Dynamics of nuclear fluid. VI. Nuclear giant resonances as elastic vibrations

Cheuk-Yin Wong
Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830

N. Azziz
University of Puerto Rico, Mayaguez, Puerto Rico 00708
and University of Rio Grande do Sul, Brazil
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Starting with the previous result that the equation of motion for some collective motion of the nuclear fluid can be approximated by the Lamé equation, we consider the nuclear giant resonances as elastic vibrations of a nucleus, the properties of elasticity being a peculiar manifestation of the quantum stress tensor. The nucleus is taken to be compressible and endowed with elastic moduli, surface tension, Coulombic charge, and two-body viscosity. Eigenenergies and widths for the isoscalar electric multipole states ($0^+,1^-,2^+,3^-,4^+,\ldots$) and the isoscalar magnetic multipoles states ($1^+,2^-,3^+,4^-,\ldots$) are obtained. The energies and widths of the 0^+, 2^+, and 3^- states agree well with those of the observed giant resonances. Such agreement lends support to the present macroscopic description of the collective excitation of a nucleus. Nuclear viscosity coefficients and the incompressibility of nuclear matter are extracted. In the present unified approach, the high-lying electric multipole "giant resonance" states and the low-lying "liquid-drop" states emerge as eigenstates of the same characteristic equation. Similarities and differences between these two types of states are assessed.

NUCLEAR STRUCTURE Giant resonances vibration of a compressible, elastic sphere with surface tension, charge, and viscosity. Electric multipole and magnetic multipole resonances. Nuclear viscosity coefficients and incompressibility extracted.

I. INTRODUCTION

This is one of a series of articles dealing with the dynamics of nuclear fluid. Other studies concern themselves with the time-dependent Hartree-Fock (TDHF) approximation from a fluid dynamical[1] viewpoint, general considerations on the kinetic theory of the nuclear fluid,[2] spin and isospin waves in the nuclear fluid,[3,4] the extension of the time-dependent Hartree-Fock approximation to include particle collisions,[5] and the time-dependent Hartree-Fock approximation from a classical viewpoint.[6] This article deals with the elastic response of the nuclear fluid and its manifestation in the form of multipole giant resonances.[7]

Our previous studies of the nuclear collective motion in terms of nuclear hydrodynamics met with success in some cases but without success in some other cases. In particular, although the isoscalar giant monopole and isovector giant electric dipole can be explained in the hydrodynamical model, the giant quadrupole and octupole resonances cannot.[4] With only a partial success in describing collective excitations, we are motivated to seek alternative descriptions. On the other hand, we showed previously that the equations of motion for a quantum many-body system are similar in form to those one encounters in classical hydrodynamics, with the exception of an additional quantum stress tensor proportional to $\hbar^2$.[2,8] Such a close analogy does not imply immediately the validity of a completely macroscopic or a hydrodynamical description. The pattern of behavior of a quantum fluid is governed in an important way by the quantum stress tensor, the extra term of quantum origin. In some types of dynamical motion where the quantum stress tensor is a function of local density and temperature, as in the case when local equilibrium can be established, the equation of motion of a Fermi liquid can be approximated by the classical Euler or Navier-Stokes equation. The dynamics are then described properly by hydrodynamics. In some other types of dynamical motion involving fast responses without local equilibrium, the quantum stress tensor is proportional to the first spatial derivatives of the displacement field.[2] The equation of motion can be approximated by the Lamé equation and the dynamics are properly described by an elastic response of the quantum fluid.[2] Thus, the pattern of behavior of a Fermi fluid is governed by the time scale of the dynamical motion as compared to the relaxation time leading to local equilibrium. Elastic response results when the relaxation time is much greater than the time scale for the dynamical motion, whereas a hydrodynamical motion occurs in the opposite extreme.

For the nuclear fluid at a collective excitation of only a few MeV per nucleon, the Pauli principle is very effective in inhibiting the collisions between nucleons. The time scale for collective motion is small compared with the relaxation time leading to local equilibrium. One expects that the nuclear fluid behaves like an elastic solid. As was first observed by Bertsch,[9] nuclear multipole giant resonances are simply the manifestation of the elastic vibration of the nuclear field, although other types of interpretation are also possible.[10] Nuclear elasticity also enters in the rotation of deformed nuclei.[11] Therefore, nuclear elasticity is of physical interest.

In the previous studies of the giant multipole resonances in terms of the elastic vibration of a nucleus, an irrotational and incompressible flow was assumed from the onset as a trial form of the displacement vector in a variational calculation. Another study[12] of the same problem also assumed an irrotational and incompressible flow of the nuclear fluid and obtained the eigenenergies by evaluating the stiffness and mass parameters. The use of a trial displacement vector of an incompressible and irrotational type only gives an upper bound to the eigenenergies and cannot encompass elastic vibrations involving compressional or rotational flows containing vorticities.

We undertake a study of the elastic vibration of a nucleus based on the solution of the Lamé equation. Here and henceforth, we shall focus our attention to isoscalar excitations; the other excitations involving spin and isospin degrees of freedom will be the subject of future investigations. The general treatment to include the rotational and compressional flows containing vorticities allows one to study both the isoscalar electric and isoscalar magnetic multipole resonances. The eigenenergies of these states can be obtained from the characteristic equations given by Lamb in the last century.[13] Among the electric multipole resonances, the 0^+, 1^-, 2^+, 3^-, and other natural parity states come out from the same characteristic equation, while the magnetic multipole states $1^+,2^-,3^+,4^-$ are solutions for rotational and incompressible flow, the 2^- vibrational state having been rediscovered recently.[14]

Our study of the elastic vibration also helps clarify different types of vibrations of the same multipolarity. The vibrational eigenenergies of a liquid-drop nucleus are well known.[15] Being rather low-lying and decreasing with mass number faster than $A^{-1/2}$, they are different from those of the high-lying electric giant multipole resonances. These vibrational states arise due to the forces of nuclear surface tension and Coulomb repulsion. What then are the differences between these low-lying "liquid-drop" states and the high-lying "giant resonance" states? Is there a unified model in which both states appear together as eigenstates of the same characteristic equation? We shall see that an inclusion of the nuclear surface tension and Coulomb interaction into the treatment of elastic vibration provides just such a unified description.

Besides allowing a more general nature of the displacement fields, our work also differs from previous work[9,12] in the introduction of an effective mass. As is well known, nucleons in a nucleus do not behave as nucleons with a bare mass. Because of the velocity dependence and the nonlocality of the nucleon-nucleon interaction, and also because of the coupling of the phonons to the single-particle motion, the effective mass is reduced inside a nucleus.[16,17] Furthermore, this effective mass depends on the collective excitation energy, being approximately the bare mass for low collective excitations and much smaller for the giant multipole resonances.[17] With the introduction of this effective mass, the giant 2^+ energy, which was found to be substantially lower than the experimental value,[9] can now be brought to agree with experiment.

The present investigation also serves another purpose. By generalizing the present treatment to include damping due to nucleon-nucleon collisions (two-body viscosity), one obtains a stress tensor which is proportional to the first spatial derivatives of the displacement field and the velocity field. We are then dealing with the nucleus as a Maxwell solid or a viscoelastic system. Comparison of the widths of the multipole giant resonances allows the extraction of the nuclear viscosity coefficients. New equations of motion for the nuclear fluid are suggested.

This paper is organized as follows. In Sec. II, we review how the Lamé equation can be approximated from the TDHF approximation when all the collective strength is concentrated at one level. In Sec. III, we show that in a more general many-body problem in which all the particles have the same displacement field, we obtain again the Lamé equation as an approximate equation of motion. We summarize in Sec. IV the solutions of the Lamé equation subject to the boundary condition that the stress tensor at the surface vanishes. How the surface tension and Coulomb repulsion can be included into the treatment of elastic vibration is discussed in Sec. V. In Sec. VI, we examine the eigenenergies of the different multipole states and compare them with experiment. We investigate the widths of the giant multipole states and extract the viscosity coefficients in Sec. VII. In Sec. VIII, we compare the similarities and differences of the liquid drop vibrational states and the elastic vibrational (giant multipole resonance) states. Section IX concludes the present discussion. For completeness, the explicit forms of the basic displacement vectors are given in the Appendix.

II. THE LAMÉ EQUATION AS A SPECIAL CASE OF THE RANDOM-PHASE APPROXIMATION

We would like to review the results obtained previously[2] in deriving the Lamé equation for a special case of the random-phase approximation. For a simple short-range, density-dependent effective interaction, we showed previously from the TDHF equations that the equation of motion for the density field $n(\vec{r},t)$ and the velocity field $u(\vec{r},t)$ satisfy the following equations [cf. Eqs. (2.1) and (2.2) of Ref. 4]:

$$\frac{\partial}{\partial t}n+\nabla\cdot n\vec{u}=0 \tag{2.1}$$

and

$$m\frac{\partial n u_i}{\partial t}+\sum_j \nabla_j\left[m n u_i u_j+p_{ij}^{(p)}+p_{ij}^{(q)}\right]=-n\nabla_i\frac{\partial(W_s n)}{\partial n}\ . \tag{2.2}$$

Here $p_{ij}^{(p)}$ is the par-thermal stress tensor arising from the deviation of the single-particle velocity from the mean velocity

$$p_{ij}^{(p)}=m\sum_{\substack{\lambda\\ \text{occ}}}\phi_\lambda^2(\nabla_i S_\lambda-u_i)(\nabla_j S_\lambda-u_j) \tag{2.3}$$

and $p_{ij}^{(q)}$ is the quantum stress tensor

$$p_{ij}^{(q)}=+\frac{\hbar^2}{4m}\nabla^2 n-\frac{\hbar^2}{m}\sum_{\substack{\lambda\\ \text{occ}}}\phi_\lambda\nabla_i\nabla_j\phi_\lambda\ , \tag{2.4}$$

where ϕ_λ and S_λ are the amplitude and phase of the single-particle wave function ψ_λ

$$\psi_\lambda(\vec{r},t)=\phi_\lambda(\vec{r},t)e^{imS_\lambda(\vec{r},t)/\hbar} \tag{2.5}$$

and m is the mass of a nucleon. In Eq. (2.2), $W_s n$ is the energy density due to short-range interactions alone. We have neglected the long-range Coulomb interaction and surface tension. They will be taken up in Sec. V.

