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Quasi-potential approach to scattering theory

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The quasi-potential procedure of Sasakawa and Austern is generalized to obtain an improved convergence of iteration series. The modified form of the quasi-potential method is compared with the earlier approaches and its effectiveness tested. Extensions to treat the general coupled-channel integro-differential equations are outlined.

[NUCLEAR REACTIONS Quasi-potential method, coupled integro-differential equations.]

I. INTRODUCTION

The quasi-potential (QP) method, also called the quasi-particle method, was proposed sometime ago by Weinberg[1] and by Sasakawa[2] to facilitate the iterative solution of a complicated scattering equation, often coupled and involving integral operators. An extension of the Sasakawa approach was then given by Austern,[3] who employed a specific choice of certain functions which appear in Sasakawa's QP method. A detailed comparison of the various formulations and numerical tests was carried out by Soper[4(a)] who noted that the Sasakawa-Austern (SA) procedure does not converge for interactions which are too strong—those resulting in phase shifts larger than π.

For many physical applications, this is too stringent a limitation[4(b)] and we propose a trivial extension of the SA procedure to improve its applicability. This extension is seen to be consistent with the general procedure outlined by Weinberg.[1]

Of special interest is the pion-nucleus scattering at medium energies around 200 MeV laboratory energy. When the (33) resonance on the πN scattering is parametrized by a separable, energy-dependent interaction, the resulting πA interaction becomes a complicated nonlocal, nonseparable interaction,[5] which has to be treated by an iterative procedure, and which may not often converge. Special procedures such as developed here and in Refs. 1–3 are needed to examine the energy dependence of the scattering cross section across the resonance energy.

II. FORMALISM

We consider the QP procedure by first reviewing briefly the Sasakawa-Austern method. For a general scattering equation

$$Du = Vu \equiv \int V(\vec{r}, \vec{r}\,')u(\vec{r}\,')d\vec{r}\,' , \tag{2.1}$$

where $D \equiv (E - H_0)$ and V can generally be a nonlocal integral interaction potential, which is to be treated by iterations. Of course H_0 may contain an additional local distortion potential U, which we assume can be treated exactly. Denoting the homogeneous solution of (2.1) by

$$D|u_0\rangle = 0, \quad DG_0 = +1 , \tag{2.2}$$

with the outgoing wave boundary conditions for G_0, we have

$$|u\rangle = |u_0\rangle + G_0 V |u\rangle . \tag{2.3}$$

(a) The Sasakawa method is to rewrite (2.3) into two parts as

$$|u\rangle = |u_0\rangle + |w\rangle t + K_0 V |u\rangle , \tag{2.4}$$

where

$$t = -\langle u_0 | V | u \rangle , \tag{2.4a}$$

$$K_0 = G_0 + |w\rangle\langle u_0| , \tag{2.4b}$$

and where ω is arbitary, except that it satisfies the boundary condition at $r = \infty$ of having purely outgoing waves. Then

$$u = |X\rangle + |Y\rangle t , \tag{2.5}$$

with

$$|X\rangle = \frac{1}{1 - K_0 V} |u_0\rangle, \quad |Y\rangle = \frac{1}{1 - K_0 V} |w\rangle ,$$

and where

$$t = -\frac{\langle u_0|V|X\rangle}{1+\langle u_0|V|Y\rangle} \ . \tag{2.6}$$

Therefore, the iteration series involves the expansion $(1-K_0V)^{-1}$, which gives in turn X and Y and thus t. The convergence of the iteration series in the case of the Sasakawa approach has been proved by Coester[6] for a local potential, but the situation is not clear in the case of nonlocal interaction.[4(b)] In any case, for a judicious choice of ω, the corresponding K_0V should be small enough to have the series converge.

(b) Austern[3] makes a specific choice of ω of the following form for the nth iteration:

$$|\tilde{\omega}_n\rangle = G_0V|\tilde{u}_{n-1}\rangle/\tilde{t}_{n-1} \tag{2.7}$$

with

$$\tilde{t}_n = -\frac{\langle u_0|V|u_0\rangle}{1+\langle u_0|V|\tilde{w}_n\rangle} \tag{2.8}$$

and

$$\tilde{t}_0 = -\langle u_0|V|u_0\rangle, \quad \tilde{u}_0 \equiv u_0 \ .$$

Equations (2.7) and (2.8) then give

$$|\tilde{u}_n\rangle = |u_0\rangle + |\tilde{\omega}_n\rangle\tilde{t}_n \ . \tag{2.9}$$

