Antisymmetrized Lippmann-Schwinger equations and optical potentials

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(Received 21 April 1981)

A recent study of the effect of the Pauli principle in elastic nuclear scattering uses the prior off-shell extension for the multichannel transition operators to derive antisymmetrized Lippmann-Schwinger equations, while such equations are shown not to exist if the off-shell extension due to Alt, Grassberger, and Sandhas is employed instead. We show that the limitations associated with this second off-shell extension are inconsequential for all physical considerations. New physically constrained antisymmetrized Lippmann-Schwinger equations are found in terms of a simple effective interaction which incorporates all of the effects of the Pauli principle and whose properties are explicit. These equations, which are based on the Alt *et al.* off-shell extension, are employed to obtain in a direct manner the antisymmetrized generalization of the Feshbach formalism for the two fragment optical potential.

[NUCLEAR REACTIONS Pauli principle in nuclear reaction theory. Comparison of different off-shell extensions for transition operators. Antisymmetrized Lippmann-Schwinger equations. Antisymmetrized optical potential.]

I. INTRODUCTION

In a recent article Picklesimer and Thaler[1] advocate a particular way of imposing the Pauli principle in elastic two-fragment nuclear scattering.[2] This method is a special case of an approach developed earlier,[3-6] but it differs from Refs. 3, 4, and 6 especially, basically on the question of the reality properties demanded of the optical potential (OP). The point of departure between Ref. 1 and Refs. 3–6 involves the commitment to a definite off-shell extension for the two-fragment transition operators. The more conventional "prior" form is recommended in Ref. 1, while the extension proposed by Alt, Sandhas, and Grassberger (AGS) (Ref. 7) is employed in Refs. 3–6.[8] The question of off-shell extension is pertinent because in the usual scattering theory the Pauli principle is imposed on a framework in which the nucleons are initially regarded as distinguishable.[9] Thus, even for elastic scattering one is necessarily involved with a multichannel problem where the relevant channels are all physically equivalent by virtue of the Pauli principle.

Various attributes of both the AGS and the prior off-shell choices have been explored in subsequent works.[10,11] However, one important aspect of Ref. 1 remains to be settled satisfactorily. Namely, in Ref. 1 considerable emphasis is placed upon the attainment of Lippmann-Schwinger (LS) equations for antisymmetrized transition and wave operators involving an effective interaction $V^{(+)}$ which incorporates all of the effects of the Pauli principle.[12] This result, which is realized in Ref. 1 using the prior off-shell extension, is interesting because of its relative uniqueness. Namely, Picklesimer and Thaler[1] find that such LS equations *cannot* be obtained using the AGS off-shell extension.

An open question is whether or not the preceding result is significant for physical applications. The work of Refs. 10 and 11 suggests it is not and suggests further that some restricted type of LS formalism involving the effective interaction $V_e^{\beta}(z)$ which is introduced in Ref. 10 may exist. In the present article we explicitly establish such an antisymmetrized LS description, which appears to possess considerable practical advantages while retaining the manipulative simplicities of the usual unsymmetrized LS equations. These simplicities appear to be the major reason for the emphasis on the attainment of such equations in Ref. 1.

II. ANTISYMMETRIZED LIPPMANN-SCHWINGER EQUATIONS

In this section we determine the reason for the no-LS-equation property in the case of the AGS extension. This study suggests a loophole out of the argument of Ref. 1 involving a set of transition operators which are both on-shell *and* off-shell equivalent to the AGS choice over the physically relevant portions of the Hilbert space. We then construct a set of antisymmetrized LS equations which possess all of the formal advantages of the equations proposed in Ref. 1 but none of their practical disadvantages. We also find that all of the desirable features associated with the AGS choice[3–6,10] remain intact. This example eliminates the attainment of LS-type equations as a distinguishing characteristic of the prior off-shell extension.

Before we introduce the Pauli principle let us consider the pertinent aspects of the multichannel scattering formalism. The *prior* form of the transition operator for scattering from channel β to channel α is defined as

$$T_{\alpha,\beta}^{(+)} = V^\alpha + V^\alpha G V^\beta \ , \tag{2.1}$$

where H is the complete Hamiltonian, $G=(z-H)^{-1}$, and z is a complex parametric energy which is equal to $E+i0$ for scattering. The various two-fragment channels are labeled by the partitions $\alpha,\beta,\gamma,\cdots$ of the N nucleon system into two clusters. The external interaction between the two fragments characterized by β is denoted as V^β. One has the decomposition

$$H = H_\beta + V^\beta \ , \tag{2.2}$$

where H_β is the channel Hamiltonian which has eigenstates $|\phi_\beta(\vec{k})\rangle$ corresponding to both fragments in their ground states with relative momentum $\vec{k}$ between the fragments.

