Confrontations between the interacting boson approximation and the Bohr-Mottelson model

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In the past year several papers were published discussing, from very different points of view, the relation between the interacting boson approximation and the Bohr-Mottelson model. We show that, under certain assumptions about the measure in the Hilbert space, these approaches are equivalent.

[NUCLEAR STRUCTURE Nuclear collective models, interacting boson approximation, Bohr-Mottelson model.]

The objective of the present Communication is to confront several papers[1-4] that appeared in the past year discussing—from very different points of view—the relation between the interacting boson approximation[5] (IBA) and the Bohr-Mottelson (BM) model[6] and its extensions.[7]

We start by discussing the paper by Moshinsky[1] whose title "Confrontation of nuclear collective models" led to the title of the present Communication. In this paper the objective was to show that, for a fixed number N of d-s bosons, to each Hamiltonian in the IBA we could find an equivalent one in the extension of the BM model developed by Greiner and collaborators.[7] This was achieved with the help of the procedure of Dzholos *et al.*[8] in which the s boson was not considered but one had instead appropriate operators in d bosons alone. Klein and Vallieres[4] pointed out that this procedure is in fact related to an old idea of Holstein and Primakoff[9] for the realization of an SU(2) algebra in terms of a creation, annihilation, and certain functions of the number operators.

We proceed now to write down the generators of U(6) in the s-d boson model in terms of the purely d type creation and annihilation operators used in the construction of the full set of states of the BM oscillator model.[10] Our notation will be the following: We denote by $s^\dagger, s$ the creation and annihilation operators corresponding to s bosons and by $d_m^\dagger, d^m$, $m = 2, 1, 0, -1, -2$ the corresponding ones for d bosons having the commutation relation $[d^{m'}, d_m^\dagger] = \delta_m^{m'}$. The creation and annihilation operators for the BM model[10] will be denoted by $\eta_m, \xi^{m'}$ with $[\xi^{m'}, \eta_m] = \delta_m^{m'}$ and $\hat{n} = \sum \eta_m \xi^m$ being the number operator associated with the BM oscillator Hamiltonian.

For a fixed number N of s-d bosons the correspondence to the 36 generators of the U(6) Lie algebra is given by[1,4]

$$s^\dagger s \rightarrow N - \hat{n} \ , \tag{1a}$$

$$d_m^\dagger s \rightarrow \eta_m (N - \hat{n})^{1/2} \ , \tag{1b}$$

$$s^\dagger d^m \rightarrow (N - \hat{n})^{1/2} \xi^m \ , \tag{1c}$$

$$d_m^\dagger d^{m'} \rightarrow \eta_m \xi^{m'} \ , \tag{1d}$$

where left and right hand side operators satisfy the same commutation relations.

Clearly (1) would allow us to write all one and two body scalar operators in the s-d boson picture in terms of corresponding functions of $\eta_m, \xi^{m'}$. As shown in Ref. 1, these functions would be linear combinations of $1, \beta^2, \beta^4, \beta^3 \cos 3\gamma$ with coefficients that depend on $\hat{n}$ and the square of the angular momentum L^2, and thus their matrix elements in a basis in which $\hat{n}, L^2$ are diagonal can be computed straightforwardly.[10]

The main purpose of the present paper is to confront the approach followed above by Moshinsky[1] and, with another emphasis, by Klein and Vallieres,[4] with that by Ginocchio and Kirson[2] and, from another standpoint, by Dieperink *et al.*[3]

To achieve our objective we note that another correspondence of operators that allows a realization of a U(6) Lie algebra is given by

$$s^\dagger s \rightarrow N - \hat{n} \ , \tag{2a}$$

$$d_m^\dagger s \rightarrow \eta_m \ , \tag{2b}$$

$$s^\dagger d^m \rightarrow (N - \hat{n}) \xi^m \ , \tag{2c}$$

$$d_m^\dagger d^{m'} \rightarrow \eta_m \xi^{m'} \ , \tag{2d}$$

as the right hand side (rhs) of (2) satisfies the same commutation relations as the rhs of (1). Furthermore with respect to eigenstates $|\nu\tau\rangle$ of $\hat{n}$, where ν is the eigenvalue of this operator and τ represents the other quantum numbers, we have

$$\langle\nu'\tau'|\eta_m(N-\hat{n})^{1/2}|\nu\tau\rangle=\langle\nu'\tau'|\eta_m|\nu\tau\rangle(N-\nu)^{1/2} \quad , \tag{3a}$$

with $\nu'=\nu+1$,

$$\langle\nu'\tau'|(N-\hat{n})^{1/2}\xi^m|\nu\tau\rangle =(N-\nu+1)^{-1/2}\langle\nu'\tau'|(N-\hat{n})\xi^m|\nu\tau\rangle \quad , \tag{3b}$$

with $\nu'=\nu-1$. Thus the matrix M_2 of any polynomial function of the rhs of (2) is related with the matrix M_1 of the *same* polynomial function of the rhs of (1) through the equation

$$M_2=D^{-1}M_1D \quad , \tag{4}$$

where D is the diagonal matrix

$$D=||[(N-\nu)!]^{-1/2}\delta_{\nu'\nu}\delta_{\tau'\tau}|| \quad , \tag{5}$$

$$\nu,\nu'=0,1\cdots N \quad .$$

Therefore the translation of the generators of U(6) in the *s-d* boson model into those on the rhs of (1) or (2) will give, for a definite IBA Hamiltonian, different functions $H_1(\eta_m,\xi^{m'})$ or $H_2(\eta_m,\xi^{m'})$ that have the same eigenvalues.