Note that the choice $|\omega\rangle = G_0V|u\rangle t^{-1}$ makes $K_0V|u\rangle = 0$ in (2.4)

(c) On the other hand, the Weinberg procedure is to choose a quasi-potential of the form

$$V \to V_s \equiv |\Gamma\rangle\langle\bar{\Gamma}| \ , \tag{2.10}$$

where $\Gamma \to V\varphi_n \equiv \Gamma_n$ and where $\{\varphi_n\}$ are, for example, solutions of the strength eigenvalue problem defined by[1,7]

$$(G_0V)\varphi_n = \lambda_n\varphi_n \tag{2.11}$$

which are the usual Sturmain functions. The potential of the form (2.10) is introduced to elimate those eigenstates φ_n with $|\lambda_n|$ larger than one in order to make the Born series converge. When more than one such state is involved, the form (2.10) should be modified into a sum of similar terms. The iteration is then directed for the residual interaction $W_s \equiv V - V_s$ as

$$(D - V_s)|u\rangle = W|u\rangle \ , \tag{2.12}$$

where the left hand side can of course be solved exactly. Here we choose a different approach which leads to a set of nonlinear equations which can be solved by iteration and which can be shown to correspond to a generalization of the SA approach.

(d) The quasi-potential procedure outlined below is based essentially on some of the features of (2.6), (2.7), and (2.12). That is, consider a particular form of a QP given by

$$V_s = V|u\rangle\frac{1}{\langle\varphi|V|u\rangle}\langle\varphi|V \tag{2.13}$$

with the W of (2.12) being zero in this case. When this form is inserted in place of V in Eq. (2.1), the original equation is recovered. In (2.13), we have the exact solution $|u\rangle$ so that V_s of (2.13) is not possible to construct *a priori*. However, for an approximate u in V_s, the resulting equation for u can be solved exactly. The function φ in (2.13) is so far completely arbitrary. (That is, $V|\varphi\rangle$ is arbitrary.) The iteraction procedure then follows from replacing u by its successive approximations. Thus, for the nth iteration, we have

$$|\chi_n\rangle = G_0V|u_{n-1}\rangle \ , \tag{2.14a}$$

$$y_n = \frac{\langle\varphi|V|u_0\rangle}{\langle\varphi|V|u_{n-1}\rangle - \langle\varphi|V|\chi_n\rangle}, \tag{2.14b}$$

and

$$|u_n\rangle = u_0 + |\chi_n\rangle y_n \ , \tag{2.15a}$$

$$t_n = -\langle u_0|V|u_{n-1}\rangle y_n \ . \tag{2.15b}$$

The form (2.13) in (2.1) provides the dynamics which is equivalent to a generalization of the Sasakawa-Austern theory. The Appendix contains the explicit proof of this equivalence for a specific choice of φ, $\varphi = u_0$. Here we write for u.

$$u = u_0 + \omega C + KVu \ , \tag{2.16}$$

where

$$K = G_0 + |\omega\rangle\langle\varphi|, \quad C = -\langle\varphi|V|u\rangle \ , \tag{2.17}$$

and where *both* ω and φ are arbitrary, except for the specified boundary conditions. In this general form (2.16), $\varphi = u_0$ gives the Sasakawa expression (2.4), while $\varphi = u_0$ and $\omega = G_0Vu/t$ result in the Austern procedure. What we have shown here is that both forms are probably too restructive. Obviously, the choice $\omega = G_0Vu/C$ gives back the result with (2.13); note that this choice of ω makes the last term in (2.16) vanish, i.e., $KVu = 0$ for an arbitrary φ.

We now consider the various special cases which depend on the choice of φ:

(i) $\varphi = u_0$(or u_0^*). This gives

$$V_s^n = V|u_n\rangle\frac{1}{\langle u_0|V|u_n\rangle}\langle u_0|V \quad (n = 0, 1, 2, \ldots), \tag{2.18}$$

which reduces immediately to the Austern procedure of (2.7), (2.8), and (2.9). The proof is

straightforward.

(ii) $\varphi = u_n$(or u_n^*). This choice is attractive from a theoretical point of view because it makes the connection between the form (2.13) and Weinberg's choice (2.10) more explicit, with

$$|\Gamma_n\rangle \rightarrow V|u_n\rangle/\langle u_n|V|u_n\rangle^{1/2} .$$

On the other hand, when u_n changes appreciably during the iteration, the procedure itself may become very unstable. This has been borne out by extensive numerical studies.