It is easily shown that $T_{\alpha,\beta}^{(+)}$ satisfies

$$T_{\alpha,\beta}^{(+)} = V^\alpha G_\alpha G_\beta^{-1} + V^\alpha G_\alpha T_{\alpha,\beta}^{(+)} \ , \tag{2.3}$$

where $G_\alpha=(z-H_\alpha)^{-1}$. One uses the resolvent identities relating G and G_α to obtain (2.3). Suppose we ask whether for some set S of two-cluster channels there are operators $v_{\alpha,\gamma}^{(+)}$ such that instead of (2.3) we have, with $\alpha,\beta,\gamma\in S$,

$$T_{\alpha,\beta}^{(+)} = v_{\alpha,\beta}^{(+)} + \sum_\gamma v_{\alpha,\gamma}^{(+)} G_\gamma T_{\gamma,\beta}^{(+)} \ . \tag{2.4}$$

In matrix form (2.4) is similar to a single-channel LS equation. Evidently, if (2.4) is well-defined then we also have

$$T_{\alpha,\beta}^{(+)} = v_{\alpha,\beta}^{(+)} + \sum_\gamma T_{\alpha,\gamma}^{(+)} G_\gamma v_{\gamma,\beta}^{(+)} \ , \tag{2.5}$$

so

$$v_{\alpha,\beta}^{(+)} = \sum_\gamma T_{\alpha,\gamma}^{(+)} [\delta_{\gamma,\beta} - G_\gamma v_{\gamma,\beta}^{(+)}] \ . \tag{2.6}$$

Thus, if in (2.3) we set $\beta=\gamma$, multiply on the right by $[\delta_{\gamma,\beta}-G_\gamma v_{\gamma,\beta}^{(+)}]$, sum over γ, and use (2.6) we find a set of Watson-type equations for $v_{\alpha,\beta}^{(+)}$, with $\alpha,\beta,\gamma\in S$,

$$v_{\alpha,\beta}^{(+)} = V^\alpha G_\alpha G_\beta^{-1} - V^\alpha G_\alpha \sum_\gamma \overline{\delta}_{\alpha,\gamma} v_{\gamma,\beta}^{(+)} \ , \tag{2.7}$$

where $\overline{\delta}_{\alpha,\beta}=1-\delta_{\alpha,\beta}$.

On the other hand, the AGS operators

$$T_{\alpha,\beta}^{\rm AGS} = \overline{\delta}_{\alpha,\beta} G_\beta^{-1} + T_{\alpha,\beta}^{(+)} \tag{2.8}$$

satisfy

$$T_{\alpha,\beta}^{\rm AGS} = \overline{\delta}_{\alpha,\beta} G_\beta^{-1} + V^\alpha \delta_{\alpha,\beta} + V^\alpha G_\alpha T_{\alpha,\beta}^{\rm AGS} \ . \tag{2.9}$$

So if we postulate operators $v_{\alpha,\beta}^{\rm AGS}$, which are such that, with $\alpha,\beta,\gamma\in S$,

$$T_{\alpha,\beta}^{\rm AGS} = v_{\alpha,\beta}^{\rm AGS} + \sum_\gamma v_{\alpha,\gamma}^{\rm AGS} G_\gamma T_{\gamma,\beta}^{\rm AGS} \ , \tag{2.10}$$

we find, following the same steps which led to (2.7), a sum rule which holds for any $\alpha,\beta,\gamma\in S$:

$$\sum_\gamma v_{\gamma,\beta}^{\rm AGS} = \overline{\delta}_{\alpha,\beta} G_\beta^{-1} + V^\alpha \delta_{\alpha,\beta} \ . \tag{2.11}$$

Since the left side of (2.11) is independent of α, we have arrived at a contradiction. We see that the AGS off-shell structure and (2.10) [and its counterpart analogous to (2.5)] are incompatible. The preceding analysis also applies to the antisymmetrized problem provided S is restricted to a set of permutation-related partitions.

The $\overline{\delta}_{\alpha,\beta}G_\beta^{-1}$ term in (2.8) which removes the channel Green's function mismatch appearing in (2.3) is responsible for the remarkable off-shell cancellations leading to the nonexistence of $v_{\alpha,\beta}^{\rm AGS}$. However, these cancellations take place over the *entire* Hilbert space. This suggests that we might introduce an off-shell extension which lacks the full symmetry of the AGS but only on physically irrelevant portions of the Hilbert space.

If we let P_β denote the projector on the space spanned by the $|\phi_\beta(\vec{k})\rangle$,[13] then the transition operator $T_{\alpha,\beta}$, which satisfies

$$T_{\alpha,\beta} = \overline{\delta}_{\alpha,\beta} P_\beta G_\beta^{-1} + V^\alpha \delta_{\alpha,\beta} + V^\alpha G_\alpha T_{\alpha,\beta} \ , \tag{2.12}$$

is both on-shell *and* off-shell equivalent to $T^{\rm AGS}_{\alpha,\beta}$ over the physically relevant parts of the Hilbert space:

$$T_{\alpha,\beta}P_\beta = T^{\rm AGS}_{\alpha,\beta}P_\beta \ . \tag{2.13}$$

In fact, one easily shows that

$$T_{\alpha,\beta} = T^{\rm AGS}_{\alpha,\beta} - \bar{\delta}_{\alpha,\beta}(G_\alpha{}^{-1}GG_\beta{}^{-1})Q_\beta \ , \tag{2.14}$$

where $Q_\beta = I - P_\beta$.