We now turn our attention to the procedure followed in Refs. 2 and 3 for finding the operators corresponding to the generators of U(6) given on the left hand side (lhs) of (1) or (2). The starting point is the introduction of a normalized intrinsic state[2,3] for N bosons outside the doubly closed shell; the latter is denoted by $|0\rangle$. We prefer to work in the frame of reference fixed in space rather than the one fixed in the body, and thus the intrinsic state becomes[2,3]

$$|N,\alpha^m\rangle=[N!(1+|\beta|^2)^N]^{-1/2}\left(s^\dagger+\sum_m\alpha^m d_m^\dagger\right)^N|0\rangle \quad . \tag{6}$$

We then obtain the correspondence

$$s^\dagger s\rightarrow(1+|\beta|^2)^{-N/2}[N-\beta\partial/\partial\beta](1+|\beta|^2)^{N/2} \quad , \tag{7a}$$

$$d_m^\dagger s\rightarrow(1+|\beta|^2)^{-N/2}[\partial/\partial\alpha^m](1+|\beta|^2)^{N/2} \quad , \tag{7b}$$

$$s^\dagger d^m\rightarrow(1+|\beta|^2)^{-N/2}[\alpha^m(N-\beta\partial/\partial\beta)](1+|\beta|^2)^{N/2} \quad , \tag{7c}$$

$$d_m^\dagger d^{m'}\rightarrow(1+|\beta|^2)^{-N/2}[\alpha^{m'}\partial/\partial\alpha^m](1+|\beta|^2)^{N/2} \quad , \tag{7d}$$

where

$$|\beta|^2=\sum_m\alpha^m(\alpha^m)^* \quad , \tag{8a}$$

$$\beta\partial/\partial\beta=\sum_m\alpha^m\partial/\partial\alpha^m \quad . \tag{8b}$$

The proof is immediate as when we apply the lhs and rhs of (7) to (6) we get the same results, taking into account that from the commutation relation we can interpret $s=\partial/\partial s^\dagger$, $d^m=\partial/\partial d_m^\dagger$.

We can easily check that the commutators of the expressions on the rhs of (7) give the same result as those on the lhs except for an overall minus sign. This is to be expected as when we operate on (6) with products of operators on the lhs of (7), we get the same result if we apply the corresponding ones on the rhs in the opposite order.

If we have now a definite IBA Hamiltonian we can write a corresponding expression for it in terms of the operators appearing in the square brackets of (7) with factors $(1+|\beta|^2)^{-N/2}$ and $(1+|\beta|^2)^{N/2}$ on the left and right of the final expression. These factors can be eliminated if we assume that the resulting expression instead of acting on a function ψ of the α^m, does it on ϕ related to it through

$$\psi=[1+|\beta|^2]^{-N/2}\phi \quad . \tag{9}$$

We get then for each IBA Hamiltonian a corresponding operator $H(\alpha^m,\partial/\partial\alpha^{m'})$ constructed from sums of products of operators appearing in the square brackets of (7). In the particular case of two body interactions associated with the O(6) and SU(3) subgroups of U(6) we get[2] Hamiltonians with many resemblances to those of the BM model[6] and its extensions.[7]

While the procedure outlined in the previous paragraph[2,3] is very elegant, we must remember that in quantum mechanics we not only wish to have operators such as $H(\alpha^m,\partial/\partial\alpha^{m'})$ and their eigenfunctions, but we must have also a way for defining a scalar product, i.e., a measure in the Hilbert space characterized by the coordinates α^m. Comparing the expressions on the rhs of (2), where because of the commutation relations we can replace ξ^m by $\partial/\partial\eta_m$, with those in the square brackets in the rhs of (7), we immediately have the impression that the role played by η_m, ξ^m in the former is played by α^m, $\partial/\partial\alpha^m$ in the latter. This suggests that the α^m must be complex variables[11,12] and that the scalar product should be defined in what is known as Bargmann Hilbert space (BHS).[11,12]

We briefly discuss the essential features of BHS in one dimension as the generalization to the five dimensional case of the α^m, $m=2,1,0,-1,-2$ will be trivial. Associated with the creation and annihilation operators η, ξ in ordinary Hilbert space we have in BHS $z,d/dz$ where $z=x+iy$ is a complex variable. The scalar product of two states $g(z),f(z)$ which are analytic functions of z, is defined by integration over the full complex plane with a measure $d\mu(z)$, i.e.,[12]

$$(g,f)=\int g^*(z)f(z)\,d\mu(z) \quad ; \tag{10a}$$

$$d\mu(z)=\pi^{-1}\exp(-zz^*)\,dx\,dy \quad . \tag{10b}$$

Because of the property that the derivative of any function of $z^* = x - iy$ with respect to z is zero, we can easily prove[11,12] that

$$(zg, f) = (g, df/dz) \ , \tag{11a}$$

$$(dg/dz, f) = (g, zf) \ . \tag{11b}$$

Finally oscillator functions of ν quanta in ordinary Hilbert space transform into the monomials $z^\nu(\nu!)^{-1/2}$ which, from the definition (10) of the scalar product, are orthonormal.[12]

