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Coulomb effects on charged particle exchange singularities

F. D. Santos*
University of Wisconsin, Madison, Wisconsin 53706
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Coulomb effects on the singularities of charged particle exchange scattering amplitudes are discussed. It is shown that the nature of the exchange singularity depends on the value of η and that the amplitude is finite for $\eta > 1$. Coulomb correction factors which include the effects of Coulomb distortion are given for a suitably defined exchange pole residue in the cross section. The implications of these results on determinations of the asymptotic D- to S-state ratio in the deuteron, ^{3}H, and ^{3}He are discussed.

[NUCLEAR REACTIONS Coulomb corrections in charged particle exchange amplitudes.]

I. INTRODUCTION

Several authors[1-3] have recently reported determinations of the asymptotic D to S state, ρ_D, of the deuteron using analytic continuations in $\cos\theta$ to the neutron exchange pole of tensor analyzing power measurements in d-p elastic scattering. The same method can also be used to determine an analogous ratio of asymptotic normalization constants for D and S states in other systems, as, for instance, in ^{3}H and ^{3}He. In order to improve the accuracy of these determinations of ρ_D it is necessary to give careful consideration to the Coulomb effects outside the physical region. In the case of neutron exchange it has been shown[4] that Coulomb distortion modifies the exchange pole residue by an energy dependent factor that is independent of the orbital angular momentum of the bound states involved in the exchange process. Here we consider the Coulomb effects on single particle exchange singularities in the case where the exchanged particle is charged. The present discussion is also applicable to direct s-channel processes in the scattering of two charged particles.

II. THE EXCHANGE SINGULARITY

The Born exchange amplitude[5] involves the product of the nuclear bound state wave functions ψ_{l_n} in momentum space at each vertex $n=1,2$ of the exchange process

$$F_{l_n}(\vec{q}_n)=\langle \vec{q}_n \mid \psi_{l_n}\rangle = 4\pi i^{-l_n}\int_0^\infty j_{l_n}(q_n r)u_{l_n}(r)r^2dr\, Y_{l_n}^{\lambda_n}(\hat{q}_n),\quad n=1,2\ . \tag{1}$$

In Eq. (1) j_{l_n} is a spherical Bessel function and u_{l_n} is the normalized radial part of ψ_{l_n} in configuration space. Since we are interested in the behavior of the scattering amplitude at the exchange singularity we can neglect the short range part of u_{l_n} and write[5]

$$u_{l_n}=[N_{l_n}/(\alpha_n r)]W_{-\eta_n,l_n+1/2}(2\alpha_n r)\ ,$$
$$n=1,2\ . \tag{2}$$

Here the quantities of interest in nuclear structure studies are the asymptotic normalization constants N_{l_n}. $W_{-\eta_n,l_n+1/2}$ is a Whittaker function, η_n are the Coulomb parameters for the repulsive Coulomb interaction at each vertex, and $\alpha_n=[(2\mu_n B_n)/\hbar^2]^{1/2}$ are the asymptotic wave numbers. B_n is the binding energy of ψ_{l_n} and μ_n are reduced masses. In the special case of neutron exchange $\eta_n=0$ and u_{l_n} in Eq. (2) is proportional to a spherical Hankel function. The position of the exchange singularity in the $\cos\theta$ (θ is the scattering angle) plane[6] is determined by the condition that the energy denominator

$$E_n+B_n=\frac{\hbar^2}{2\mu_n}({q_n}^2+{\alpha_n}^2),\ \ n=1,2 \tag{3}$$

in the propagator is equal to zero. Hence at the exchange singularity $\cos\theta=z_p$

$$\beta_n={q_n}^2+{\alpha_n}^2=0,\ \ n=1,2\ . \tag{4}$$

Because of energy conservation in the exchange process $\beta_1\mu_2=\beta_2\mu_1$. By substituting Eq. (2) into Eq. (1) the result of the radial integration can be expressed[7] as a sum of hypergeometric functions in two variables. Using this analytical form it becomes possible to study the behavior of $F_l(\vec{q})$ at $\cos\theta=z_p$. We can follow an equivalent but simpler approach by noting that because $F_l(\vec{q})$ is a Fourier transform its behavior as $\beta\to 0$ is determined by the asymptotic form at large r of the radial wave function. In the asymptotic region[7]

$$\underset{r\to\infty}{W_{-\eta,l+1/2}(2\alpha r)}=(2\alpha r)^{-\eta}e^{-\alpha r}\ . \tag{5}$$