(iii) Now consider the choice

$$\varphi = u_0 f . \tag{2.19}$$

This is the improvement we propose of the SA approach, where f is always taken to be $f=1$, while, in the Weinberg's approach, V_s is taken as a slightly different form, i.e., $\varphi = u_n$ of (ii).

III. EXTENSIONS OF THE QUASI-POTENTIAL METHOD

We consider several extensions of the QP procedure of (2.19) in (2.13). Since a simple one-component equation with a local interaction can be solved trivially numerically, the QP method is useful when either V is a nonlocal, integral operator, or Eq. (2.1) represents a set of complicated coupled equations of integro-differential form.

A. Distortion potential

Consider (2.1) in which one has some prior knowledge about the general features of the solution u. Then we can add a local pseudopotential U to both sides of (2.1) so that

$$(D-U)u = (V-U)u \equiv Wu . \tag{3.1}$$

The left hand side can of course be handled exactly, while W should be small enough to be treated by a form similar to (2.13), i.e., $W_s = W|u\rangle\langle\varphi|W|u\rangle^{-1}\langle\varphi W$. Construction of a suitable U is more difficult, but several useful procedures[8] are available. A simple guess based on physical argument often suffices, however.

B. Variational procedure for φ_n and φ

The discussion of V_s in Sec. II assumed that a suitable form of φ can always be found. A more systematic way is to generate a small set of φ_n defined by (2.11), but this may be as complicated as the original scattering problem. A more practical procedure is to simultaneously diagonalize[9] a matrix $\langle\varphi_n{}^t|G_0V|\varphi_m{}^t\rangle$ and the normalization matrix $\langle\varphi_n{}^t|\varphi_m{}^t\rangle$ (or $\langle\varphi_n{}^t|VG_0V|\varphi_m{}^t\rangle$ and $\langle\varphi_n{}^t|V|\varphi_m{}^t\rangle$). This will give a set of pseudo-Sturmian functions $\tilde{\varphi}_n$ and $\tilde{\lambda}_n$, which represent variationally the entire spectrum of G_0V, and the resulting V_s will be a sum of several terms of the form (2.13).

C. Coupled equations

In the case of coupled equations, V in (2.1) necessarily represents the coupling matrix potential, with or without the diagonal elements. On the other hand, U in (3.1) will introduce additional diagonal terms in W. In any case, we can still construct, either with W or V,

$$V_s^n = W|\vec{u}_n^T\rangle \frac{1}{\langle\vec{\varphi}|W^T|\vec{u}_n^T\rangle}\langle\vec{\varphi}|W , \tag{3.2}$$

where $\vec{u}_n$ is a column vector of the nth iterated solution and $\langle\vec{\varphi}|$ is a column vector and its transpose $\langle\vec{\varphi}^T|$. The rearrangement and inelastic collisions can be treated along the line described by Austern[3].

D. Iterations on $z = u - u_0$

From (2.1) and (2.3) we have

$$Dz = Vu_0 + Vz , \tag{3.3}$$

where

$$z \equiv u - u_0 = G_0Vu .$$

Since the Vz term is to be iterated on, we may set

$$V_s = V|z\rangle \frac{1}{\langle\varphi|V|z\rangle}\langle\varphi|V , \tag{3.4}$$

which is to be inserted into V of the Vz term in (3.3), not in Vu_0. For the nth iteration, we then obtain

$$|z_n\rangle = |z_0\rangle + G_0V|z_{n-1}\rangle h_{n-1} ,$$

where

$$h_n = \frac{\langle\varphi|V|z_0\rangle}{\langle\varphi|V|z_n\rangle - \langle\varphi|VG_0V|z_n\rangle}$$

and

$$|u_n\rangle = |u_0\rangle + |z_n\rangle ,$$

$$t_n = t_B + t_0 + \langle u_0|VG_0V|z_{n-1}\rangle h_{n-1} ,$$

with

$$t_B = -\langle u_0|V|u_0\rangle ,$$

$$t_0 = -\langle u_0|V|z_0\rangle, \quad z_0 = G_0Vu_0 .$$

IV. NUMERICAL EXAMPLES

For ready comparison of the various procedures discussed above, we consider the same example studied by Soper.[4] Take the operator D to be the $l=0$ free Hamilton and the energy $E = 15$ MeV nucleon on an oxygen target. V is chosen here to be a real Woods-Saxon potential,

$$V = -V_0(1+e^{(r-R_0)/d})^{-1} ,$$

$$R_0 = r_0A^{1/3}, \quad A = 16, \quad r_0 = 1.2 \text{ fm},$$

$d = 0.5$ fm.