Evidently $T_{\alpha,\beta}$ lacks the post-prior symmetry possessed by $T^{\rm AGS}_{\alpha,\beta}$ *only* on physically irrelevant parts of Hilbert space. However, the lack of the full symmetry presents a loophole out of the LS-equation contradiction (2.11). We find that

$$T_{\alpha,\beta} = v_{\alpha,\beta} + \sum_\gamma v_{\alpha,\gamma}G_\gamma T_{\gamma,\beta} \tag{2.15}$$

and

$$v_{\alpha,\beta} = \bar{\delta}_{\alpha,\beta}P_\beta G_\beta{}^{-1} + V^\alpha\delta_{\alpha,\beta} - \sum_\gamma \bar{\delta}_{\alpha,\gamma}P_\gamma v_{\gamma,\beta} \ . \tag{2.16}$$

An important aspect of (2.16) is that it represents a set of relatively simple integral equations which have compact, nonsingular kernels. The matrix inversions needed to solve (2.16) are very similar to those involved in coupled-reaction-channel problems with the additional simplification of having no boundary conditions. When (2.15) and (2.16) are applied to the problem of elastic scattering with the Pauli principle imposed, the solution of (2.16) can be expressed very simply. By way of contrast, within (2.7) reside the pathologies and complexities representative of multiparticle scattering problems when they are approached frontally using conventional techniques. For example, the kernels of (2.7) are both noncompact and singular; the significance of this is made evident later in this section. Evidently, the attainment of the solution of (2.7) is virtually equivalent to solving the full N-particle scattering problem.

Let us next impose the Pauli principle which implies the invariance of the physics with respect to permutations of the nucleons. Let β represent an arbitrary (but fixed) partition from a set $\hat{\beta}$ of physically equivalent partitions which are related by permutations of identical nucleons. We suppose that

$$\bar{A}(\beta)\,|\phi_\beta(\vec{k})\rangle = |\phi_\beta(\vec{k})\rangle \ , \tag{2.17}$$

where $\bar{A}(\beta)$ is the antisymmetrizer (internal to the fragments) with respect to all permutations which map β into itself. We note that $\bar{A}(\beta)$ is Hermitian and

$$\bar{A}(\beta)^2 = \bar{A}(\beta) \ . \tag{2.18}$$

Because of (2.17) we have $\bar{A}(\beta)P_\beta = P_\beta\bar{A}_\beta = P_\beta$. Also, it will become clear that we can regard the problem as restricted to the Hilbert space projected out by $\bar{A}(\beta)$. Thus, we have $Q_\beta\bar{A}_\beta = \bar{\mathcal{A}}_\beta Q_\beta = Q_\beta$ as well. All essential features of our development, in particular the physical matrix elements, are independent of the choice of β.[3–6,14] The antisymmetrized prior and AGS operators are given by

$$T^{(+)}(\hat{\beta}) = V^\beta\mathcal{A}(\hat{\beta})GG_\beta{}^{-1} \tag{2.19}$$

and

$$T^{\rm AGS}(\hat{\beta}) = T^{(+)}(\hat{\beta}) + \hat{\mathcal{A}}(\hat{\beta})G_\beta{}^{-1} \ , \tag{2.20}$$

respectively.[15,16] Here[9]

$$\mathcal{A}(\hat{\beta}) = \sum_{\beta'\epsilon\hat{\beta}}\bar{\mathcal{A}}(\beta)\hat{U}(\beta',\beta)^\dagger \ , \tag{2.21}$$

$$\hat{\mathcal{A}}(\hat{\beta}) = \mathcal{A}(\hat{\beta}) - \bar{\mathcal{A}}(\beta) \ , \tag{2.22}$$

where $\hat{U}(\beta',\beta)$ is the parity-weighted unitary permutation operator corresponding to $\beta \rightarrow \beta'$. The distinct natures of $\bar{\mathcal{A}}(\beta)$, $\mathcal{A}(\hat{\beta})$, and $\hat{\mathcal{A}}(\hat{\beta})$ should be clearly kept in mind. Also $[\bar{\mathcal{A}}(\beta),\mathcal{A}(\hat{\beta})] = 0$.