Hence the substitution of Eqs. (5) and (2) in Eq. (1) leads us to consider the function

$$X_{l\eta}(q)=\int_0^\infty j_l(qr)(2\alpha r)^{-\eta}e^{-\alpha r}r\,dr\ .$$

The result of the radial integration is

$$X_{l\eta}(q)=\frac{\sqrt{\pi}q^l}{2^{l+2}\alpha^{\eta+l+1}}\frac{\Gamma(l+2-\eta)}{\Gamma(l+\frac{3}{2})}\times F\left[\frac{l+3-\eta}{2},\frac{l+2-\eta}{2},l+\tfrac{3}{2};x\right]\ , \tag{6a}$$

where

$$x=-q^2/\alpha^2=1-\beta/\alpha^2\ . \tag{6b}$$

The hypergeometric function in Eq. (6a) has a branch point at $\beta=0$. However, as the argument x approaches the value 1 the function $F[(l+3-\eta)/2,\ (l+2-\eta)/2,\ l+\frac{3}{2};x]$ goes to a finite value for $\eta>1$. Therefore for $\eta>1$, $F_l(\vec{q})$ does not become infinite when $\beta\to 0$. This result has been noted by Andrews *et al.*[8] in the particular case of (d,n) reactions. The modified nature of the exchange singularity is a consequence of the fact that Coulomb repulsion makes the bound state wave function (2) decrease faster than $r^{-1}e^{-\alpha r}$ at large r. The bound particle tends to avoid the region of space where the long range part of the Coulomb potential is acting and there is no nuclear force. For $\eta>1$, $u_l(r)$ decreases faster than $r^{-2}e^{-\alpha r}$ and therefore the integral $X_{l\eta}(q)$, as $\beta\to 0$, converges to the value

$$\underset{q\to i\alpha}{X_{l\eta}(q)}\frac{\sqrt{\pi}i^l}{2^{l+2}\alpha^{\eta+1}}\frac{\Gamma(l+2-\eta)\Gamma(\eta-1)}{\Gamma\left[\frac{l+\eta}{2}\right]\Gamma\left[\frac{l+1+\eta}{2}\right]}\ . \tag{7}$$

For $\eta<1$ the exchange singularity is a branch point, since in the neighborhood of $\beta=0$

$$X_{l\eta}(q)=\left[\frac{q}{\alpha}\right]^l(2\alpha)^{-2\eta}\Gamma(1-\eta)\beta^{\eta-1}\ . \tag{8}$$

Only in the case of neutron transfer the exchange singularity is a simple pole

$$X_{l0}(q)=\left[\frac{q}{\alpha}\right]^l\beta^{-1}\ . \tag{9}$$

In the particular case where $\eta=1$ the Born amplitude has a logarithmic singularity as $\beta\to 0$. As an example of how restrictive is the condition $\eta<1$ we note that a proton state in a $Z=18$ nucleus with a binding energy of 8 MeV has $\eta>1$.

The preceding discussion can also be applied to the singularities in the scattering amplitude of elastic scattering of two charged particles through the direct (or s-channel) bound state ψ_{l_n}. The contribution from this process to the elastic scattering amplitude involves[9] the limit of $\langle\chi^{(-)}(\vec{q}_n)\,|\,\psi_{l_n}\rangle$ as $q_n\to i\alpha_n$, where $\chi^{(-)}$ is a Coulomb scattering wave function. The effect of the Coulomb interaction on ψ_{l_n} in the asymptotic region of large r modifies the behavior of $\langle\chi^{(-)}(\vec{q}_n)\,|\,\psi_{l_n}\rangle$ as $q_n\to i\alpha_n$. It can be shown that this behavior depends on the particular value of η and is described by equations essentially analogous to Eqs. (7)–(9).