The potential strength V_0 is increased slowly to study the convergence property of each iteration series.

As shown by Soper,[4] The Austern series converges remarkably well in the case of $V_0 = 10$ MeV, as compared with the usual Born series. Instead of more than 15 iterations of the Born type, the Austern method converged in approximately three iterations.

The case with $V_0 = 50$ MeV was more questionable, and the Austern series starts to diverge for $V_0 \geq 55$ MeV. In Table I, we show the phase shift δ_n at nth iterations and for $V_0 = 16.6$ MeV. Apparently, the form (2.13) with $\varphi = u_0$ or its variations with $f(r)$ all seem to produce the correct phase shift $\delta = 1.141$. However, $f(r) = e^{-ar}$ with $a = 0.2$, 0.4, and 0.8 all improved the convergence. We especially note the overshot of δ_n at $n = 2 \sim 3$ in the case of $a = 0.4$ and 0.8, which seems to guarantee that the cases $a = 0.0$ and 0.2 had converged to the correct phase.

Table II contains the comparison between the different choices of the function f in $\varphi = u_0 f$ for $V_0 = 72.63$ MeV. Evidently, the Austern series (with $f = 1$) and $f = \exp(-ar)$ with $a = 0.1$ do not converge, while $a = 0.3$ may eventually converge to the correct phase shift $\delta = 3.82$ with more iterations. Numerical study of the model shows that the $f = 1$ series always converges to $\delta' = \pi$ when the actual $\delta > \pi$. On the other hand, for $a \geq 0.5$, the iteration series converged after $6 \sim 10$ iterations. More important for practical applications is the behavior of δ_n when $a \geq 0.75$; in this model they *fluctuate* about the correct value by first shooting *over* the mark. This feature is very important, since the nonconvergence of the series may often be difficult to discern, as is shown by the $a = 0$, 0.1 (and 0.3) cases. For practical situations, therefore, it is important to find at least two sets of $f(r)$ for which the δ_n converges to the same δ *nonmonotonically*. This is a practical way of determining whether the converged phase shift with a particular choice of φ is in fact the correct one and not spurious.

Table III contains the values of y_n and the amplitude t_n for the case $V_0 = 72.63$ MeV and $a = 0.75$ fm^{-1}. As the iteration series converges, $y_n \to 1$ and is real. The phase shift is very large and u_n has an extra node developed compared with the original u_0. Applications of the above procedure to $\pi\alpha$ scattering in the resonance region are in progress.

TABLE I. Real part of the phase shifts δ_n for the nth iterations are compared for $V_0 = 16.60$ MeV in the real Woods-Saxon potential and $E = 15$ MeV in a model proton-oxygen scattering. The function φ is given by $\varphi = u_0 f$, with $f = \exp(-an)$ for various values of a. The exact phase shift $\delta = 1.141$ rad and is real. The last column with $\varphi = u_0$ gives δ_n, which seems to oscillate around the correct value.

n \ a	0 (Austern)	0.2	0.4	0.8	u_n
1	0.831	0.850	0.870	0.908	0.841
2	1.098	1.126	1.153	1.204	1.115
3	1.123	1.133	1.140	1.145	1.136
4	1.136	1.139	*1.141*	*1.141*	1.147
5	1.139	*1.141*	1.141	1.141	1.151
6	1.140	1.141			1.154
7	1.141	1.141			1.156
8	1.141				

TABLE II. Real part of the phase shifts δ_n for the nth iterations are compared for $V_0 = 72.63$ MeV in the real Woods-Saxon potential and $E = 15$ MeV. Note the false convergence at $a = 0.0$ and 0.1 which implies nonconvergence. The exact phase shift is $\delta = 3.82$ rad. The imaginary part of δ_n remains small throughout the iteration.