Evidently the antisymmetrized counterpart of $T_{\alpha,\beta}$ is

$$T(\hat{\beta}) = T^{\rm AGS}(\hat{\beta}) - G_\beta{}^{-1}\hat{\mathcal{A}}(\hat{\beta})GQ_\beta G_\beta{}^{-1} . \tag{2.23}$$

Equation (2.23) still preserves (2.13):

$$T(\hat{\beta})P_\beta = T^{\rm AGS}(\hat{\beta})P_\beta \ . \tag{2.24}$$

We note, e.g., that

$$T(\hat{\beta}) = \sum_{\beta'\epsilon\hat{\beta}}\bar{\mathcal{A}}(\beta)\hat{U}(\beta',\beta)^\dagger T_{\beta',\beta} \ . \tag{2.25}$$

We can then immediately transform (2.12) into

$$T(\hat{\beta}) = \mathfrak{N}G_\beta{}^{-1} + V^\beta\bar{\mathcal{A}}(\beta) + V^\beta G_\beta T(\hat{\beta}) \ , \tag{2.26}$$

where[15]

$$\mathfrak{N} = \hat{\mathcal{A}}(\hat{\beta})P_\beta \ . \tag{2.27}$$

One can either apply the same procedures to Eqs. (2.15) and (2.16) or proceed directly from (2.26) to deduce the antisymmetrized LS equation

$$T(\hat{\beta}) = V^\beta_e(z) + V^\beta_e(z)G_\beta T(\hat{\beta}) \ , \tag{2.28}$$

where[10]

$$V^\beta_e(z) = (1+\mathfrak{N})^{-1}[V^\beta + \mathfrak{N}G_\beta(z)^{-1}] \ . \tag{2.29}$$

The effective interaction $V^\beta_e(z)$ is the antisymmetrized form of $v_{\alpha,\beta}$. In this case the solution of (2.16) is expressed as the right side of (2.29). The existence of $(1+\mathfrak{N})^{-1}$ has been established previously.[10] The z dependence of $V_e^\beta(z)$ is relatively simple but it is, nevertheless, a crucial determinant of the unitarity properties of $T(\hat{\beta})$ and other auxiliary operators.[10] We have ignored explicit $\bar{\mathcal{A}}(\beta)$ factors in (2.29).

Equations (2.28) and (2.29) achieve the stated goal of Ref. 1 of an LS equation wherein all the effects of antisymmetry are contained in the effective interaction. There is, however, one major advantage in the present realization of that goal: *The effective interaction (2.29) can be regarded as explicitly known.* On the other hand, the antisymmetrized effective interaction $V^{(+)}(z)$ found in Ref. 1 is[17–19]

$$V^{(+)}(z)=V^\beta\mathcal{A}(\hat{\beta})[1+G_\beta(z)V^\beta\mathcal{A}(\hat{\beta})]^{-1} \ . \tag{2.30}$$

Related to our previous remarks concerning Eq. (2.7), we observe that calculating the inverse[17] in (2.30) is equivalent to solving the N-particle problem. In this instance in place of (2.28) we have[1]

$$T^{(+)}(\hat{\beta})=V^{(+)}(z)+V^{(+)}(z)G_\beta T^{(+)}(\hat{\beta}) \ . \tag{2.31}$$

Closely related to the preceding is the question of the existence of LS equations for antisymmetrized wave operators. If we let $|\psi_\beta^{(+)}(\vec{k})\rangle$ denote the antisymmetrized eigenstate of H which corresponds to $\mathcal{A}(\hat{\beta})|\phi_\beta(\vec{k})\rangle$ in the infinite past, then

$$|\psi_\beta^{(+)}(\vec{k})\rangle=\mathcal{A}(\hat{\beta})GG_\beta{}^{-1}|\phi_\beta(\vec{k})\rangle \ . \tag{2.32}$$

This suggests the introduction of an antisymmetrized wave operator[15]

$$\Omega^{\mathrm{AGS}}(\hat{\beta})\equiv\mathcal{A}(\hat{\beta})GG_\beta{}^{-1} \ . \tag{2.33}$$

One finds that

$$\Omega^{\mathrm{AGS}}(\hat{\beta})=\bar{\mathcal{A}}(\beta)+G_\beta T^{\mathrm{AGS}}(\hat{\beta}) \ . \tag{2.34}$$

Because of the operator $\mathcal{A}(\hat{\beta})$, which is proportional to the projector onto the antisymmetrized states, the inverse $[\Omega^{\mathrm{AGS}}(\hat{\beta})]^{-1}$ does not exist on the full *unsymmetrized* Hilbert space. We note that the existence of $[\Omega^{\mathrm{AGS}}(\hat{\beta})]^{-1}$ is equivalent to the requirement that $T^{\mathrm{AGS}}(\hat{\beta})$ satisfy an LS equation of the form (2.28) or (2.31). As a consequence, $\Omega^{\mathrm{AGS}}(\hat{\beta})$ does not satisfy an LS equation either.[1]