III. COULOMB DISTORTION

Photon exchange in the entrance and exit channels of the scattering process give rise to Coulomb distortion effects in the exchange amplitude. To calculate these effects we use the Coulomb wave Born approximation. In this model the T matrix in the "post" representation for a general binary reaction $a+A \to b+B$, where $a=b+x$ and x is the exchanged particle, involves the amplitudes

$$I_{l_2\lambda_2 l_1\lambda_1}=\int d^3R\, d^3r \chi_b^{(-)*}(\eta_b,\vec{k}_b,\vec{r}_b) \times[u_{l_2}(R)Y_{l_2}^{\lambda_2}(\hat{R})]^* V_{bx} u_{l_1}(r) Y_{l_1}^{\lambda_1}(\hat{r}) \times \chi_a^{(+)}(\eta_a,\vec{k}_a,\vec{r}_a)\ . \tag{10}$$

The indexes 1 and 2 correspond to the vertices $a \to b+x$ and $x+A \to B$, respectively. $\chi_a^{(+)}, \chi_b^{(-)}$ are Coulomb wave functions, η_a, η_b are Coulomb parameters, and $\vec{k}_a, \vec{k}_b$ are the asymptotic momenta. In Eq. (10) $\vec{r}$ is the argument of the bound state wave function in the $a \to b+x$ vertex and $\vec{R}$ is the argument of the bound state wave function in the $x+A \to B$ vertex. Hence the relation between $\vec{r}$, $\vec{R}$, $\vec{r}_a$, and $\vec{r}_b$ is

$$\vec{r}_a=\vec{R}+\frac{m_b}{m_a}\vec{r},\quad \vec{r}_b=\frac{m_A}{m_B}\vec{R}+\vec{r}\ ,$$

where m_i is the mass of particle i. It is important to realize that the amplitude (10) has a simple pole at $\cos\theta=z_p$ only when (a) the exchanged particle has no charge; and (b) the scattering wave functions are approximated by plane waves. In this case I is the well known exchange pole Born amplitude[5] [Eq. (A13) of Ref. 5],

$$I_{l_2\lambda_2 l_1\lambda_1}^{\mathrm{PW}}=-\frac{(4\pi\hbar^2)^2}{2\mu_1}\frac{N_{l_1}N_{l_2}}{\beta_2\alpha_1\alpha_2}\left[\frac{iq_1}{\alpha_1}\right]^{l_1}\left[\frac{iq_2}{\alpha_2}\right]^{l_2} \times Y_{l_1}^{\lambda_1}(\hat{q}_1)Y_{l_2}^{\lambda_2^*}(\hat{q}_2)\ . \tag{11}$$

The momenta $\vec{q}_1$ and $\vec{q}_2$ are given by

$$\vec{q}_1=\vec{k}_a'-\vec{k}_b,\quad \vec{k}_a'=\frac{m_b}{m_a}\vec{k}_a\ , \tag{12a}$$

$$\vec{q}_2=\vec{k}_a-\vec{k}_b',\quad \vec{k}_b'=\frac{m_A}{m_B}\vec{k}_b\ , \tag{12b}$$

and β_2 is as previously defined in Eq. (4). The nature of the exchange singularity in the amplitude (10) is modified by Coulomb distortion effects and also, in the case of charged particle exchange, by Coulomb effects in the bound state wave functions. Hence it is convenient to choose as the definition of the exchange pole residue in the cross section (for a polarized or unpolarized incident beam)

$$L=(2\alpha_1)^{4\eta_1}(2\alpha_2)^{4\eta_2}\lim_{\beta_2\to 0}\beta_2^{2(1-\eta_1-\eta_2)} \times I_{l_2\lambda_2 l_1\lambda_1} I^*_{l_2'\lambda_2' l_1'\lambda_1'}\ . \tag{13}$$