We shall examine the quantum and the par-thermal stress tensors for a special case of the random-phase approximation. For this purpose, we wish to write the single-particle wave functions in the form of Eq. (2.5). It was shown previously that in the random-phase approximation,[9] a single-particle wave function $\psi_\lambda(\vec{r},t)$ can be quite generally written in terms of a displacement vector $\vec{D}_\lambda(\vec{r},t)$ and a phase $S_\lambda(\vec{r},t)$

$$\psi_\lambda(r,t)=\frac{\psi_\lambda^{(0)}(\vec{r}-\vec{D}_\lambda(\vec{r},t),0)\exp[imS_\lambda(\vec{r},t)/\hbar+i\epsilon_\lambda(t)t/\hbar]}{[1+\nabla\cdot\vec{D}_\lambda(\vec{r},t)]^{1/2}}\ , \tag{2.6}$$

where $\psi_\lambda^{(0)}(\vec{r},t)$ is the single-particle wave function in the absence of dynamical motion. A collective state of the system, if it ever occurs, is characterized by a concentration of the collective strength at one level and by having the same displacement vector and velocity potential for all the single-particle states

$$\vec{D}_\lambda(\vec{r},t)=\vec{D}(\vec{r},t) \tag{2.7}$$

and

$$S_\lambda(\vec{r},t)=S(\vec{r},t)\ . \tag{2.8}$$

These relations greatly simplify the evaluation of the quantum and par-thermal stress tensor. Because all the single-particle states have the same velocity field, we have

$$p_{ij}^{(p)}=0 . \tag{2.9}$$

A straightforward evaluation of the quantum stress tensor gives

$$p_{ij}^{(q)}=p_{ij}^{(0)}-2\tau_{ij}^{(0)}\nabla\cdot\vec{D} - \sum_{\gamma}[\nabla_i(p_{j\gamma}^{(0)}D_\gamma)+\nabla_j(p_{i\gamma}^{(0)}D_\gamma)]+\mathscr{O}(\nabla^3 D) , \tag{2.10}$$

where $p_{ij}^{(0)}$ and $\tau_{ij}^{(0)}$ are the equilibrium quantum stress tensor and kinetic energy density, respectively. They are given by

$$p_{ij}^{(0)}=+\frac{\hbar^2}{4m}\nabla_i\nabla_j\sum_{\substack{\lambda\\ \text{occ}}}|\psi_\lambda^{(0)}|^2 -\frac{\hbar^2}{m}\sum_{\substack{\lambda\\ \text{occ}}}\psi_\lambda^{(0)}\nabla_i\nabla_j\psi_\lambda^{(0)} \tag{2.11}$$

and

$$\tau_{ij}^{(0)}=-\frac{\hbar^2}{2m}\sum_{\substack{\lambda\\ \text{occ}}}\psi_\lambda^{(0)}\nabla_i\nabla_j\psi_\lambda^{(0)} . \tag{2.12}$$

Using the Thomas-Fermi approximation for the above equilibrium quantities and neglecting higher-order terms and terms involving $\nabla^3 D$, we have

$$p_{ij}^{(q)}=p_0^{(q)}\delta_{ij}-\lambda^{(q)}\delta_{ij}\nabla\cdot\vec{D} -\mu^{(q)}(\nabla_i D_j+\nabla_j D_i) , \tag{2.13}$$

where

$$\lambda^{(q)}=\mu^{(q)}=p_0^{(q)}=\frac{1}{5}\frac{\hbar^2}{m}k_f^2 n , \tag{2.14}$$

and k_f is the Fermi momentum. As the stress tensor depends on the first spatial derivatives, the coefficients $\lambda^{(q)}$ and $\mu^{(q)}$ are the Lamé coefficients arising from the quantum stress tensor.

We consider now small derivations of the density from the equilibrium density n_0

$$n=n_0+\delta n \cong n_0-n_0\nabla\cdot\vec{D}(\vec{r},t) . \tag{2.15}$$

From the equation of continuity (2.1) which now reads

$$\vec{u}=\frac{\partial\vec{D}}{\partial t} , \tag{2.16}$$

we obtain from (2.2) the Lamé equation

$$mn_0\frac{\partial^2\vec{D}}{\partial t^2}=\left[\lambda^{(q)}+\mu^{(q)}+n_0{}^2\frac{\partial^2(W_s n)}{\partial n^2}\bigg|_{n_0}\right]\nabla(\nabla\cdot\vec{D}) +\mu^{(q)}\nabla^2\vec{D} . \tag{2.17}$$

Upon introducing the total Lamé coefficients λ,μ:

$$\lambda=\lambda^{(q)}+n_0{}^2\frac{\partial^2(W_s n)}{\partial n^2}\bigg|_{n_0} \tag{2.18}$$

and

$$\mu=\mu^{(q)} , \tag{2.19}$$

the equation of motion (2.17) becomes the standard form

$$mn_0\frac{\partial^2\vec{D}}{\partial t^2}=(\lambda+\mu)\nabla(\nabla\cdot\vec{D})+\mu\nabla^2\vec{D} . \tag{2.20}$$

Noting that the nuclear matter energy density Wn is related to $W_s n$ by

$$Wn=\frac{3}{10}\frac{\hbar^2}{m}k_f{}^2 n+W_s n , \tag{2.21}$$

we can write the Lamé coefficient λ in terms of the nuclear incompressibility K

$$\lambda=\frac{n_0 K}{9}-\frac{2}{15}\frac{\hbar^2}{m}k_f{}^2 n_0 . \tag{2.22}$$

There is an additional amendment to the equations of motion before we can apply it to the discussion of nuclear dynamics. We know that because of the momentum dependence and the nonlocality of the nucleon-nucleon interaction and also because of the coupling of phonon to the single-particle motion, the mass of a nucleon can be effectively modified.[16,17] Since we have not included these effects explicitly into our consideration, we are well advised to use an effective mass m^* in our equation of motion. The effective mass m^* is approximately constant in the interior of a nucleus, but depends on the energy of the collective excitation.[17] Equation (2.20) is thus modified to be

$$m^* n_0\partial^2\vec{D}/\partial t^2=(\lambda+\mu)\nabla(\nabla\cdot\vec{D})+\mu\nabla^2\vec{D} , \tag{2.23}$$

where

$$\mu=(\hbar^2/5m^*)k_f{}^2 n \tag{2.24}$$

and

$$\lambda = n_0 K/9 - (2\hbar^2/15m^*)k_f{}^2 n_0 \ . \tag{2.25}$$

Before leaving this section, it is of interest to estimate the speeds of elastic waves in a nuclear medium. For the compressional elastic waves, the speed is

$$a_1 = \left[\frac{\lambda + 2\mu}{m^* n_0} \right]^{1/2} , \tag{2.26}$$

and for the shear waves, the speed is

$$a_2 = \left[\frac{\mu}{m^* n_0} \right]^{1/2} . \tag{2.27}$$

Upon taking $k_f = 1.3$ fm^{-1}, $K = 220$ MeV, and $m^*/m = 0.8$, we find the speed of compressional elastic waves

$$a_1 = 0.2523c \ , \tag{2.28}$$

and the speed of shear waves

$$a_2 = 0.1527c \ . \tag{2.29}$$

These speeds can be compared with the speed of hydrodynamical waves given by

$$a_{\text{hydro}} = (K/9m^*)^{1/2} = 0.1804c \ . \tag{2.30}$$

We see that the speed of compressional elastic waves is greater than the speed of hydrodynamical waves which in turn is greater than the speed of shear elastic waves.

III. THE LAMÉ EQUATION AS A MORE GENERAL EQUATION FOR COLLECTIVE MOTION

The Lamé equation we have obtained admits both rotational and irrotational flows. On the other hand, the derivation of the Lamé equation from the TDHF approximation made use of only irrotational velocity fields. Must the flow be irrotational always? It should be realized that the flow can be irrotational if and only if: (1) a single Slater determinant adequately describes the dynamics of the collective motion completely, and (2) all the single-particle states have the same velocity field. While a single Slater determinant is often used to describe collective dynamics, there is no *a priori* reason to expect that the description can be complete in view of the large configuration mixing that occurs even for the ground state of a nucleus.[18] When the static ground state is represented by a more general wave function that is not a single Slater determinant, rotational flows can be admitted. We shall see that for a general wave function and the case when the displacement fields of all the particles are the same, the Lamé equation is again obtained as an approximate equation of motion. It follows then that the Lamé equation is a more general approximate result for a Fermi liquid in collective motion and has a range of validity greater than the TDHF approximation of one Slater determinant.

We consider a many-body system having a many-body wave function $\Psi^{(0)}(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N; t)$ for the system at equilibrium. For simplicity, we have suppressed the spin and isospin degrees of freedom. We consider a collective motion as one in which all the nucleons have the same displacement field $\vec{D}$ and represent the many-body wave function by

$$\Psi(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N, t) = \frac{\Psi^{(0)}(\vec{r}_1 - \vec{D}(\vec{r}_1, t), \vec{r}_2 - \vec{D}(\vec{r}_2, t), \ldots ; 0)}{\prod_{\alpha=1}^{N} [1 + \nabla_{r_\alpha} \cdot \vec{D}(\vec{r}_\alpha t)]^{1/2}} e^{imS(\vec{r}_1 \ldots \vec{r}_N; t)/\hbar} \ . \tag{3.1}$$

A collective eigenstate is one in which $\vec{D}$ is periodic and satisfies the proper boundary conditions. The equation of continuity gives [cf. Eq. (3.13) of Ref. 2]

$$\frac{\partial n(\vec{r}_1, t)}{\partial t} + \nabla_{r_1} \cdot [n(\vec{r}_1, r)\vec{u}(\vec{r}_1, t)] = 0 \ , \tag{3.2}$$

where

$$n(\vec{r}_1, t) = N \int d\vec{r}_2 \ldots d\vec{r}_N \phi^2(\vec{r}_1 \ldots \vec{r}_N, t) \ , \tag{3.3}$$

$$\vec{u}(\vec{r}_1, t) = N \int d\vec{r}_2 \ldots d\vec{r}_N \phi^2(\vec{r}_1 \ldots \vec{r}_N, t) \nabla_{r_1} S(\vec{r}_1 \ldots \vec{r}_N, t) \Big/ n(\vec{r}_1, t) \ , \tag{3.4}$$

and

$$\phi(\vec{r}_1\ldots\vec{r}_N,t)=\frac{\Psi^{(0)}(\vec{r}_1-\vec{D}(\vec{r}_1,t),\vec{r}_2-\vec{D}(\vec{r}_2,t),\ldots;0)}{\prod_{\alpha=1}^{N}[1+\nabla_{r_\alpha}\cdot\vec{D}(\vec{r}_\alpha,t)]^{1/2}} . \tag{3.5}$$

Keeping terms up to the lowest order in the displacement vector, one finds from Eq. (3.2) that

$$\frac{\partial\vec{D}(\vec{r}_1,t)}{\partial t}=\vec{u}(\vec{r}_1,t) . \tag{3.6}$$

Because $\vec{u}(\vec{r}_1 t)$ as given by Eq. (3.4) is not necessarily irrotational, the displacement vector $\vec{D}(\vec{r}_1 t)$ need not be irrotational.