We now return to our problem. Instead of the five variables α^m we prefer to use α_m^* with a lower index defined by $\alpha_m = (\alpha^m)^*$ as this will establish a complete parallelism between the procedure followed on the rhs of (2) in ordinary Hilbert space and that on the rhs of (7) in BHS. We then define a measure associated with α_m by

$$d\mu(\alpha_m) = \pi^{-5}\exp(-|\beta|^2) \prod_{m=-2}^{2} d\,\mathrm{Re}\alpha_m \, d\,\mathrm{Im}\alpha_m \ . \tag{12}$$

If we have a wave function $\psi(\alpha_m)$ in our BHS we can define a corresponding state in the IBA, which we designate by the capital Ψ, through the relation

$$\Psi = \int |N\alpha_m^*\rangle \psi(\alpha_m)(1+|\beta|^2)^N d\mu(\alpha_m) \ . \tag{13}$$

The appearance of the extra factor $(1+|\beta|^2)^N$ is due to the fact that both the intrinsic state (6) and the wave function of (9) have factors $(1+|\beta|^2)^{-N/2}$ that are canceled by it. We are then left with $(s^\dagger + \sum \alpha_m^* d_m^\dagger)^N|0\rangle$ and $\phi(\alpha_m)$ which are analytic functions, respectively, of α_m^* and α_m and thus with respect to the measure $d\mu(\alpha_m)$ we have a bonafide scalar product.

We wish now to apply the generators of U(6) on the lhs of (7) to the IBA states Ψ of (13), to see how they are reflected in their operation on $\phi(\alpha_m)$. Taking, for example, $d_m^\dagger d^{m'}$ of (7d), we obtain

$$d_m^\dagger d^{m'}\Psi = \int [(\alpha_{m'}^* \partial/\partial\alpha_m^*)(1+|\beta|^2)^{N/2}|N\alpha_m^*\rangle]\phi(\alpha_m)\, d\mu(\alpha_m) = \int [(1+|\beta|^2)^{N/2}|N\alpha_m^*\rangle][\alpha_m(\partial\phi/\partial\alpha_{m'})]\, d\mu(\alpha_m) \ , \tag{14}$$

where we made use of the rhs of (7d) and of (11). Thus to $d_m^\dagger d^{m'}$ in IBA corresponds $\alpha_m \partial/\partial\alpha_{m'}$ in the BHS. In a similar fashion we obtain the set of correspondences

$$s^\dagger s \to (N - \beta\partial/\partial\beta) \ , \tag{15a}$$

$$d_m^\dagger s \to \alpha_m \ , \tag{15b}$$

$$s^\dagger d^m \to (N - \beta\partial/\partial\beta)\partial/\partial\alpha_m \ , \tag{15c}$$

$$d_m^\dagger d^{m'} \to \alpha_m \partial/\partial\alpha_{m'} \ , \tag{15d}$$

(where now $\beta\partial/\partial\beta = \sum \alpha_m \partial/\partial\alpha_m$) which are clearly the equivalent of (2) in a BHS whose measure is (12).

We have established, by going to BHS, that the confrontation between the IBA and BM model developed by Ginocchio and Kirson,[2] is equivalent to the one obtained by Moshinsky[1] and, from another angle, by Klein and Vallieres.[4] We hasten to add though that Ginocchio and Kirson[2] *do not* work in BHS. In fact for them the α_m's are not general complex variables like we assumed here, but have the relations $\alpha_m^* = (-1)^m\alpha_{-m}$. Thus when defining [in Eq. (4) of the first paper in Ref. 2] a state equivalent to (13), they do not integrate over the Bargmann measure (12) but rather over the standard volume element of the BM model. This, however, causes them problems for the eigenstates associated with the U(5) ⊃O(5) ⊃O(3) chain of groups, which are homogeneous polynomials of degree ν in the α_m and would certainly not be orthonormal over the standard volume element of the BM model, while they will have this property with the Bargmann measure (12), as we indicated above for the one dimensional case.

The use of complex α_m's, which has also been proposed by Gilmore *et al.*,[13] allows the introduction of a well defined scalar product. However, a very serious drawback is the lack of a straightforward relation to the *real* β, γ, and Euler angles required in the BM model. It is thus of great interest to be able to define a scalar product satisfying the restriction $\alpha_m^* = (-1)^m\alpha_{-m}$ in the "intrinsic state" approach[2,3] relating the IBA with the BM model. This is still an open problem which requires the full attention of those interested in establishing such a connection.

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