The wave operators

$$\Omega^{(+)}(\hat{\beta})\equiv\bar{\mathcal{A}}(\beta)+G_\beta T^{(+)}(\hat{\beta}) \ , \tag{2.35}$$

$$\Omega(\hat{\beta})\equiv\bar{\mathcal{A}}(\beta)+G_\beta T(\hat{\beta}) \tag{2.36}$$

are half-on-shell equivalent to $\Omega^{\mathrm{AGS}}(\hat{\beta})$ and may be used instead in (2.32).[20–22] The inverses of these operators are not manifestly nonexistent as is the case with $\Omega^{\mathrm{AGS}}(\hat{\beta})$, so we find using (2.28) and (2.31):

$$\Omega^{(+)}(\hat{\beta})=\bar{\mathcal{A}}(\beta)+G_\beta V^{(+)}(z)\Omega^{(+)}(\hat{\beta}) \ , \tag{2.37}$$

$$T^{(+)}(\hat{\beta})=V^{(+)}(z)\Omega^{(+)}(\hat{\beta}) \ , \tag{2.38}$$

$$\Omega(\hat{\beta})=\bar{\mathcal{A}}(\beta)+G_\beta V_e^\beta(z)\Omega(\hat{\beta}) \ , \tag{2.39}$$

$$T(\hat{\beta})=V_e^\beta(z)\Omega(\hat{\beta}) \ . \tag{2.40}$$

Equations (2.35), (2.37), and (2.38) are obtained in Ref. 1. As with the comparison of (2.28) and (2.31) we see that (2.39) is of precisely the same *form* as (2.37) and therefore shares with (2.37) all the possible benefits which accrue thereby. On the other hand, (2.39) involves the significantly less complicated effective potential $V_e^\beta(\hat{\beta})$. Since

$$\Omega(\hat{\beta})P_\beta=\Omega^{\mathrm{AGS}}(\hat{\beta})P_\beta \ , \tag{2.41}$$

we obtain from (2.39)

$$\Omega^{\mathrm{AGS}}(\hat{\beta})P_\beta=P_\beta+G_\beta V_e^\beta(z)\Omega^{\mathrm{AGS}}(\hat{\beta})P_\beta, \tag{2.42}$$

and therefore

$$|\psi_\beta^{(+)}(\vec{k})\rangle=|\phi_\beta(\vec{k})\rangle+G_\beta V_e^\beta(z)|\psi_\beta^{(+)}(\vec{k})\rangle \ . \tag{2.43}$$

Equations (2.42) and (2.43) reflect the fact that for *all physical considerations* the limitations of $\Omega^{\mathrm{AGS}}(\hat{\beta})$ noted in Ref. 1 are entirely inconsequential. If one, for whatever purposes, requires a wave operator defined on the *entire* Hilbert space it would appear that the quasi-AGS operator $\Omega(\hat{\beta})$ is preferable to $\Omega^{(+)}(\hat{\beta})$ in any case.

We have seen that when the various scattering operators are restricted to the domains relevant to their physical application, the LS analogies proposed in Ref. 1 realize in the AGS case as well. The AGS example also fulfills one of the claims made in Ref. 1 of attaining an LS equation [e.g., (2.42)] which is written in terms of an effective interaction $V_e^\beta(z)$ which incorporates all of the effects of antisymmetrization and whose properties are explicit. It cannot be fairly stated that the properties of $V^{(+)}(z)$ are explicated in Ref. 1 as claimed there.

Picklesimer[11] has pointed out that for the purposes of treating the antisymmetrized *elastic* scattering problem the Schrödinger equation

$$G^{-1}|\psi_\beta^{(+)}(\vec{k})\rangle=0 \ , \tag{2.44}$$

can be replaced by

$$[G_\beta^{-1} - V_e^\beta(z)]\,|\psi_\beta^{(+)}(\vec{k})\rangle = 0 \ , \qquad (2.45)$$

since

$$(1+\mathfrak{N})^{-1}G^{-1} = G_e^{-1} \ , \qquad (2.46a)$$

$$G_e^{-1} \equiv G_\beta^{-1} - V_e^\beta(z) \ . \qquad (2.46b)$$

Equation (2.45) is certainly consistent with (2.43), but we have also learned that (2.43) and (2.45) are valid for *any* process which is initiated from the state $\mathcal{A}(\hat{\beta})\,|\phi_\beta(\vec{k})\rangle$. Moreover, in (2.43) we find an explicit answer to the question of the appropriate boundary conditions to be imposed upon the solution of (2.45). We see that $V_e^\beta(z)$ really does behave as an effective interaction, because even though $|\psi_\beta^{(+)}(\vec{k})\rangle$ is asymptotic to $A(\hat{\beta})\,|\phi_\beta(\vec{k})\rangle$ initially, in solving (2.45) the "unperturbed" form of the equation is obtained by setting $V_e^\beta(z)$ equal to zero [cf. (2.43)].