The equation of motion for the system is [Eq. (5.4) of Ref. 2]

$$\frac{\partial}{\partial t}mn(\vec{r}_1)u_i(\vec{r}_1)+\sum_j\frac{\partial}{\partial r_{ij}}[n(\vec{r}_1)u_i(\vec{r}_1)u_j(\vec{r}_1)+p_{ij}^{(\delta)}(\vec{r}_1)+p_{ij}^{(p)}(\vec{r}_1)+p_{ij}^{(q)}(\vec{r}_1)]$$
$$=-\int d^3r_2 n^{(2)}(\vec{r}_1,\vec{r}_2)\frac{\partial}{\partial r_{12}}v_L(\vec{r}_1,\vec{r}_2) . \tag{3.7}$$

Here $p^{(\delta)}$ is the stress tensor due to the short-range interaction, $p^{(p)}(\vec{r}_1)$ is the par-thermal stress tensor, and $p^{(q)}$ is the quantum stress tensor. They are defined by

$$p_{ij}^{(p)}(\vec{r}_1)=m\int d\mu_1\phi^2[\nabla_{x_j}S-u_i(\vec{r}_1)][\nabla_{x_j}S-u_j(\vec{r}_1)] , \tag{3.8}$$

and

$$p_{ij}^{(q)}(\vec{r}_1)=\int d\mu_1\left[\frac{\hbar^2}{4m}\nabla_{x_i}\nabla_{x_j}\phi^2-\frac{\hbar^2}{m}\phi\nabla_{x_i}\nabla_{x_j}\phi\right] , \tag{3.9}$$

where, $d\mu_1=Nd\vec{r}_2\ldots d\vec{r}_N$, x_i is a Cartesian component of the radii vector $\vec{r}_1$, and the arguments in ϕ and S are $(\vec{r}_1\ldots\vec{r}_N,t)$. In Eq. (3.7), $v_L(\vec{r}_1,\vec{r}_2)$ is the long-range interaction between nucleons, $n^{(2)}(\vec{r}_1,\vec{r}_2)$ is the two-body distribution function, and the term on the right hand side is the force density due to the long-range interaction. A straightforward evaluation of Eq. (3.9) gives

$$p_{ij}^{(q)}(r_1)=p_{ij}^{(0)}(r_1)-2\tau_{ij}^{(0)}\nabla\cdot\vec{D}(\vec{r}_1)-\sum_\gamma[\nabla_{x_i}(p_{j\gamma}^{(0)}D_\gamma)+\nabla_{x_j}(p_{i\gamma}^{(0)}D_\gamma)]+\mathscr{O}(\nabla^3D) , \tag{3.10}$$

where

$$p_{ij}^{(0)}=\int d\mu_1\left[\frac{\hbar^2}{4m}\nabla_{x_i}\nabla_{x_j}|\Psi^{(0)}|^2-\frac{\hbar^2}{m}\Psi^{(0)*}\nabla_{x_i}\nabla_{x_j}\Psi^{(0)}\right] \tag{3.11}$$

and

$$\tau_{ij}^{(0)}=-\frac{\hbar^2}{2m}\int d\mu_1\Psi^{(0)*}\nabla_{x_i}\nabla_{x_j}\Psi^{(0)} . \tag{3.12}$$

As $\Psi^{(0)}$ is the many-body wave function for the system at equilibrium, $p_{ij}^{(0)}$ and $\tau_{ij}^{(0)}$ are the quantum stress tensor and kinetic energy density for the system at equilibrium. These quantities can be adequately represented by using the Thomas-Fermi approximation. Equation (3.10) then leads one to the same Lamé constants as before. We can neglect the par-thermal stress tensor which is small for low-energy phenomena. Upon using a proper parametrization of $p_{ij}^{(\delta)}$ in terms of $\partial^2(W_s n)/\partial n^2$ and neglecting the long-range interaction, we obtain again the Lamé equation (2.20) for nuclear fluid with a flow that is not restricted to be irrotational. The introduction of an effective mass then leads again to Eqs. (2.23)–(2.25) for the dynamics of nuclear fluid at low excitation energies.

IV. MULTIPOLE GIANT RESONANCES AS ELASTIC VIBRATIONS OF A NUCLEUS

We have shown in the last two sections that the equation of motion for low-excitation collective motion in a nucleus can be approximated by that for the propagation of elastic waves. On the other hand, some of the most notable nuclear collective states are the multipole giant resonances. As was first observed by Bertsch,[9] these giant resonances are manifestations of the elastic vibration of a nucleus. It is therefore of interest to examine the eigenfrequencies and displacement vectors for the elastic vibrations of a spherical nucleus.

The mathematical procedures to obtain the eigenenergies were presented by Lamb in the last century[13] and need not be repeated again. We hereby summarize Lamb's solution with a slight change of notations.

We look for solutions of the displacement vector in the form

$$\vec{D}(\vec{r},t)=\vec{\mathscr{D}}(\vec{r})e^{i\omega t} . \quad (4.1)$$

The Lamé equation, (2.23), becomes

$$-m^*n_0\omega^2\vec{\mathscr{D}}(\vec{r})=(\lambda+\mu)\nabla[\nabla\cdot\vec{\mathscr{D}}(\vec{r})]+\mu\nabla^2\vec{\mathscr{D}}(\vec{r}) . \quad (4.2)$$

A general solution of $\vec{\mathscr{D}}(\vec{r})$ is a linear combination of three different types of displacement $\vec{\mathscr{D}}_1(\vec{r})$, $\vec{\mathscr{D}}_2(\vec{r})$, and $\vec{D}_3(\vec{r})$ satisfying

$$\nabla\cdot\vec{\mathscr{D}}_1(\vec{R})\neq 0 \quad (4.3)$$

and

$$\nabla\cdot\vec{\mathscr{D}}_2(\vec{r})=\nabla\cdot\vec{D}_3(\vec{r})=0 . \quad (4.4)$$

Among the displacement fields $\vec{\mathscr{D}}_2(\vec{r})$, there are two different solutions $\vec{\mathscr{D}}_2$ and $\vec{\mathscr{D}}_3$ having the properties of Eq. (4.4) and satisfying Eq. (4.2). In consequence, the most general solutions of the displacement vector are of the form

$$\vec{D}(\vec{r},t)=[A_1\vec{\mathscr{D}}_1(\vec{r})+A_2\vec{\mathscr{D}}_2(\vec{r})+A_3\vec{\mathscr{D}}_3(\vec{r})]e^{i\omega t} , \quad (4.5)$$

where the A's are constants and the $\vec{\mathscr{D}}$ functions are the displacement vectors satisfying Eqs. (4.2)–(4.4):

$$\vec{\mathscr{D}}_1(\vec{r})=-\frac{1}{h^2}\nabla\sum_{lm}\psi_l(hr)\Omega_{lm} , \quad (4.6)$$

$$\vec{\mathscr{D}}_2(\vec{r})=\frac{1}{k}\sum_{lm}\psi_l(kr)(\vec{r}\times\nabla)\Omega_{lm} , \quad (4.7)$$

and

$$\vec{\mathscr{D}}_3(\vec{r})=\frac{1}{k^2}\nabla\times\sum_{lm}\psi_l(kr)(\vec{r}\times\nabla)\Omega_{lm} . \quad (4.8)$$

Here, $\psi_l(x)$ is defined in terms of the spherical Bessel function j_l by

$$\psi_l(x)=(-1)^l j_l(x)/x^l , \quad (4.9)$$

and $\Omega_{lm}(r,\theta,\phi)$ is the solid spherical harmonic

$$\Omega_{lm}(r,\theta,\phi)=r^l Y_{lm}(\theta,\phi) . \quad (4.10)$$

The quantities h and k are related to the frequency ω by

$$h^2=m^*n_0\omega^2/(\lambda+2\mu) \quad (4.11)$$

and

$$k^2=m^*n_0\omega^2/\mu . \quad (4.12)$$

It is easy to prove that the $\vec{\mathscr{D}}_i(\vec{r})$ functions in Eqs. (4.6)–(4.8) satisfy Eqs. (4.2)–(4.4) by direct substitution.

Since $\nabla(\nabla\cdot\vec{\mathscr{D}}_1)\propto\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_1$ is the gradient of a scalar, the solution $\vec{\mathscr{D}}_1$ leads to a compressional irrotational flow. The solution $\vec{\mathscr{D}}_2$ and $\vec{\mathscr{D}}_3$ lead to rotational flows with vorticities but because of Eq. (4.4) they are also isovolumetric. In the case of a constant density, the solution $\vec{\mathscr{D}}_2$ and $\vec{\mathscr{D}}_3$ give rise to incompressible but rotational flows.

From the general form of the solution Eq. (4.5) which satisfies the Lamé equation, one determines the eigenfrequencies by imposing the appropriate boundary condition. For a free vibration, the unbalanced stress tensor δp_{ij} (which is the deviation of the stress tensor from the equilibrium value) must vanish at the surface $r=a$ so that there is no residual force acting across the boundary. Instead of working with δp_{ij}, it is more convenient to construct the vector $\vec{T}$ with components

$$T_i=-\frac{1}{\mu}\sum_j x_j\delta p_{ij} , \quad (4.13)$$

and to require $\vec{T}$ to vanish at the surface as the boundary condition. We shall first consider the case with no surface tension and Coulomb interaction. The unbalance stress tensor δp_{ij} comes only from the displacement fields, as given by Eqs. (2.13), (2.18), and (2.19)

$$\delta p_{ij}=-\mu\left[\frac{\partial D_i}{\partial x_j}+\frac{\partial D_j}{\partial x_i}\right]-\lambda\nabla\cdot\vec{D}\delta_{ij} . \quad (4.14)$$

Substituting Eqs. (4.14) and (4.5) into Eq. (4.13), we obtain the vector $\vec{T}$:

$$\vec{T}=\sum_{lm}[(A_1a_l+A_3c_l/k^2)\nabla\Omega_{lm} +(A_1b_l+A_3d_l/k^2)r^{2l+3}\nabla(\Omega_{lm}/r^{2l+1}) +A_2e_l(\vec{r}\times\nabla)\Omega_{lm}/k]e^{i\omega t}, \quad (4.15)$$

where the coefficients a_l, b_l, c_l, d_l, and d_l at the surface $r=a$ will be given below. As the functions $\nabla\Omega_{lm}$, $\nabla(\Omega_{lm}/r^{2l+1})$, and $\vec{r}\times\nabla\Omega_{lm}$ are different functions of θ and ϕ, the requirement that $\vec{T}$ vanish at the surface will be satisfed if their coefficients are simultaneously zero

$$A_1a_l+A_3c_l/k^2=0, \quad (4.16)$$

$$A_1b_l+A_3d_l/k^2=0, \quad (4.17)$$

and

$$A_2e_l=0. \quad (4.18)$$

The above conditions lead to two classes of solutions.[13] In the first class of solution, which we call electric multipole states, the coefficients $A_1\neq 0$, $A_3\neq 0$, but $A_3=0$. The characteristic equation as obtained from Eqs. (4.16)–(4.18) is

$$a_ld_l-b_lc_l=0, \quad (4.19)$$

where

$$a_l=\frac{1}{(2l+1)h^2}[k^2a^2\psi_l(ha) +2(l-1)\psi_{l-1}(ha)], \quad (4.20)$$

$$b_l=-\frac{1}{2l+1}\left[\frac{k^2}{h^2}\psi_l(ha) +\frac{2(l+2)}{ha}\psi_l'(ha)\right], \quad (4.21)$$

$$c_l=k^2a^2\psi_l(ka)+2(l-1)\psi_{l-1}(ka), \quad (4.22)$$

and

$$d_l=\frac{lk^2}{l+1}\left[\psi_l(ka)+\frac{2(l+2)}{ka}\psi_l'(ka)\right]. \quad (4.23)$$

