|_{\rm avg}^2 = \sum_{d_i M_i d_j M_j} \frac{|\langle \vec{Q}_i d_i J_i M_i | U_{ij} | \vec{Q}_j d_j J_j M_j \rangle|^2}{(2s_j+1)(2J_i+1)},$$

which, using Eqs. (9) and (24), yields on simplification

$$|\langle \vec{Q}_i d_i J_i | U_{ij}(E) | \vec{Q}_j d_j J_j \rangle|_{\rm avg}^2 = \left(\frac{N_i N_j}{4\pi}\right)^2 \left(\frac{Q_i Q_j}{4}\right) \sum_{l_i l_i'} [T_{ij}^R(l_i) T_{ij}^R(l_i') + T_{ij}^{(Im)}(l_i) T_{ij}^{(Im)}(l_i')] \times \frac{(2l_i+1)(2l_i'+1) P_{l_i}(\cos\theta) P_{l_i'}(\cos\theta)}{16\pi^2}, \tag{27}$$

where $\cos\theta = \hat{Q}_i \cdot \hat{Q}_j$. This equation, in conjunction with Eq. (10), determines the differential cross section of the transfer reaction $j \to i$.

V. RESULTS

The absolute values of the differential cross section of the (^{6}Li, d) reaction on ^{40}Ca leading to

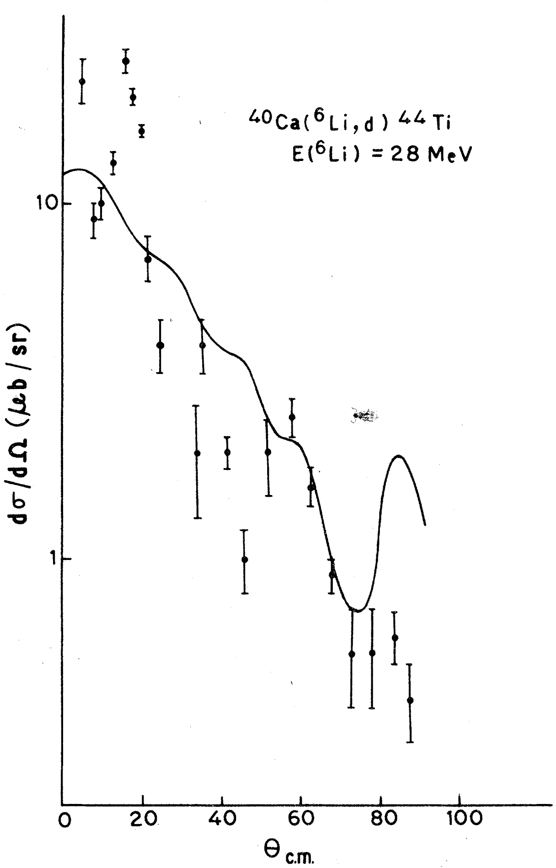

FIG. 1. Calculated differential cross section of the (^{6}Li, d) reaction on ^{40}Ca, leading to the ground state of ^{44}Ti, for $E(^6\text{Li}) = 28$ MeV. In this case the calculated values were reduced by a factor of 2 for comparison with experimental points which are due to Fulbright *et al.*