We also note from (2.28) that

$$T(\hat{\beta}) = V_e^\beta(z) + V_e^\beta(z)G_eV_e^\beta(z) \ , \qquad (2.47)$$

or

$$T(\hat{\beta}) = G_\beta^{-1}G_eG_\beta^{-1} - G_\beta^{-1} \ . \qquad (2.48)$$

It has been noted in Ref. 11 that

$$P_\beta \mathcal{T} P_\beta = P_\beta G_\beta^{-1}G_eG_\beta^{-1}P_\beta - P_\beta G_\beta^{-1} \qquad (2.49)$$

for any transition operator $\mathcal{T}$ such that

$$P_\beta \mathcal{T} P_\beta = P_\beta T^{\rm AGS}(\hat{\beta})P_\beta \ . \qquad (2.50)$$

Evidently, $T(\hat{\beta})$ not only satisfies (2.50), but it also satisfies the much more general relation (2.13). It is this last property which makes the LS analogies for $T(\hat{\beta})$ meaningful, with regard to the AGS case. We remark that only the fully P_β-projected form (2.50) is exploited in Ref. 11; this fact is very important in distinguishing some of the results of the present and the next section from those in Ref. 11.

III. ANTISYMMETRIZED OPTICAL POTENTIAL

The LS equations for $T^{(+)}(\hat{\beta})$ and $\Omega^{(+)}(\hat{\beta})$ are used in Ref. 1 to recover the definition of the OP proposed in Refs. 3 and 4 appropriate to this off-shell extension. Then expressions for the OP which are formally similar to those found by Feshbach[23] in the unsymmetrized case are obtained.[1] In these expressions the role of V^β is taken over by $V^{(+)}(\hat{\beta})$. Nearly all of this is done over again in Ref. 1 *without* using the LS equations in order to deal with the AGS case. The results for the latter appear to lack the formal resemblance to the Feshbach formalism found in the (+) case. Under circumstances of physical interest this apparent deficiency of the AGS case is now known to be illusory.[10,11] In this section we show that with the quasi-AGS operator $T(\hat{\beta})$ one actually does achieve a very simple realization of the Feshbach formalism generalized to include Pauli effects, in accord with the results of Refs. 10 and 11.

The physical idea behind the definition of the OP operator proposed in Refs. 3 and 4 is that the elastic two-fragment scattering equations be manifestly of the standard two-body type. However, given this stipulation, the OP operator then depends upon the off-shell extension of the transition operator employed. For the cases at hand, we have as defining integral equations for the OP operators[3,4]

$$U^{(+)}(\hat{\beta}) = T^{(+)}(\hat{\beta}) - T^{(+)}(\hat{\beta})G_\beta P_\beta U^{(+)}(\hat{\beta}) \ , \qquad (3.1)$$

$$U^{\rm AGS}(\hat{\beta}) = T^{\rm AGS}(\hat{\beta}) - T^{\rm AGS}(\hat{\beta})G_\beta P_\beta U^{\rm AGS}(\hat{\beta}) \ , \qquad (3.2)$$

$$U(\hat{\beta}) = T(\hat{\beta}) - T(\hat{\beta})G_\beta P_\beta U(\hat{\beta}) \ . \qquad (3.3)$$

Equation (3.3), for example, is equivalent to

$$T(\hat{\beta}) = U(\hat{\beta}) + U(\hat{\beta})G_\beta P_\beta T(\hat{\beta}) \ , \qquad (3.4)$$

with similar equations for the (+) and AGS operators. The P_β-space projection of (3.4) is a two-body LS equation; given $P_\beta U(\hat{\beta})P_\beta$, the solution of this equation determines the physical on-shell elastic scattering amplitudes $\langle\phi_\beta(\vec{k}')\,|\,T(\hat{\beta})\,|\,\phi_\beta(\vec{k})\rangle$. Analogous remarks apply to the (+) and AGS cases. The two-body wave function associated with (3.4) and its counterparts for (+) and AGS is $P_\beta\,|\psi_\beta^{(+)}(\vec{k})\rangle$ in *all three* cases.[10,24] We note that (2.24) along with (3.2)–(3.4) implies that

$$U(\hat{\beta})P_\beta = U^{\rm AGS}(\hat{\beta})P_\beta \ . \qquad (3.5)$$

From (2.28) we infer that

$$[1 - V_e^\beta(z)G_\beta]T(\hat{\beta}) = V_e^\beta(z) \ . \qquad (3.6)$$

If we then multiply (3.3) on the left by $[1 - V_e^\beta(z)G_\beta]$ and use (3.6) we obtain the LS equation[15,25]

$$U(\hat{\beta}) = V_e^{\beta}(z) + V_e^{\beta}(z)Q_{\beta}G_{\beta}U(\hat{\beta}) \ , \qquad (3.7)$$

and so

$$U(\hat{\beta}) = V_e^{\beta}(z) + U(\hat{\beta})Q_{\beta}G_{\beta}V_e^{\beta}(z) \ . \qquad (3.8)$$

Then using standard manipulations[22,23] we obtain from (3.7) and (3.8) the closed-form expression which holds on the entire Hilbert space:

$$U(\hat{\beta}) = V_e^{\beta}(z) + V_e^{\beta}(z)Q_{\beta}[G_{\beta}{}^{-1} - Q_{\beta}V_e^{\beta}(z)Q_{\beta}]^{-1} \times Q_{\beta}V_e^{\beta}(z) \ . \qquad (3.9)$$