In this class of vibration, only terms involving $\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_3$ are nonvanishing. The displacement has, in general, both transverse and radial components. A nucleus undergoing vibrations of this type with multipolarity l produces only electric radiation of multipolarity l. Hence, we call these states electric multipole states. These states have natural parities given by $(-1)^l$. The displacement $\vec{\mathscr{D}}_1$ is irrotational but compressional. However, $\vec{\mathscr{D}}_3$ is rotational but isovolumetric. It leads to a flow containing vorticities. A given eigenstate is a mixture of both $\vec{\mathscr{D}}_1$ and $\mathscr{D}_3$ displacements whose coefficients A_1 and A_3 are related through Eq. (4.16) by

$$\frac{A_3}{A_1}=-\frac{a_lk^2}{c_l}. \quad (4.24)$$

In the second class of solutions, which we call the magnetic multipole states, the coefficients $A_1=A_3=0$ but $A_2\neq 0$. The characteristic equation as obtained from Eqs. (4.16)–(4.18) is[13]

$$e_l=(l-1)\psi_l(ka)+ka\psi_l'(ka)=0. \quad (4.25)$$

In this class of solutions, only the term involving $\vec{\mathscr{D}}_2$ is nonvanishing. The displacement for this class is rotational but isovolumetric. It leads to a flow containing vorticities. The radial displacement vanishes so that the displacement at any point is directed at right angles to the radius drawn from the center of the sphere. it is also directed at right angles to the normal of the surface of constant Ω_{lm}. Vibration of this type is a purely shear vibration. A nucleus undergoing this class of vibrations of multipolarity l produces only magnetic radiation of multipolarity l. Hence, we call these states magnetic multipole states. They have unnatural parities given by $(-1)^{l+1}$.

For completeness, we list the explicit form of the displacement vectors in terms of their components for a multipolarity of l with a z component m in the Appendix. They are useful in exhibiting the currents of the nuclear fluid for different eigenmodes of vibration.

The eigenvalue equations (4.19)–(4.23) assume a simple form for the radial vibration with $l=0$. Because of the interest in giant electric monopole resonances, we write the eigenvalue equation due originally to Lamb[13]

$$\frac{\tan ha}{ha}=\frac{1}{1-\frac{1}{4}\left[\frac{k}{h}\right]^2h^2a^2}. \quad (4.26)$$

This equation can be solved for ha provided the ratio

$$\frac{k^2}{h^2}=\frac{\lambda}{\mu}+2 \tag{4.27}$$

is known. The eigenenergy is then given in terms of ha by

$$\hbar\omega=\frac{1}{a}\left[\frac{\hbar^2}{m^*}\right]^{1/2}\cdot\frac{ha}{3^{1/2}}\left[K+\frac{12}{5}\frac{\hbar^2k_f{}^2}{m^*}\right]^{1/2} .$$

For finite nuclei, ha ranges from 1.85 to 2.2 and $\hbar\omega_0 A^{1/3}$ increases from 62.4 to 82.8 MeV as A increases from 50 to 200.

V. SURFACE TENSION AND COULOMB INTERACTIONS

The characteristic equations obtained by Lamb are appropriate for the vibration of an elastic sphere in the absence of surface tension and Coulomb interaction. The equations need to be modified if we want to apply them to a nucleus. Although the effects of surface tension and Coulomb interaction on elastic vibrations are expected to be small, their introduction allows one to uncover the low-lying vibrational states arising mainly from the forces of surface tension and Coulomb interaction. The new characteristic equation will take into account the elastic response, surface tension, and Coulomb interaction, and will provide a unified description of all the collective vibrational states of a nucleus. It will become possible to compare and contrast the high-lying giant multipole states with the low-lying "liquid-drop" states because both will come out as solutions of the same equation.

The results of Lamb can be very simply generalized. The displacement field changes the shape of the nucleus which now becomes

$$R(\theta,\phi,t)=a+D_r(a,\theta,\phi)e^{i\omega t} , \tag{5.1}$$

where $D_r(a,\theta,\phi)$ is the radial component of the displacement vector at the nuclear surface. The angular displacement gives a second-order correction and is therefore neglected. A change in shape gives rise to a change in the local radii of curvature and thus, because of the surface tension, a subsequent change in the stress tensor at the surface. The change in surface stress tensor due to D_r is given by [cf. Eq. (298) of Chap. 10 in Ref. 19]

$$\delta p_{rr}=\frac{T}{a^2}(l-1)(l+2)D_r(a,\theta,\phi) , \tag{5.2}$$

where T is the surface tension coefficient and a single multipolarity l is assumed for the displacement D_r. The change in shape also produces a change in the Coulomb potential which in turn changes the stress tensor at the surface. As is well known,[15] the additional effect of Coulomb interaction for an incompressible flow is to modify Eq. (5.2) by a factor involving the fissility parameter x

$$\delta p_{rr}=\frac{T}{a^2}(l-1)(l+2)D_r(a,\theta,\phi) \times\left[1-\frac{20x}{(2l+1)(l+2)}\right] , \tag{5.3}$$

where

$$x=(Z^2/A)/(Z^2/A)\mathrm{crit} . \tag{5.4}$$

There is an additional effect of the Coulomb interaction in a compressional oscillation. This is the tendency of a uniformly charged medium to restore a volume-type density variation with a characteristic frequency ω_p depending on the charge density of the nuclear medium. For the nuclear fluid, this frequency leads to a characteristic energy of $\hbar\omega_p=7.25$ MeV.[4] It is an important effect in the discussion of monopole oscillations. We first consider the case without this "plasma oscillation" and shall return to it in a future investigation.

The boundary condition for a free vibration is that the total stress tensor at the spherical surface $r=a$ is zero. Besides the stress tensor due to elastic responses, the total stress tensor must include the stress tensor due to surface tension and Coulomb interaction.

For the magnetic multipole states, as the radial component of $\vec{\mathscr{D}}_2$ is zero, the surface shape does not change. Hence, surface tension and Coulomb interaction have no effect on the vibration of magnetic multipole states.

For the electric multipole states, the displacement is a combination of $\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_3$. We cast the stress tensor (5.3) in the same form as that of Lamb and collect corresponding terms. The characteristic equation now reads

$$(a_l+\alpha_l)(d_l+\delta_l)-(c_l+\gamma_l)(b_l+\beta_l)=0 , \tag{5.5}$$

where a_l, b_l, c_l, and d_l are given previously in Eqs. (4.20)–(4.23),

$$\alpha_l = -\beta_l a^2 = -\frac{T}{\mu a}\frac{(l-1)(l+2)}{(2l+1)ha} \times \left[1-\frac{20x}{(2l+1)(l+2)}\right] \times \left[\psi_l'(ha)+\frac{l\psi(ha)}{ha}\right]a^2 , \quad (5.6)$$

and

$$\gamma_l = -\delta_l a^2 = -\frac{T}{\mu a}\frac{(l-1)(l+2)}{(2l+1)ha} \times \left[1-\frac{20x}{(2l+1)(l+2)}\right] \times l(l+1)\frac{\psi(ka)}{ka}k^2a^2 . \quad (5.7)$$

The solution of the characteristic equation gives the eigenenergies of vibration. When the eigenenergies are found, the mixtures of $\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_3$ are determined by

$$\frac{A_3}{A_1} = -\frac{(a_l+\alpha_l)k^2}{c_l+\gamma_l} , \quad (5.8)$$

and the displacement vector for each vibrational state can be determined up to a normalization constant.

VI. EIGENFREQUENCIES OF COLLECTIVE VIBRATIONS

Given a set of Lamé constants for the nuclear medium, the eigenenergies of collective vibration can be obtained. The Lamé constant μ depends on k_f and m^*, while the Lamé constant λ depends additionally on K. In our model, we are using a sharp cutoff density distribution, whereas the proper density has a transitional region of finite thickness. We are therefore advised to use an effective compressible K and μ which are obtained by averaging over the whole nucleus. Such an averaging will lead to an effective compressibility which contains a surface term proportional to $A^{-1/3}$. Furthermore, as the nuclear energy density depends on the density difference of neutron and proton, the nuclear incompressibility also depends on $(N-Z)^2/A^2$.[4] We can parametrize the nuclear incompressibility as[4,20]

$$K(N,Z) = K_\infty + \frac{K_s}{A^{1/3}} + K_T\frac{(N-Z)^2}{A^2} . \quad (6.1)$$

The surface tension coefficient T is taken to be[21]

$$T = 1.017 \text{ MeV/fm}^2 . \quad (6.2)$$

To obtain the fissility parameter for a given mass number, we parametrize the β-stability line by[22]

$$Z = A/(2+0.0146A^{2/3}) , \quad (6.3)$$

and get the fissility parameter as[23]

$$x = \frac{(Z^2/A)}{45} . \quad (6.4)$$

The effective mass depends on the energy of the collective state $\hbar\omega_{\text{vib}}$. We choose it according to the prescription of Brown and Specht[17]

$$\frac{m^*}{m} = 0.64 + 0.36 \bigg/ \left[1+\frac{\hbar\omega_{\text{vib}}(\text{MeV})}{82/A^{1/3}}\right]^2 , \quad (6.5)$$

where the energy dependence is such that the effective mass is approximately unity for nucleons on the top of the Fermi sea but decreases when the particle-hole energy increases.

With most of the parameters determined by other methods, the only parameters at our disposal are k_f, K_∞, K_s, and K_T. The equilibrium density n_0 and the radius parameter r_0 are related to k_f by the well-known relations

$$n_0 = \frac{k_f^3}{1.5\pi^2} \quad (6.6)$$

and

$$r_0 = \left[\frac{3n_0}{4\pi}\right]^{1/3} . \quad (6.7)$$

A search is made of the best set of parameters of k_f, K_∞, K_s, and K_T to reproduce the energies of the 0^+, 2^+, and 3^- states. There is ambiguity in determining the coefficients of K_s and K_T. We found that the set of parameters $k_f = 1.25 \text{ fm}^{-1}$, $K_\infty = 220$ MeV, $K_s = -550$ MeV, and $K_T = 0$ gives a reasonably good description of the experimental data. Good agreement with experimental data can also be obtained when K_s and K_T are changed to $K_s = -500$ MeV and $K_T = -300$ MeV. In what follows, we shall present results only with the first set of parameters with $K_s = -550$ MeV and $K_T = 0$.

