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Higher-order WKB phase shifts for the heavy-ion optical potential

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It is shown that a higher-order correction improves the accuracy of the zeroth order WKB phase for a heavy-ion optical potential. The WKB phase shifts, correct to the first order in $\hbar^2$, agree with those obtained from a quantal optical model code.

[NUCLEAR REACTIONS Higher-order WKB calculation ^{58}Ni(^{18}O, ^{18}O'); E_{lab} = 60 MeV; calculated reflection coefficients.]

The WKB method is used extensively to study the elastic scattering of heavy ions. For example, Landowne *et al.*[1] used the WKB phases to calculate the elastic cross sections for the system ^{18}O + ^{58}Ni (E_{lab}=60 MeV). In spite of the discrepancies between the WKB and quantal phase shifts, the elastic cross sections agree with the experimental values to many orders of magnitude. The objective of this note is to show that the discrepancies can be minimized by higher-order corrections.

The asymptotic series for the WKB phase is

$$\delta(E,l)=\delta_0(E,l)+\hbar^2\delta_1(E,l)+\cdots . \tag{1}$$

For an ion-ion potential, the zeroth order nuclear phase[1,2] is

$$\delta_0(E,l)=\int_{r_t}^{\infty} k\,dr[1-u(r,l)]^{1/2} - \int_{r_c}^{\infty} k\,dr[1-u_c(r,l)]^{1/2} , \tag{2}$$

where

$$u(r,l)=U(r)/E+2\eta/kr+l(l+1)/k^2r^2 ,$$
$$u_c(r,l)=2\eta/kr+l(l+1)/k^2r^2 . \tag{3}$$

In Eq. (2), r_t and r_c are the turning points, $k^2=2mE/\hbar^2$, η is the Coulomb parameter ($mZZ'e^2/\hbar^2k$), and $U(r)$ is the Woods-Saxon potential,

$$U(r)=-V_0/\{1+\exp[(r-R_R)/a_R]\} - iW_0/\{1+\exp[(r-R_I)/a_I]\} . \tag{4}$$

To the first order in $\hbar^2$, from Refs. 2 and 3,

$$\hbar^2\delta_1(E,l)=\frac{1}{96k^{-1}}\left\{\oint u''(r,l)[1-u(r,l)]^{-3/2}dr-\oint u_c''(r,l)[1-u_c(r,l)]^{-3/2}dr\right\} , \tag{5}$$

where the contour encloses the branch cuts which extend from the respective turning points to infinity. Using the same contour, we can rewrite Eq. (2) as

$$\delta_0(E,l)=\tfrac{1}{2}k\left\{\oint[1-u(r,l)]^{1/2}dr - \oint[1-u_c(r,l)]^{1/2}dr\right\} . \tag{6}$$

For the system ^{18}O + ^{58}Ni (E_{lab}=60 MeV), the parameters of the optical potential as given in Ref. 4 are $a_R=a_I=a=0.50$ fm, $R_R=R_I=R=7.920$ fm, and $U_0=V_0+iW_0$ = (90.1 + i42.9)MeV.

To compare our WKB phases with the published results,[1] we calculated the imaginary parts of the phase shifts and the corresponding reflection coefficients in the transition region $22\le l\le 40$. η_l, the nuclear reflection coefficient is defined as

$$\eta_l=|\exp[2i\delta(E,l)]| . \tag{7}$$

The contour was chosen to be the line segments $y=0.3$ fm, $x=8.5$ fm, and $y=-1.5$ fm to circumvent the outermost turning points. It is also necessary to avoid the poles of the Woods-Saxon potential. For a detailed plot of the turning points and poles one can refer to Fig. 4 of Ref. 1. The infinite interval was truncated to [8.5, 15.0] and [8.5, 20.0]

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TABLE I. Real part and imaginary part of $\delta(E,l)$ for $22 \leq l \leq 40$. (a) WKB results correct to the zeroth order in $\hbar^2$. (b) WKB results correct to the first order in $\hbar^2$. (c) Quantal results.