This is of the same form as obtained by Feshbach[23] in the unsymmetrized case.[26–28] Indeed, it is clear that

$$\lim_{\epsilon \to 0} U(\hat{\beta}) = [V^{\beta} + V^{\beta}Q_{\beta}(G_{\beta}{}^{-1} - Q_{\beta}V^{\beta}Q_{\beta})^{-1}Q_{\beta}V^{\beta}]\bar{A}(\beta) \ . \qquad (3.10)$$

We remind ourselves that the intermediate sums in (3.9) and (3.10) are carried out over the space projected by $\bar{A}(\beta)$.

Physically one only requires the P_{β} projections of the various OP operators. Equation (3.5) indicates that one should be able to obtain the P_{β} projection of (3.9) without the intermediary of LS equations and this is indeed the case.[10]

An expression similar to (3.9) obtains for $U^{(+)}(\hat{\beta})$ but with $V^{(+)}$ in place of $V_e^{\beta}(z)$.[1] In view of our previous remarks concerning the relative complexity of $V^{(+)}(z)$, the antisymmetrized version of the Feshbach formalism proposed in Ref. 1 for $U^{(+)}(\hat{\beta})$ appears to possess no advantages and many disadvantages compared to what can be achieved in the AGS case. In this connection we point out that the relationship of the operator $U^{(+)}(\hat{\beta})$ to antisymmetrized multiple scattering formalisms which is investigated in Ref. 1 appears to be entirely independent of the existence of $V^{(+)}(z)$ or any of its properties. Other advantages associated with the AGS choice for the OP operator are explored at length in Refs. 3–5, 10 and 28. Of particular note is that the AGS form of the OP leads to the Feshbach resonance structure in the antisymmetrized case while this is almost certainly not the case for $U^{(+)}(\hat{\beta})$.[10,11] With the OP operator $U(\hat{\beta})$ we have the dual advantages of the Feshbach resonance structure for elastic scattering along with the LS analogies.

IV. SUMMARY

We have shown that for all physical considerations the limitations associated with the AGS off-shell extension found in Ref. 1 are inconsequential. In contrast to the results of Ref. 1, we have obtained Lippmann-Schwinger equations for the AGS case which are expressed in terms of an effective interaction which contains all of the effects of nucleon antisymmetrization and whose properties are explicit. The effective interaction which appears in the case of the AGS off-shell extension is very simple in structure. We show how the AGS Lippmann-Schwinger equations derived in the present article lead very simply to the antisymmetrized generalization of the Feshbach[23] OP formalism. These results supplement those obtained in Refs. 10 and 11 in providing operator and wave function equations valid over the *entire* Hilbert space. We have demonstrated how one can combine the advantages of the AGS off-shell extension associated with the symmetrical treatment of Pauli-equivalent channels along with a convenient operator formalism which allows one to exploit the familiar strategies of standard nuclear reaction theory.

ACKNOWLEDGMENTS

I would like to thank R. Goldflam, A. Picklesimer, R. M. Thaler, and W. Tobocman for many discussions concerning the topics of this paper. This work was supported in part by the National Science Foundation under the Grant No. PHY78-26595.

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[1] A. Picklesimer and R. M. Thaler, Phys. Rev. C 23, 42 (1981).

[2] For the sake of simplicity we regard all the nucleons as identical.

[3] R. Goldflam and K. L. Kowalski, Phys. Rev. Lett. 44, 1044 (1980).

[4] R. Goldflam and K. L. Kowalski, Phys. Rev. C 22, 949 (1980).

[5] A. Picklesimer and K. L. Kowalski, Phys. Lett. 95B, 1 (1980).

[6] R. Goldflam and K. L. Kowalski, Phys. Rev. C 22, 2341 (1980).

[7] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. 2B, 167 (1967); P. Grassberger and W. Sandhas, *ibid.*

2B, 181 (1967).

[8]It should be emphasized that the basic approach of Refs. 3–6 is stated in such a way as to hold for *any* off-shell extension. This is especially pertinent for the definition of the antisymmetrized optical potential proposed in Refs. 3 and 4. The prior form is examined and discarded as a suitable choice in Refs. 3–6 on the basis of unfavorable unitarity properties for the OP. Although both the AGS and prior forms are considered in Ref. 1, it is only the latter which is featured in the applications contemplated there.

[9]G. Bencze and E. F. Redish, Nucl. Phys. A238, 240 (1975); J. Math. Phys. 19, 1909 (1978). See also G. Bencze, in *Few-Body Nuclear Physics,* edited by G. Pisent, V. Vanzani, and L. Fonda (IAEA, Vienna, 1978), p. 113. The theoretical techniques for incorporating permutation symmetries into certain types of (label transforming) multiparticle scattering theories are developed in these references. This is done with careful attention to the appropriate identification of scattering amplitudes and the attendant normalizations. What can be referred to as the "conventional" statement of scattering theory falls into the label-transforming class. See Ref. 5 for some remarks in this connection.