The eigenenergies of the lowest electric multipole states are shown in Fig. 1, given in terms of $\hbar\omega_l A^{1/3}$ as a function of A. One observes that there is a 0^+ state at an energy of $63/A^{1/3}$ MeV for $A = 50$ which increases up to $82/A^{1/3}$ MeV for $A = 200$. A more detailed fit to the experimental

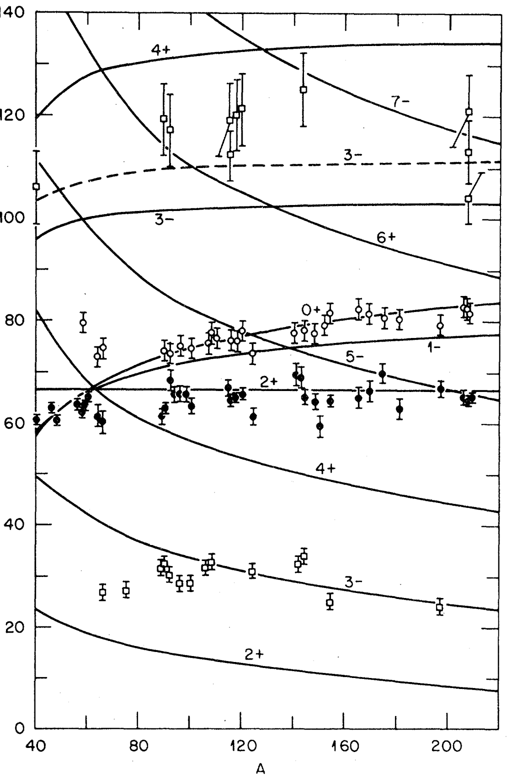

FIG. 1. The eigenenergies of electric multipole states multiplied by $A^{1/3}$ as a function of the mass number. The data points are from the compilation of Ref. 24. Solid circles are data points for the 2^+ states, open circles for 0^+, and square data points for 3^- states. The solid curves give the eigenenergies calculated with the effective mass parametrization of Brown and Specht [Eq. (6.5)]. The dashed curve is the eigenenergies for the giant octupole state calculated with the effective mass parametrization of Eq. (6.9).

data is given in Fig. 2(a). The experimental data of 0^+ are quite well represented by the 0^+ elastic vibration of a nucleus. There is a 2^+ state at an energy of $\hbar\omega_2=67/A^{1/3}$ MeV which agrees well with the giant quadrupole state observed experimentally[24] [Fig. 3(a)]. The agreement for smaller masses can be improved if one assumes that μ, similarly to K, varies with mass according to

$$\mu(A)=\mu_\infty+\mu_s/A^{1/3}\ , \tag{6.8}$$

where the last term is due to the presence of the surface. A small negative value of μ_s will give a better fit to the experimental data. The theoretical 3^- state at $\hbar\omega_3\sim 102/A^{1/3}$ MeV is slightly lower than the observed 3^- octupole state at $\hbar\omega_l\simeq 108/A^{1/3}$ (Refs. 24 and 25) [Fig. 4(a)]. A slightly smaller effective mass will give a better fit

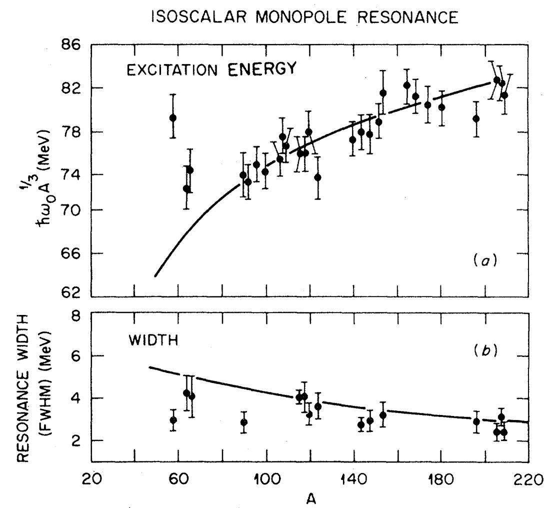

FIG. 2. Comparison of the calculated energies and widths (solid curves) for the isoscalar giant monopole resonances with the experimental data. The data points are from the compilation of Ref. 24.

to the data. The theoretical results also give a 4^+ state at an energy of $\hbar\omega_4\simeq 134$ MeV$/A^{1/3}$, which is yet to be observed.

It should be pointed out that the effective mass parametrization of Brown and Speth [Eq. (6.5)] was an estimate based on its values at $\hbar\omega_{\rm vib}=0$ and $\hbar\omega_{\rm vib}\sim 67/A$ MeV. For other regions of energy, it is not well determined. Recent quantitative determination of the effective mass by Mahaux and Ngo[26] gives an effective mass which is about equal to the estimate of Brown and Speth at low energies but becomes smaller at higher energies. It may be reasonable to readjust the effective mass so that the experimental $\hbar\omega_3$ states are well accounted for. In

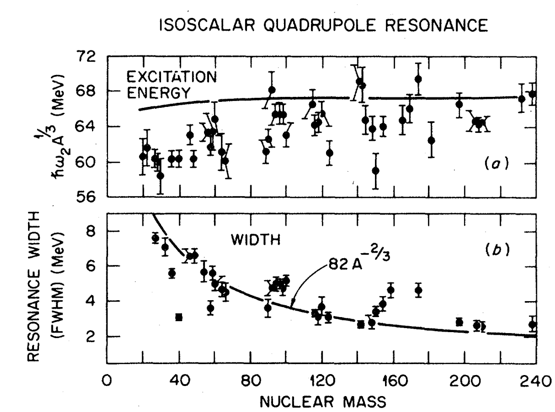

FIG. 3. Comparison of the calculated energies and widths (solid curves) for the isoscalar giant resonances with the experimental data. The data points are from the compilation of Ref. 24.

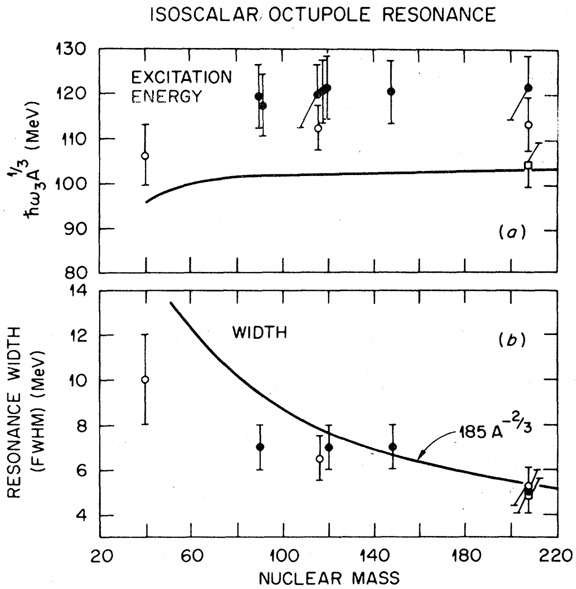

FIG. 4. Comparison of the calculated energies and widths (solid curves) for the isoscalar giant octupole resonances with the experimental data. The data points are from the compilation of Ref. 24.

fact, if we parametrize m^*/m by

$$m^*/m = [1+\hbar\omega_{\rm vib}({\rm MeV})/(82/A^{1/3})]^{-1/2} \qquad (6.9)$$

we obtain $\hbar\omega_3 A^{1/3}=105$, 110, 112 MeV for $A=50$, 100, and 200, respectively. These results are in good agreement with experiment (see the dashed curve in Fig. 1). The effective mass for the lower states remains about the same as in the parametrization of Eq. (6.5); the new parametrization does not affect the good agreement for the lower 0^+, 2^+, and 3^- states. The high-lying 4^+ state is now raised to $\hbar\omega_4 \simeq 154/A^{1/3}$ MeV.

It is worth noting that as the mass number varies from $A=100$ to $A=250$, the effective incompressibility varies by as much as 30%. However, the positions of $\hbar\omega_l A^{1/3}$ for the 2^+, 3^-, and 4^+ "giant" resonances remain essentially independent of A and hence independent of the incompressibility K. One concludes therefore that the positions of these giant resonances depend essentially on μ and are insensitive to λ (or K). This result has important consequences on the positions of the giant resonances involving spin and isospin degrees of freedom.

Within the present model of elastic vibration, there are also isoscalar 1^- states obtained from the characteristic equation (5.5). They lie at an energy of $58.8/A^{1/3}$ MeV for $A=40$, increasing in energy to $71.8/A^{1/3}$ MeV for $A=100$ and to $76.7/A^{1/3}$ MeV for $A=200$ (Fig. 1). These states are due to elastic vibration of the nucleus as the surface tension and Coulomb interaction give no contributions for $l=1$. The flow pattern of the nuclear fluid in the nucleus indicates that these states are not spurious states because different layers of the nucleus are displayed by different amounts. Such differential displacement gives rise to compressional and shear distortions. The vibrational motion is a result of the tendency to drive towards equilibrium due to the presence of these distortions. Thus, while an isoscalar 1^- state cannot arise from the restoring force of surface tension and Coulomb repulsion, the isoscalar electric dipole state can be an independent state in the elastic vibration of a nucleus. Future experimental search and identification of the isoscalar 1^- state is of interest. It may also be important to examine theoretically the degree of spuriosity in this 1^- state. Experimentally, there is some evidence for the presence of an isoscalar 1^- state at an energy between 13.3 to 16.7 MeV of ^{40}Ca,[27] which is not far from the energy of 17.2 MeV ($58.8/A^{1/3}$ MeV) predicted from the present elastic model. More data points for heavier nuclei will be of great interest in mapping out the systematics of the isoscalar electric dipole state.