The factors $(2\alpha)^{4\eta}$ are introduced to ensure that L has the same dimensions for charged and uncharged exchange. The definition (13) is chosen because the behavior of the scattering observables in the neighborhood of the exchange singularity depends only on these limits. It should be noticed that Eq. (13) applies only to cases where $\eta_1+\eta_2<1$. When $\eta_1+\eta_2>1$ the cross section is not infinite at $\cos\theta=z_p$ and the method of analytical continuation in the $\cos\theta$ plane cannot be used. In the particular case of (d,n) reactions $\eta_1=0$. Hence the cross section is finite at $\cos\theta=z_p$ for $\eta_2>1$ as shown in Ref. 8.

Using the methods developed in Ref. 4 it is proved that the leading term of a power series expansion in β_2 of the Coulomb amplitude (10) is

$$I_{l_2\lambda_2 l_1\lambda_1}=A_c\left[\frac{\beta_1}{4\alpha_1^2}\right]^{\eta_1}\left[\frac{\beta_2}{4\alpha_2^2}\right]^{\eta_2} I_{l_2\lambda_2 l_1\lambda_1}^{\mathrm{PW}}\ . \tag{14}$$

The Coulomb factor A_c in this equation is given by

$$A_c=\Gamma(1-\eta_1)\Gamma(1-\eta_2+i\eta_a+i\eta_b)e^{-(\pi/2)(\eta_a+\eta_b)} \times\left[\frac{g_a}{\beta_2}\right]^{i\eta_a}\left[\frac{g_b}{\beta_2}\right]^{i\eta_b}, \tag{15}$$

where, at $\cos\theta=z_p$,

$$g_a=\alpha_2^2+(k_b'^2-k_a^2)-2i\alpha_2 k_a\ , \tag{16a}$$

$$g_b=\alpha_2^2-(k_b'^2-k_a^2)-2i\alpha_a k_a'\ . \tag{16b}$$

IV. DISCUSSION OF COULOMB CORRECTIONS

The factors that multiply the plane wave amplitude in Eq. (14) are independent of l_1 and l_2. Hence the Coulomb correction to the exchange pole residue ρ in the cross section is independent of whether the incident beam is unpolarized or polarized. Using Eqs. (11), (13), and (14) we obtain

$$\rho=(2\alpha_1)^{4\eta_1}(2\alpha_2)^{4\eta_2}\lim_{\beta_2\to 0}\beta_2^{2(1-\eta_1-\eta_2)}\sigma^{\exp}(\theta)$$

$$=R_c\lim_{\beta_2\to 0}\beta_2^{2}\sigma^{\mathrm{PW}}(\theta)\ . \qquad (17)$$

Here $\sigma^{\exp}(\theta)$ is the cross section for a polarized or unpolarized beam, as determined by experiment, and $\sigma^{\mathrm{PW}}(\theta)$ is the corresponding cross section calculated with the amplitude (11). In particular, $\sigma(\theta)$ in Eq. (17) can be taken as the product $\sigma_0(\theta)T_{kq}(\theta)$, where σ_0 is the unpolarized cross section and $T_{kq}(\theta)$ is an analyzing power. In Eq. (17)

$$R_c=|A_c|^2\ . \qquad (18)$$

It is noted that the relation between ρ of Eq. (17) and L of Eq. (13) is determined by the particular polarized cross section present in Eq. (17). The polarized cross section $\sigma^{\mathrm{PW}}(\theta)$ is a bilinear form in the amplitudes $I^{\mathrm{PW}}_{l_2\lambda_2 l_1\lambda_1}$ defined by Eq. (11).