[10]K. L. Kowalski and A. Picklesimer, Phys. Rev. Lett. 46, 228 (1981), and CWRU report, 1980.

[11]A Picklesimer, CWRU report, 1981.

[12]In Ref. 1, $V^{(+)}$ is denoted as $\hat{V}$. In the present paper we maintain a general consistency with the notational conventions of Ref. 10. A translation code between the notation used in Ref. 1 for the transition $(\hat{T},\tilde{T})$ and the OP operators $(\hat{U},\tilde{U})$, and that used in the present article and in Refs. 3–6 is useful in comparing these works. We have $\hat{T} = T^{(+)}(\hat{\beta})$, $\tilde{T} = T^{\rm AGS}(\hat{\beta})$, $\hat{U} = U^{(+)}(\hat{\beta})$, and $\tilde{U} = U^{\rm AGS}(\hat{\beta})$. The operators on the right-hand sides of each of these equations are those originally defined in Refs. 3 and 4.

[13]It is trivial to extend what follows to let P_β include the excited bound states of the fragments.

[14]K. L. Kowalski, Phys. Rev. C 23, 597 (1981).

[15]For the sake of simplicity of notation we have suppressed the β dependence of some of the operators in this paper.

[16]We remark that the passage from $T^{(+)}_{\alpha,\beta}$ and $T^{\rm AGS}_{\alpha,\beta}$ to their properly antisymmetrized counterparts $T^{(+)}(\hat{\beta})$ and $T^{\rm AGS}(\hat{\beta})$, respectively, is far from obvious. Equations (2.19) and (2.20) are established using the techniques of Ref. 9; see also Refs. 5 and 10 in this regard.

[17]The existence of the inverse in (2.30) is open to question. The essential point concerns the occurrence of nontrivial states $|\chi\rangle$ in the null space of $\bar{A}(\beta)[1 + G_\beta(z)V^\beta\hat{A}(\hat{\beta})]$. If $|\chi\rangle$ is Pauli forbidden, $\hat{A}(\hat{\beta})|\chi\rangle = -|\chi\rangle$, e.g., the pertinent eigenvalue equation is $|\chi\rangle = G_\beta V^\beta|\chi\rangle$ and we have a situation very similar to the nonuniqueness of the standard multichannel LS equations. See Ref. 18. Often, however, these suspect formal constructions are of considerable heuristic value in generating perfectly well-defined quantities. See Ref. 19 for a discussion of how this can come about.

[18]L. L. Foldy and W. Tobocman, Phys. Rev. 105, 1099 (1957).

[19]K. L. Kowalski, Ann. Phys. (N.Y.) 120, 328 (1979).

[20]This statement is true only to within the modifications of the Lippmann identity (Refs. 21 and 22) which occur when the state (2.32) is operated upon by *singular* operators which can cancel on-shell zeros.

[21]B. A. Lippmann, Phys. Rev. 102, 264 (1956).

[22]P. Benoist-Gueutal, M. L'Huillier, E. F. Redish, and P. C. Tandy, Phys. Rev. C 17, 1924 (1978).

[23]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958).

[24]R. Goldflam (unpublished).

[25]Antisymmetrized LS-type equations for the (AGS) optical potential operator first appear in Ref. 5.

[26]Our results apparently differ from those obtained by Feshbach in dealing with the Pauli principle (Ref. 27). In Ref. 27 the problem is formulated on the space $\mathcal{H}_\Lambda$ of fully antisymmetrized states. Thus, projectors $\mathcal{P}$ and $Q = \Lambda - \mathcal{P}$ onto subspaces of $\mathcal{H}_\Lambda$ are employed. Here Λ is the projector onto $\mathcal{H}_\Lambda$. The operator $\mathcal{P}$ introduced in Ref. 27 is the counterpart of P_β and is given by $\mathcal{P} = (\mathfrak{a}_\beta)^{-1}\mathcal{A}(\hat{\beta})P_\beta(1+\mathfrak{N})^{-1}P_\beta A(\hat{\beta})$, where $\mathfrak{a}_{\hat{\beta}}$ is the constant such that $\mathcal{A}(\beta) = \mathfrak{a}_{\hat{\beta}}\Lambda$, as shown in Ref. 28. The occurrence of these somewhat more complicated projectors is incidental to the fact that a theory of the antisymmetrized optical potential is not provided by the formalism of Ref. 27. The intimate relationship between the antisymmetrized projection operator formalism of Feshbach (Ref. 27) and the AGS approach to the optical potential is established in Ref. 28.

[27]H. Feshbach, Ann. Phys. (N.Y.) 19, 287 (1962).

[28]K. L. Kowalski and A. Picklesimer, CWRU report, 1981.