Besides these "giant resonances" whose energies behave approximately as $A^{-1/3}$, there are low-energy vibrational states lying lower than these giant resonances. They have energies which decrease faster than $A^{-1/2}$ (Fig. 1). These are the liquid-drop vibrational states of Bohr and Wheeler[15] which arise from the restoring force of surface tension and the disruptive Coulomb repulsion. The energies of the liquid-drop vibrational state of multipolarity is given by[15,28]

$$\hbar\omega_l=\hbar\left[\frac{l(l-1)(l+2)T}{m^*na^3}\times\left[1-\frac{20x}{(2l+1)(l+2)}\right]\right]^{1/2} . \qquad (6.10)$$

The low-lying multipole states in Fig. 1 obtained by solving the characteristic equation (5.5) agree well with the formula given above. If there were no Coulomb repulsion, the energies of these states will vary as $A^{-1/2}$. Because of Coulomb interaction, the energy decreases with A faster than $A^{-1/2}$ (for example, approximately as A^{-1} for the quadrupole state, $A^{-0.74}$ for the octupole state, and $A^{-0.69}$ for the hexadecapole state). As is well known, the position of the low-energy vibrational state is greatly influenced by the nuclear shell effect

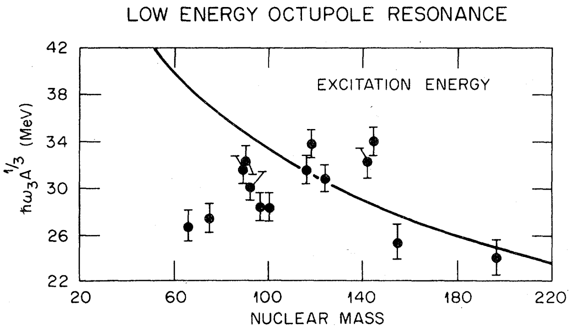

FIG. 5. Comparison of the calculated energies of the liquid-drop octupole state (solid curve) with the experimental data of the isoscalar low-energy octupole resonances. The data points are from the compilation of Ref. 30.

which alters the underlying collective liquid-drop potential.[23] In many cases, the alteration leads to permanent deformations in nuclei. In other cases, the shell effects change the vibrational energy to $A^{-1/3}$ or $A^{-2/3}$ as was known from the random-phase approximation and Bohr-Mottelson treatment.[23,28,29] It would be of interest to study the positions of the identifiable vibrational states that can be excited from the ground state to discern the liquid-drop behavior and the shell effects in collective vibrations. In this respect, a systematic study of these low-energy quadrupole states has so far not been carried out but deserves special attention. For the low-energy 3^- state, there is evidence for its identification in inelastic scattering experiments at an excitation energy of 32 $A^{-1/3}$ MeV, interpreted as a $1\hbar\omega$ transition between major shells.[30] It is tempting to suggest that the observed low-energy octupole state is perhaps built on the liquid-drop vibrational state with modifications due to nuclear shell effects. In this interpretation, the data points at ^{197}Au can be brought to be consistent with those at $A \sim 100$ (Fig. 5). The data points at $A \lesssim 70$ are lower than these systematics because of the fragmentation of the low-energy collective strength. Clearly, more data points for $160 \leq A \leq 230$ are needed to confirm or deny the validity of such an interpretation and to reveal the importance of the shell effect.

The low-energy 4^+ state decreases with A approximately as $A^{-0.67}$. The systematics of this liquid-drop 4^+ state are very different from what one expects of a $2\hbar\omega \sim 60$ to 80 MeV/$A^{1/3}$ behavior. In the region of $A > 100$, this state lies distinctly separated from the other resonances (Fig. 1). It is higher than the low-energy octupole state but lower than the giant quadrupole state. A future search for this state in this region will be of interest.

What we have discussed are the lowest states of a given multipolarity. There are also higher states which differ from these by a more complicated flow pattern. For completeness and future reference, we list the eigenenergies of these states for $A = 200$ in Table I. With the exception of the 0^+ states and the lowest of the states with $l \neq 0$, the product $\hbar\omega_l A^{1/3}$ is essentially independent of the mass number A. Thus, the results in Table I can be used for nuclei in other mass regions.

Besides the electric multipole states, we also obtain the magnetic multipole states from Eq. (4.25). The eigenenergies of these states depend only on k_f and m^*. They are independent of the other nuclear parameters. These are the rotational vibrations of Lamb.[13] The flow pattern of the different states is illustrated in Fig. 6, where only the surface current is exhibited. For the 1^+ state, the current is independent of θ and ϕ but depends on the radial distance. Thus, there is a differential rotation of one layer against another. The vibrational motion results from the restoring force due to the shear distortion. For the 2^- state, the top part of the nucleus rotates against the bottom part in a twisting motion, in addition to the differential rotation for the different layers. This is the twisting mode recently rediscovered by Holzwarth *et al.*[14] For the 3^+ state, the flow pattern is such that the equatorial section rotates against the top and the bottom parts of the nucleus (Fig. 6). We list in Table II the energies of these vibrational states. As of now, these states have not been observed.

TABLE I. Isoscalar electric multipole states for $A = 200$.

l^π	0^+	1^-	2^+	3^-	4^+	5^-	6^+	7^-
$\hbar\omega_l A^{1/3}$	82.8	76.7	8.36	24.58	44.08	66.37	90.97	117.52
(MeV)	244.4	157.7	66.49	102.87	133.88	162.77	190.48	217.31
		213.9	114.8	160.34	207.51	254.50		

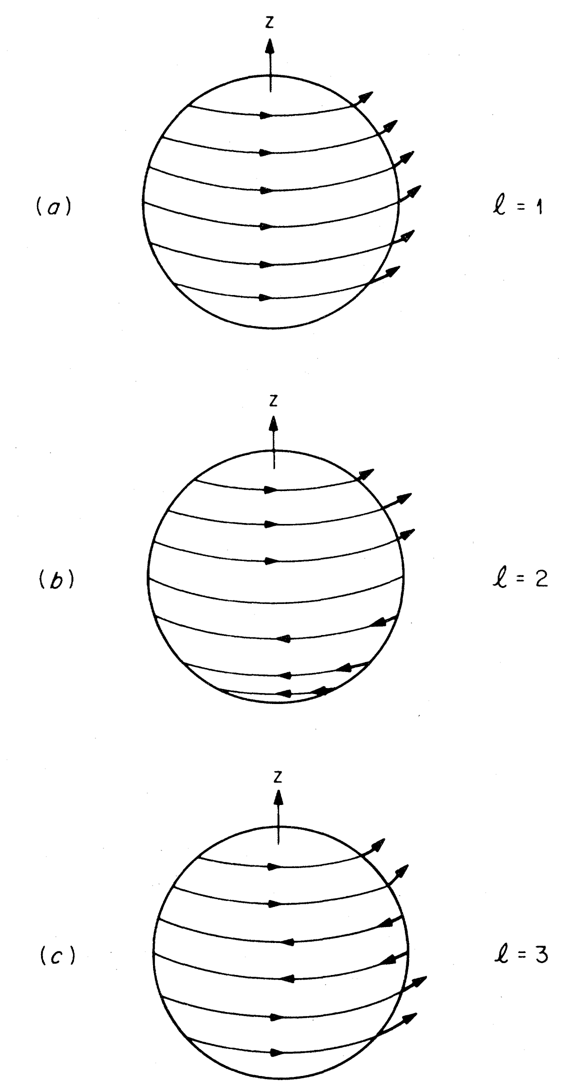

FIG. 6. The displacement vector $\vec{\mathscr{D}}(\vec{r})$ at the nuclear surface for magnetic multipole vibrations of $l=1^+$, 2^-, and 3^+. The complete displacement field is given by the product of $\vec{\mathscr{D}}(\vec{r})$ with a sinusoidal temporal factor $\cos\omega_l t$.

It is of interest to investigate the effects of surface tension and Coulomb repulsion. We found that for the giant resonances which are already present in the absence of surface tension and Coulomb repulsion, the effect of surface tension raises the eigenenergies only by $1-2\,\%$. We see therefore that these effects have a small influence on elastic vibrations but are important in providing a new degree of freedom for a new group of liquid-drop vibrations.

VII. DAMPING OF THE GIANT RESONANCES

We shall consider the damping of the giant resonances due to the two-body viscosity alone. The introduction of one-body dissipation into the macroscopic picture of the present type requires a different global approach of the dynamics[12] and will not be considered here.

Before we introduce the two-body viscosity in the elastic model, we wish to clarify an apparent peculiarity concerning the shear viscosity of the nuclear fluid. On the one hand, theoretical estimates[31] of the shear viscosity η of a Fermi fluid gives a temperature dependence of $\eta \sim 1/T^2$; this is indeed confirmed by experimental macroscopic measurements in the case of liquid ^{3}He.[32] As nuclear fluid is a Fermi fluid, one expects that the shear viscosity goes to infinity as temperatures decrease, and a shear flow for the ground state of nuclear matter is impossible. How can one understand such a singular behavior of the shear viscosity? For an elastic medium, a macroscopic shear strain with a scale much greater than the interparticle space produces a correspondingly large restoring stress. A continuous shear flow of matter is therefore impossible. Hence, the "apparent" shear viscosity near $T=0$ for a continuous shear flow of very large scale is infinite. However, this apparent singularity of shear viscosity is an artifact of treating an elastic medium as a hydrodynamical medium with viscosity and without elasticity.[33] The apparent singularity of the shear viscosity arising from treating an elastic medium as a hydrodynamical medium need not be present when the elastic property of the medium is taken into account.

When the temperature is decreased further below a critical temperature T_C, many Fermi fluids become superfluids.[34] Theoretical estimates of the

TABLE II. Isoscalar magnetic multipole states.

l^π	1^+	2^-	3^+	4^-	5^+	6^-	7^+
$\hbar\omega_l A^{1/3}$	161.0	63.0	103.5	140.7	176.2	210.8	244.6
(MeV)	262.0	202.6	242.3				

viscosity of such a fluid gives a temperature dependence of $\eta \sim T^{1/2}$, reaching a constant value at the critical temperature.[35] As a finite nucleus in its ground state has very strong pairing correlations, viscosity in a nucleus near its ground state is expected to be small. It is reasonable to treat the viscosity stress tensor phenomenologically in terms of the Navier-Stokes stress tensor with the viscosity coefficients adjusted to fit experimental data.

We introduce the Navier-Stokes stress tensor given by

$$p'_{ij} = -\eta \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma} \delta_{ij} \right] - \zeta \sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma} \delta_{ij} \; . \tag{7.1}$$

The inclusion of the Navier-Stokes viscous tensor into the equation of motion leads to the Lamé-Navier-Stokes equation

$$mn \frac{\partial^2 D_i}{\partial t^2} = (\lambda + \mu) \nabla_i (\nabla \cdot \vec{D}) + \mu \nabla \vec{D}_i + \eta \sum_j \nabla_j \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma} \right] + \zeta \nabla (\nabla \cdot \vec{u}) \; . \tag{7.2}$$

We shall treat the viscous terms as perturbation and substitute $\vec{u} = i\omega_0 \vec{D}$ on the right hand side of Eq. (7.2), where ω_0 is the vibration frequency in the absence of viscosity. The equation of motion becomes

$$mn \frac{\partial^2 \vec{D}}{\partial t^2} = [\lambda + \mu + i\omega_0 (\zeta + \tfrac{1}{3}\eta)] \nabla (\nabla \cdot \vec{D}) + (\mu + i\omega_0 \eta) \nabla^2 \vec{D} \; . \tag{7.3}$$

We can determine the complex eigenfrequency ω from (7.3) very simply if the motion is such that only $\vec{\mathscr{D}}_1$, $\vec{\mathscr{D}}_2$, or $\vec{\mathscr{D}}_3$ is present, such as the magnetic multipole states which contain a purely $\vec{\mathscr{D}}_2$ displacement, and the 0^+ compressional state which contains a purely $\vec{\mathscr{D}}_1$ displacement.