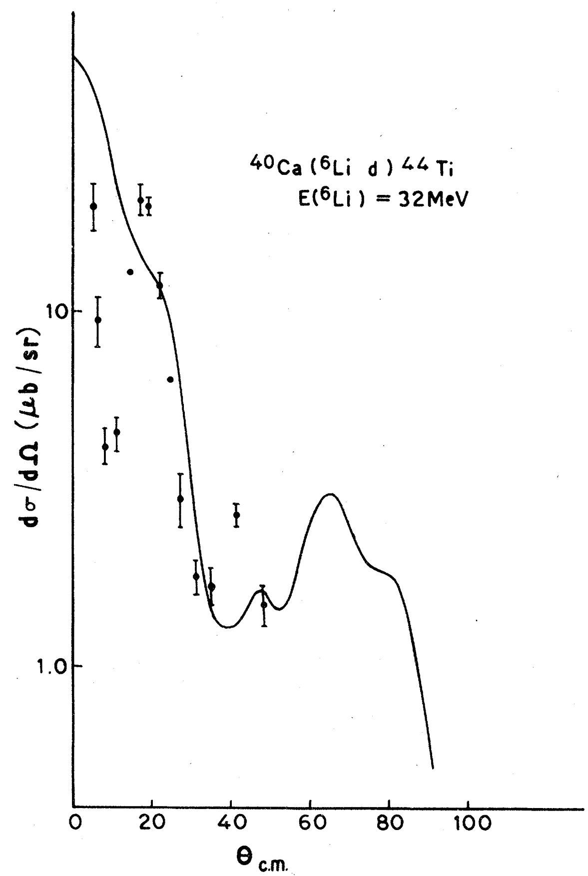

FIG. 2. Calculated differential cross section of the (^{6}Li, d) reaction on ^{40}Ca leading to the ground state of ^{44}Ti for $E(^6\text{Li}) = 32$ MeV. The corresponding experimental points are due to Fulbright *et al.* In this case no scaling of calculated values was needed.

the ground state of ^{44}Ti are computed for two different incident energies, viz., 28 and 32 MeV, and are compared with the corresponding experimental values due to Fulbright *et al.*[14] (Figs. 1 and 2). It is rather encouraging to note that the calculated values of the absolute cross section show a general agreement with the experimental values although the finer features of the angular distribution of the outgoing particles are not reproduced. One may note, for instance, that for higher energy [$E(^6\text{Li}) = 32$ MeV)] the experimental cross section falls off, on the average, more rapidly with θ than in the case of lower energy [$E(^6\text{Li}) = 28$ MeV]—which is also borne out by the three-body calculations in BSA. The general agreement of the absolute values of the cross section in BSA is to be expected because in this calculation the form factors of the separable t are normalized, resulting in normalized bound state wave functions of the two particle systems. However, this calculation at the present stage cannot reproduce the finer details of the angular distribution for the following reasons: Our inputs are rather crude. (A better method of constructing the form factor of the separable interaction of two clusters would be through the use of the wave function of relative motion of the pair of clusters in their bound state, obtained from the two center model using the generator coordinate method. Such calculations, however, are currently not available.) Secondly, we have neglected Coulomb effects in the three-body formalism which, though secondary in a transfer reaction, will not be insignificant. Lastly, even in the framework of the three-body model, we have not considered all the possible channels in that we consider only one bound state in each pair, thus ignoring the effect of the channels corresponding to the excited states of the bound pairs. Nevertheless, it may be said that the three-body model holds the promise of explaining the features of reactions involving transfers of clusters of nuclei, if more accurate input is used.

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[16]Doleschall's expression for the overlap also reduces to this expression for ^{40}Ca, α, and the d system.

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[21]It may easily be seen that when both q_i^2 and u_k^2 are $\leq E$, then $|D_{ik}|$ may be <1; i.e., D_{ik} may lie between $+1$ and -1, resulting in logarithmic singularities in the kernel of integral equation (5), defined by Eq. (8). On the other hand, when either q_i^2 or u_k^2, or both, are greater than E, then D_{ik} is either greater than $+1$ or less than -1 and no logarithmic singularity occurs in this case.

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[23]Obviously, the mesh points in the 2nd and 3rd intervals are slightly different in the three partitions so that in all three cases, the eighth quandrature point is the on-shell momentum.

[24]Since the unknown amplitude $\Gamma_{kj}(u_k Q_j l_k l_j; J)$ is an even or odd function of u_k (or z), depending on whether l_k is even or odd, and this is to be multiplied in the integrand by u_k apart from $Q_{l_k}(D_{ik})$ and other functions of u_k which are even, the parity of the integrand, apart from $Q_{l_k}(D_{ik})$, is $(-)^{l_k+1}$.