Using the "prior" representation of the Coulomb Born amplitude Eq. (14) remains valid except for the replacement of A_c by

$$A_c^{\mathrm{prior}}=\Gamma(1-\eta_2)\Gamma(1-\eta_1+i\eta_a+i\eta_b)e^{-(\pi/2)(\eta_a+\eta_b)}\times\left[\frac{f_a}{\beta_1}\right]^{i\eta_a}\left[\frac{f_b}{\beta_1}\right]^{i\eta_b}, \qquad (19)$$

where, at $\cos\theta=z_p$,

$$f_a=\alpha_1^2-(k_a'^2-k_b^2)-2i\alpha_1k_a'\ , \qquad (20a)$$

$$f_b=\alpha_1^2+(k_a'^2-k_b^2)-2i\alpha_1k_b\ . \qquad (20b)$$

The difference between A_c and A_c^{prior} results from the post-prior ambiguity in the distorted wave Born amplitude and is a manifestation of neglected Coulomb stretching or polarizability effects.[4] However, this ambiguity does not manifest itself at the exchange singularity in elastic scattering since in this case $A_c=A_c^{\mathrm{prior}}$ and

$$R_c=|\Gamma(1-\eta')\Gamma(1-\eta'+2i\eta)|^2\times\exp\left[2\eta\left[\arctan\frac{2\alpha k'}{\alpha^2-(k'^2-k^2)}+\arctan\frac{2\alpha k}{\alpha^2+(k'^2-k^2)}-\pi\right]\right].$$

Here $\eta'=\eta_1=\eta_2$, $\eta=\eta_a=\eta_b$, $0\le\arctan x<0$ and the channel identifying subscripts were dropped since they are unnecessary in elastic scattering. R_c is a positive quantity that increases monotonically with energy. In the limit $k\to\infty$, $R_c=\Gamma(1-\eta')^4$. As we approach zero energy the Coulomb repulsion becomes dominant and in the limit $k\to 0$, $R_c=0$.

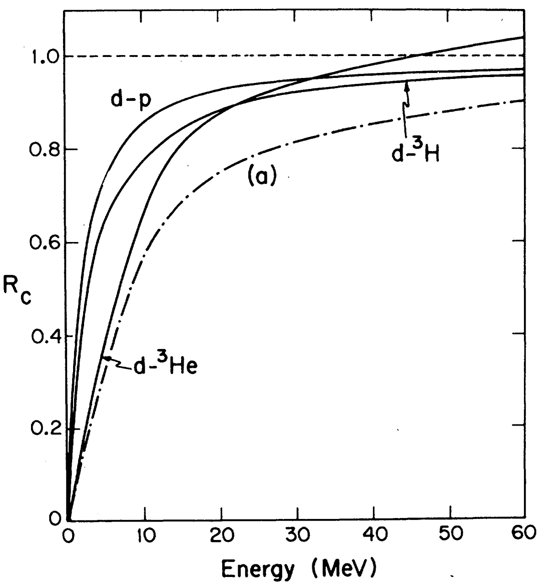

FIG. 1. Coulomb correction factor R_c to the nucleon exchange pole residue in d-p, d-^{3}H, and d-^{3}He elastic scattering as a function of deuteron energy. The dash-point curve (a) represents R_c in d-^{3}He scattering calculated assuming that the exchanged particle has no charge. The broken line is the asymptote of R_c curves with only Coulomb distortion effects.

In the case of d-p elastic scattering the inclusion of Coulomb effects increases the value of ρ_D obtained by Gruebler *et al.*[2] by about 7% to $\rho_D=0.0277$. This value is in good agreement with a recent theoretical estimate[10] of ρ_D based on one-pion exchange dominance at large distances. As shown in Fig. 1 larger Coulomb effects are present in d-^{3}H and d-^{3}He elastic scattering. The higher value of η makes R_c increase slower with energy in the case of d-^{3}He scattering. However, Fig. 1 shows that for $E_d>10$ MeV there is some cancellation between the Coulomb effects in the bound state and scattering wave functions. The value obtained for ρ_D in ^{3}H using the ^{2}H($\vec{d}$,p)^{3}H reaction[3] at 13 MeV is increased in absolute value by about 10% to $\rho_D=-0.053$, by the inclusion of the Coulomb correction factor R_c. The difference between R_c calculated in the past and prior representations is, in this case, smaller than 1%.

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*On leave from Laboratório de Física, Universidade de Lisboa, 1294 Lisboa Codex, Portugal.

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