For a purely $\vec{\mathscr{D}}_1$ displacement, the effect of viscosity is to change the frequency from ω_0 to

$$\omega = \omega_0 \left[1 + \frac{i\omega_0 (\zeta + \frac{4}{3}\eta)}{2(\lambda + 2\mu)} \right] \tag{7.4}$$

which corresponds to a width of

$$\Gamma = \frac{(\hbar\omega_0)^2 (\zeta + \frac{4}{3}\eta)}{\hbar(\lambda + 2\mu)} \; . \tag{7.5}$$

For a giant resonance with a $\vec{\mathscr{D}}_1$ displacement, when $\hbar\omega_0 A^{1/3}$ is a constant, we can write

$$\Gamma = \frac{c}{A^{2/3}} \; , \tag{7.6}$$

where

$$c = \frac{(\hbar\omega_0 A^{1/3})^2 (\zeta + \frac{4}{3}\eta)}{\hbar(\lambda + \mu)} \; . \tag{7.7}$$

Similarly, for a purely $\vec{\mathscr{D}}_2$ or $\vec{\mathscr{D}}_3$ displacement, the shear viscosity changes the frequency from ω_0 to

$$\omega = \omega_0 \left[1 + \frac{i\omega_0 \eta}{2\mu} \right] . \tag{7.8}$$

Consequently, the state has a width given by

$$\Gamma = \frac{(\hbar\omega_0)^2 \eta}{\hbar\mu} \; , \tag{7.9}$$

which can be written as

$$\Gamma = \frac{c'}{A^{2/3}} \tag{7.10}$$

with

$$c' = \frac{(\hbar\omega_0 A^{1/3})^2 \eta}{\hbar\mu} \; . \tag{7.11}$$

Equations (7.5)–(7.7) and (7.9)–(7.11) allow one to extract the viscosity coefficients from the widths of the giant resonances in an approximate way. The isoscalar monopole resonance involves only compressional flows represented by a $\vec{\mathscr{D}}_1$-type displacement. The width of the monopole resonance is therefore proportional to $(\zeta + \frac{4}{3}\eta)$. On the other hand, even though the giant quadrupole and octupole resonances involve a mixture of $\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_3$ displacements, the flow is mainly incompressible. It is reasonable to treat these modes as a purely $\vec{\mathscr{D}}_3$ displacement in evaluating the effect of damping. The widths of these resonances then gives an estimate of the shear viscosity η. The second viscosity ζ can then be obtained from the widths of the monopole giant resonances.

It is interesting to note that as Γ is proportional to $(\hbar\omega_0)^2$, the width is proportional to the square of the energies for the same type of vibrations. This is in accord with the observation that the widths of the giant octupole states are substantially greater than that of the giant quadrupole states.

With a value of $\eta = 8.3$ MeV/fm^2c, we find from Eqs. (7.9)–(7.11) for the giant quadrupole vibration

$$\Gamma(l=2) = \frac{82}{A^{2/3}}\ \text{MeV} \tag{7.12}$$

and for the giant octupole vibration

$$\Gamma(l=3) = \frac{185}{A^{2/3}}\ \text{MeV}\ , \tag{7.13}$$

which are in good agreement with experimental data (Figs. 3 and 4). This value of η is considerably greater than the values of 2.7–5.7 MeV obtained previously by using very different assumptions of the collective dynamics.[36]

As $\hbar\omega_0$ for the monopole resonance does not vary as $A^{-1/3}$, the widths need to be calculated from Eq. (7.5). Using a value of $\zeta = 3.79$ MeV/fm^2c, we obtain the widths of the monopole resonance which agree well with experiment (Fig. 2).

We have found that the elastic model with a two-body viscosity can properly describe the dynamics of collective nuclear phenomena. It will be of interest to apply it to other phenomena such as heavy-ion collisions. There, it is more convenient to deal with the velocity field rather than the displacement field. We return to Eqs. (2.1) and (2.2) and add the long-range interaction $\mathscr{V}_L$ and the Navier-Stokes stress tensor p'_{ij} [Eq. (7.1)] to obtain

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0\ , \tag{7.14}$$

$$m^*\frac{\partial n u_i}{\partial t} + \sum_j \nabla_j [m^* n u_i u_j + p_{ij}^{(q)} + p'_{ij} + p_{ij}^{(p)}] = -n\nabla_i \left[\frac{\partial (W_s n)}{\partial n} + \mathscr{V}_l \right]\ , \tag{7.15}$$

where $p_{ij}^{(q)}$ is the elastic stress tensor satisfying

$$\frac{\partial}{\partial t} p_{ij}^{(q)} = -\mu^{(q)}(\nabla_i u_j + \nabla_j u_i) - \lambda^{(q)}\delta_{ij}\nabla\cdot\vec{u}\ . \tag{7.16}$$

It is more convenient to introduce the elastic force density F_i

$$F_i = -\sum_j \nabla_j p_{ij}^{(q)}\ . \tag{7.17}$$

In terms of F_i, the equation of motion (7.15) becomes

$$m^*\frac{\partial n u_i}{\partial t} + \sum_j \nabla_j [m^* n u_i u_j + p'_{ij} + p_{ij}^{(p)}] = F_i - n\nabla_i \left[\frac{\partial (w_s n)}{\partial n} + \mathscr{V}_L \right] \tag{7.18}$$

and

$$\frac{\partial}{\partial t} F_i = -[\lambda^{(q)} + \mu^{(q)}]\nabla_i(\nabla\cdot\vec{u}) - \mu^{(q)}\nabla^2 u_i\ . \tag{7.19}$$

The corresponding thermal energy equation is

$$\frac{\partial}{\partial t} n E_T + \nabla\cdot(n E_T \vec{u}) = -p_T \nabla\cdot\vec{u} + \nabla\cdot(\kappa\nabla T) - \sum_{ij} p'_{ij}\frac{\partial u_i}{\partial x_j}\ , \tag{7.20}$$

where E_T is the thermal energy density, κ the thermal conductivity, T the local temperature, and we have assumed

$$p_{ij}^{(p)} = p_T \delta_{ij}\ .$$

Equations (7.14) and (7.18)–(7.20) constitute the nuclear field equation appropriate for low-energy nuclear dynamics, its application to nuclear giant resonances through Eq. (7.18) having been shown to be successful. It will be of interest to see whether this set of equations may be useful in studying heavy-ion collisions.

VIII. CHARACTERISTICS OF LIQUID-DROP VIBRATIONS AND ELASTIC VIBRATIONS

The coexistence of the low-energy collective electric multipole state and the giant resonances at a higher energy poses an interesting question. What are the differences between these states aside from their differences in energy? Can these differences be put into a pictorial form? Clearly, we can find answers to these questions when we can put these two types of states within the same model, as we have done here. An examination of the eigenvectors appropriate for these two types of states then provides insight as to why and how these two types of states are different.

We wish to examine the states of multipolarity l and natural parity $(-1)^l$ (electric multipole states). For simplicity, we shall consider only $m=0$. The spatial part of the displacement vector is a linear combination of $\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_3$. Explicitly, it is given by

$$A_1\vec{\mathscr{D}}_1+A_3\vec{\mathscr{D}}_3=\frac{A_1N_l(-1)^l}{h^{l+1}}\left[\vec{e}_r\left[-j_l'(hr)-\frac{h^{l+1}}{k^{l+1}}\frac{A_3}{A_1}\frac{j_l(kr)}{kr}l(l+1)\right]P_l(\mu)\right.$$
$$\left.+\vec{e}_\theta\left\{\frac{j_l(hr)}{hr}+\frac{h^{l+1}}{k^{l+1}}\frac{A_3}{A_1}\left[\frac{j_l(kr)}{kr}+j_l'(kr)\right]\right\}(1-\mu^2)^{1/2}\frac{\partial P_l(\mu)}{\partial\mu}\right], \qquad (8.1)$$

where $\mu=\cos\theta$ (not to be confused with the Lamé constant), and $N_l=[4\pi/(2l+1)]^{1/2}$.

When the eigenenergies are obtained, we can determine the ratio A_3/A_1 with Eq. (5.8) and obtain the spatial part of the displacement vector up to an overall normalization constant.

We plot in Figs. 7—9 the displacement vectors for quadrupole, octupole, and hexadecapole vibrations. The length of the arrows represent the magnitude of the displacement. Only the displacement vectors at the surface $r=a$ and at $r=a/2$ are exhibited.

From Fig. 7(a), we see that in the liquid-drop vibration, which lies at a low energy, the displacement vectors in the interior, and those at the surface are generally pointing in the same direction. Furthermore, the displacement vectors around 0° and 180° point toward the direction of expansion. As an expansion in these directions leads to a greater local area, these displacement vectors do not lead to a compressional or expansional distor-

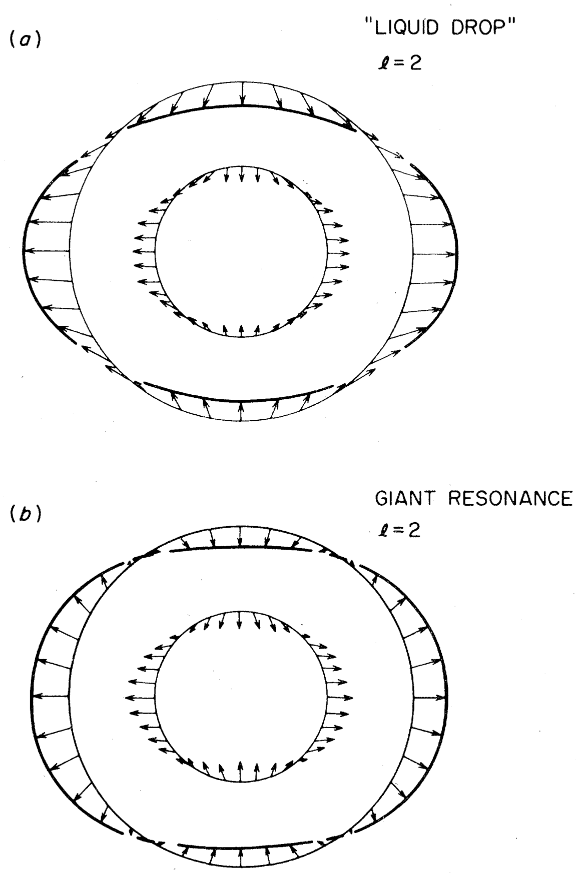

FIG. 7. The displacement vector $\vec{\mathscr{D}}(\vec{r})$ for the two different types of quadrupole vibrations. Figure 7(a) is for the liquid-drop vibrations. Displacement vector is given for points on $r=a$ and $r=a/2$ shown as the light circles. The dark curve is the envelope of the deformed surface after displacement. Figure 7(b) is for the giant quadrupole resonance oscillations at $\hbar\omega_2\sim 67$ MeV/$A^{1/3}$. The complete spatial and temporal variation of the displacement field is given by $\vec{\mathscr{D}}(\vec{r})\cos\omega_2 t$.