	(a)		(b)		(c)	
l	δ_l(real)	δ_l(imag.)	δ_l(real)	δ_l(imag.)	δ_l(real)	δ_l(imag.)
22	−0.3351	1.5774	−0.3702	1.5815	−0.3708	1.5819
23	−0.1725	1.4715	−0.2088	1.4220	−0.2095	1.4223
24	−0.0276	1.2506	−0.0654	1.2558	−0.0663	1.2571
25	0.0964	1.0775	0.0564	1.0840	0.0555	1.0861
26	0.1963	0.8993	0.1531	0.9082	0.1527	0.9117
27	0.2676	0.7184	0.2199	0.7320	0.2216	0.7376
28	0.3052	0.5393	0.2519	0.5631	0.2591	0.5693
29	0.3033	0.3710	0.2492	0.4169	0.2645	0.4155
30	0.2616	0.2316	0.2354	0.3015	0.2421	0.2860
31	0.1994	0.1396	0.2148	0.1977	0.2023	0.1878
32	0.1442	0.0878	0.1663	0.1181	0.1578	0.1206
33	0.1033	0.0579	0.1206	0.0738	0.1179	0.0777
34	0.0740	0.0394	0.0863	0.0485	0.0860	0.0510
35	0.0531	0.0273	0.0618	0.0330	0.0622	0.0343
36	0.0382	0.0192	0.0444	0.0229	0.0448	0.0236
37	0.0275	0.0136	0.0319	0.0161	0.0323	0.0164
38	0.0198	0.0097	0.0230	0.0114	0.0232	0.0116
39	0.0143	0.0069	0.0165	0.0081	0.0167	0.0082
40	0.0103	0.0049	0.0119	0.0058	0.0121	0.0058

in two separate calculations and the error of truncation was found to be negligible. All integrals were evaluated using the DCADRE routine[5] with a relative error criterion of 10^{-6}. The quantal phases were calculated using the optical model code A−THREE.[6] We chose the mesh size and the

TABLE II. Real part, imaginary part of S_l, and reflection coefficients $|S(E,l)|$ for $22 \leq l \leq 40$. (a) WKB results correct to the zeroth order in $\hbar^2$. (b) WKB results correct to the first order in $\hbar^2$. (c) Quantal results.

	(a)			(b)			(c)		
l	S_l(real)	S_l(imag.)	S_l(abs.)	S_l(real)	S_l(imag.)	S_l(abs.)	S_l(real)	S_l(imag.)	S_l(abs.)
22	0.033 43	−0.026 49	0.042 64	0.031 20	−0.02851	0.042 27	0.031 16	−0.028 55	0.0422 61
23	0.055 26	−0.019 86	0.058 72	0.053 19	−0.02360	0.058 19	0.053 08	−0.023 64	0.0581 03
24	0.081 85	−0.004 53	0.081 98	0.080 45	−0.01059	0.081 14	0.080 21	−0.010 71	0.0809 25
25	0.1138	0.0222	0.1159	0.1136	0.0128	0.1144	0.1132	0.0126	0.1139
26	0.1529	0.0633	0.1655	0.1550	0.0490	0.1626	0.1540	0.0485	0.1614
27	0.2044	0.1212	0.2377	0.2093	0.0984	0.2313	0.2266	0.0980	0.2287
28	0.2787	0.1949	0.3401	0.2840	0.1566	0.3243	0.2782	0.1586	0.3202
29	0.3912	0.2715	0.4762	0.3816	0.2077	0.4344	0.3760	0.2198	0.4355
30	0.5451	0.3144	0.6293	0.4877	0.2482	0.5472	0.4994	0.2627	0.5643
31	0.6970	0.2973	0.7560	0.6122	0.2805	0.6734	0.6313	0.2704	0.6867
32	0.8043	0.2386	0.8389	0.7463	0.2578	0.7896	0.7468	0.2439	0.7856
33	0.8717	0.1827	0.8910	0.8377	0.2061	0.8627	0.8324	0.2000	0.8560
34	0.9141	0.1363	0.9242	0.8939	0.1559	0.9074	0.8896	0.1547	0.9029
35	0.9414	0.1005	0.9467	0.9289	0.1155	0.9361	0.9264	0.1159	0.9336
36	0.9594	0.0735	0.9622	0.9514	0.0847	0.9552	0.9501	0.0854	0.9538
37	0.9716	0.0536	0.9731	0.9663	0.0618	0.9683	0.9656	0.0625	0.9676
38	0.9799	0.0389	0.9807	0.9764	0.0449	0.9774	0.9760	0.0454	0.9770
39	0.9858	0.0282	0.9862	0.9833	0.0326	0.9838	0.9831	0.0330	0.9836
40	0.9899	0.0204	0.9901	0.9882	0.0236	0.9884	0.9880	0.0239	0.9883

matching radius to be 0.036 and 25 fm, respectively. The results are shown in Tables I and II.

The higher order correction improves the accuracy by an order of magnitude. In the worst case $l=30$, the discrepancy in the reflection coefficients is reduced from 11% to 3%. In the first order WKB calculations, the inclusion of penetration effects improves the agreement with exact calculations.[7] This effect is neglected in Eqs. (5) and (6) since only the outermost turning point is used. From the turning point plot in Ref. 1 (Fig. 4), one can see that for the partial wave $l=30$, the energy is just below the barrier. So it is reasonable that the discrepancies are considerably larger in the vicinity of this particular partial wave.

In conclusion, we have shown that the higher-order correction improves the accuracy of the WKB phase shifts calculated using a single turning point.

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