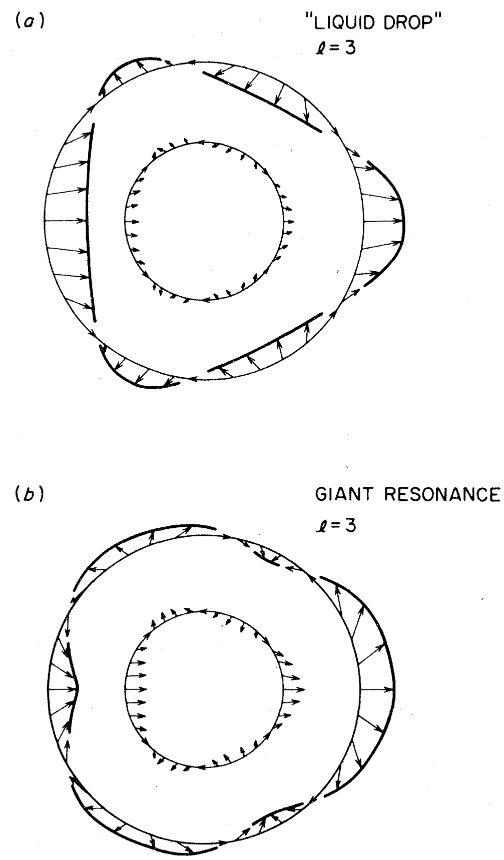

FIG. 8. The displacement vector $\vec{\mathscr{D}}(\vec{r})$ for the two different types of octupole vibrations. Figure 8(a) is for the liquid-drop vibration and Fig. 8(b) is for the giant octupole resonance at $\hbar\omega_3\sim 103$ MeV/$A^{1/3}$.

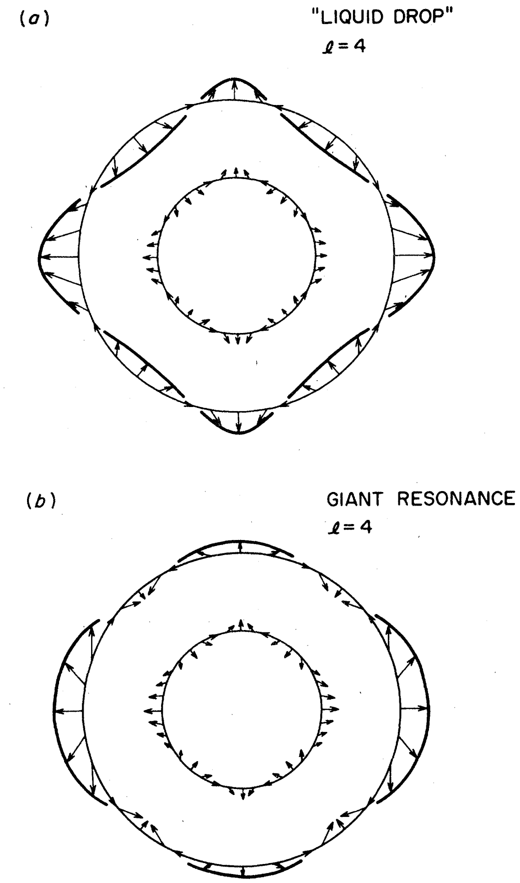

FIG. 9. The displacement vector $\vec{\mathscr{D}}(\vec{r})$ for the two different types of hexadecapole vibrations. Figure 9(a) is for the liquid-drop vibration and Fig. 9(b) is for the hexadecapole vibration at $\hbar\omega_4 \sim 133$ MeV/$A^{1/3}$.

tion of the surface. Similarly, the displacement vectors point outward at 90° and 270° and do not lead to a compressional or expansional distortion at these points.

The displacement vectors for the giant resonance ($l=2$, $\hbar\omega_2 A^{1/3} \simeq 67$ MeV) are quite different [Fig. 7(b)]. With the exception of the displacement vectors near 0°, 90°, 180°, and 270°, the displacement vectors in the interior are generally pointed in a direction different from those at the surface. For example, at 50°, the displacement vector in the interior is nearly opposite to that at the surface. This difference generates a shear distortion which causes the nucleus to vibrate. The flow is highly rotational and contains vorticities. The displacement vector around 0° and 180° at the nuclear surface point away from the direction of expansion and produce compressional or expansional distortion at the surface, in contrast to the case of liquid-drop vibration. Similar differences can be found for the octupole and hexadecapole vibrations (Figs. 8 and 9).

We see that even though the surface shape generated by these two types of vibrations can be approximately the same, the pattern of flow for the two are different. The low-energy vibration is a liquid-drop vibration, while the giant resonance is an elastic vibration involving vorticities.

IX. SUMMARY AND CONCLUSIONS

In a collective motion when all the displacement fields for all the particles are the same, the equation of motion for the nuclear fluid is that of the Lamé equation, appropriate for the propagation of elastic waves, with the Lamé constants depending on the nuclear properties. We explore whether the elastic vibrations of a finite nucleus can properly be identified as the observed nuclear giant resonances. Making use of the solutions obtained by Lamb in the last century, we found that these resonances can indeed be so identified, as was first suggested by Bertsch.

The generalization of Lamb's solution to include the effects of surface tension and Coulomb repulsion does not produce much change in the energies and characteristics of the elastic vibrational states. What emerges is a new group of liquid-drop vibrations of Bohr and Wheeler arising from surface tension and Coulomb repulsion alone. The coexistence of the liquid-drop vibration and elastic vibration in the same model provides one a pictorial way to spell out the similarities and differences between the low-energy electric multipole states and the high-energy states. The changes in surface shapes are similar but the flow patterns are different. The flow pattern for the elastic vibration is such that shear and compressional distortions are created.

The coexistence of the two types of vibrations also has an important implication. Because the high-lying elastic vibrations have been successfully identified, the low-energy multipole states which belong to the same model should also show up in the spectra. They go as the liquid-drop multipole energies which decrease with A faster than $A^{-1/2}$. It is suggested that the low-energy 3^- state observed at an energy approximately 32 $A^{-1/3}$ is perhaps built on the liquid-drop 3^- state with modifications due to nuclear shell effects. For its proper identifications, it is important to explore the 3^- states in the region around $150 \leq A \leq 230$ for which not much experimental data are available.

The inclusion of the Navier-Stokes tensor into the discussion of the dynamics leads to the treatment of a nucleus as a viscoelastic system. The widths of the giant quadrupole and octupole and monopole states can be utilized to give an estimate of the nuclear viscosity coefficients. New equations of motion for the nuclear fluid are also suggested.

Besides the natural parity electric multipole states, the solutions of Lamb[13] also include magnetic multipole states of unnatural parity, the 2^- state of which has recently been rediscovered theoretically. These are rotatory vibrations in which one layer of nuclear matter rotates against another. As the surface shape does not change, these states are unaffected by surface tension. These states have not been observed, although methods for their excitation have been suggested.[37]

The effective nuclear incompressibility deduced from the experimental data turns out to be a relatively sensitive function of the mass number A. It increases by about 30% when A varies from 100 to 250. Such a sensitivity is also the findings of similar investigations.[38–40] The nuclear incompressibility extrapolated to infinite matter K_∞ is found to be 220 MeV. One expects that when the plasma frequency is taken into account, the extrapolated nuclear incompressibility will be slightly lower.[4]

Although the present model gives good agreement with the energy systematics of observed giant quadrupole states, it remains to be shown whether the transition strengths are properly accounted for. It will be of interest to examine the effective mass associated with each of the multipole states for some properly chosen collective coordinates. The ratio of the effective mass to the effective mass for irrotational flow will give the fraction of sum rule exhaustion as pointed out by Bohr and Mottelson.[28]

With regard to the question of nuclear hydrodynamics versus nuclear elasticity, the evidence so far suggests the validity of nuclear elasticity and not nuclear hydrodynamics in describing the giant resonances. Similar conclusions are also reached by other workers.[12,40] This seems reasonable because for these low-energy phenomena the Pauli exclusion principle may be so restrictive as to inhibit the possibility of local equilibrium, which is a necessary condition for hydrodynamics. Seen in this light, previous treatment of the isovector giant dipole states in terms of hydrodynamics needs to be reexamined. It is therefore of interest to include the spin and isospin degrees of freedom in the present model; a similar treatment in hydrodynamics has already been performed.[4]

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APPENDIX

We list below the displacement vectors for the solutions $\vec{\mathscr{D}}_1$, $\vec{\mathscr{D}}_2$, and $\vec{\mathscr{D}}_3$ for a multipolarity of order (lm). The solution $\vec{\mathscr{D}}_1(\vec{r})$ is given explicitly by

$$\vec{\mathscr{D}}_1(\vec{r})=-\frac{1}{h^2}\nabla\psi_l(ha)\Omega_{lm}(\theta,\phi)$$
$$=\frac{(-1)^l N_{lm}e^{im\phi}}{h^{l+1}}\left\{-\vec{e}_r j_l'(hr)P_{lm}(\mu)+\frac{j_l(hr)}{hr}\left[\vec{e}_\theta(1-\mu^2)^{1/2}\frac{\partial}{\partial\mu}P_{lm}(\mu)-\vec{e}_\phi\frac{im}{(1-\mu^2)^{1/2}}P_{lm}(\mu)\right]\right\}, \quad (A1)$$

where

$$N_{lm}=\left[\frac{4\pi}{2l+1}\frac{(l-m)!}{(l+m)!}\right]^{1/2}, \quad (A2)$$

$$j_l'(x)=\frac{d}{dx}j_l(x), \quad (A3)$$

and $P_{lm}(\mu)$ is the associated Legendre polynomial of order (lm). The solution $\vec{\mathscr{D}}_2(\vec{r})$ is given by

$$\vec{\mathscr{D}}_2 = \frac{1}{k}\psi_l(kr)(\vec{r}\times\nabla)\Omega_{lm}(\theta,\phi)$$
$$= \frac{(-1)^l N_{lm} e^{im\phi}}{k^{l+1}} j_l(kr)\left[\vec{e}_\theta \frac{im P_{lm}(\mu)}{(1-\mu^2)^{1/2}} + \vec{e}_\phi (1-\mu^2)^{1/2}\frac{\partial}{\partial\mu}P_{lm}(\mu)\right] . \quad \text{(A4)}$$

The displacement vector $\vec{\mathscr{D}}_3$ is

$$\vec{\mathscr{D}}_3(\vec{r}) = \frac{1}{k}\nabla\times\vec{\mathscr{D}}_2$$
$$= \frac{(-1)^l N_{lm} e^{im\phi}}{k^{l+1}}\left\{ -\vec{e}_r l(l+1)\frac{j_l(kr)}{kr}P_{lm}(\mu) + \left[\frac{j_l(kr)}{kr} + j_l'(kr)\right]\left[\vec{e}_\theta (1-\mu^2)^{1/2}\frac{\partial}{\partial\mu}P_{lm}(\mu) - \vec{e}_\phi \frac{im}{(1-\mu^2)^{1/2}}P_{lm}(\mu)\right]\right\} .$$

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