n \ a	0	0.1	0.3	0.5	0.75	1.00	1.20
1	1.19	1.20	1.23	1.26	1.30	1.33	1.36
2	2.50	2.58	2.74	2.89	3.05	3.18	3.27
3	2.67	2.79	3.04	3.26	3.55	3.86	4.17
4	2.85	2.97	3.22	3.45	3.73	4.01	4.25
5	2.94	3.08	3.35	3.60	3.84	3.96	3.89
6	3.00	3.15	3.44	3.68	3.87	3.86	3.79
7	3.04	3.20	3.51	3.74	3.86	3.83	3.81
8	3.07	3.24	3.56	3.78	3.84	*3.82*	3.83
9	3.09	3.27	3.60	3.80	3.83		3.82
10	3.10	3.29	3.63	3.81	3.83		*3.82*
11	3.11	3.31	3.66	3.82	*3.82*		
12	3.12	3.32	3.68	*3.82*			
13	3.13	3.33	3.70				
14	3.13	3.33	3.72				

TABLE III. The coefficient y_n and the amplitude t_n are given for the case $V_0 = 72.63$ MeV and $a = 0.75$ in $\varphi = u_0 e^{-ar}$. If the iteration converges, $y_n \to 1$ and this provides a good indication of the quality of convergence.

n	ReY_n	ImY_n	Ret_n	Imt_n
1	0.039	0.140	0.258	0.928
2	−0.103	0.571	−0.091	0.008
3	1.317	0.715	0.361	0.157
4	1.148	0.243	0.472	0.311
5	1.179	0.086	0.520	0.415
6	1.074	0.027	0.508	0.439
7	1.046	−0.008	0.505	0.432
8	1.015	−0.013	0.501	0.416
9	1.005	−0.011	0.500	0.404
10	1.000	−0.005	0.499	0.400
11	1.000	−0.001	0.500	0.398

V. DISCUSSION

The improvement of the Sasakawa-Austern procedure considered here should be useful for many physical applications in which complicated coupled equations and nonlocal, integral operators are involved. Application of this method to pion-nucleus scattering near the (3.3) resonance will be reported later.

We have shown in Sec. IV that a simple choice for the function f in φ can improve the convergence of the iteration series (2.14)–(2.15) drastically. We have not been able to formulate a systematic way to construct the optimum form for φ. Preliminary study indicates,[10] however, that a specific form for φ in (2.16) leads to the Padé approximants.

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APPENDIX

We explicitly show the equivalence between the two iteration procedures as formulated by (2.7)–(2.9) and (2.13)–(2.15) for the special choice of $\varphi = u_0$ and $\omega = G_0 V u / t$.

For the SA procedure given by (2.7)–(2.9), we have for

$n=0$:

$$|\tilde{u}_0\rangle = |u_0\rangle \text{ and } \tilde{t}_0 = -\langle u_0|V|u_0\rangle;$$

$n=1$:

$$|\tilde{\omega}_1\rangle = G_0 V |u_0\rangle / \tilde{t}_0 ,$$
$$\tilde{t}_1 = -\langle u_0|V|u_0\rangle / (1 + \langle u_0|V|\omega_1\rangle) ,$$
$$|\tilde{u}_1\rangle = |u_0\rangle + |\tilde{\omega}_1\rangle \tilde{t}_1 ;$$

$n=2$:

$$|\tilde{\omega}_2\rangle = G_0 V |\tilde{u}_1\rangle / \tilde{t}_1 ,$$
$$\tilde{t}_2 = -\langle u_0|V|u_0\rangle / (1 + \langle u_0|V|\tilde{\omega}_2\rangle) ,$$
$$|\tilde{u}_2\rangle = |u_0\rangle + |\tilde{\omega}_2\rangle \tilde{t}_2 , \text{ etc.}$$

On the other hand, the QP procedure given by (2.14)–(2.15) generates for

$n=0$:

$$|u_0\rangle = |u_0\rangle ;$$

$n=1$:

$$|\chi_1\rangle = G_0 V |u_0\rangle ,$$
$$y_1 = \langle u_0|V|u_0\rangle / (\langle u_0|V|u_0\rangle - \langle u_0|V|\chi_1\rangle) ,$$
$$t_1 = -\langle u_0|V|u_0\rangle y_1, \quad |u_1\rangle = |u_0\rangle + |\chi_1\rangle y_1 ;$$

$n=2$:

$$|\chi_2\rangle = G_0 V |u_1\rangle ,$$
$$y_2 = \langle u_0\ V|u_0\rangle / (\langle u_0|V|u_1\rangle - \langle u_0|V|\chi_2\rangle) ,$$
$$t_2 = -\langle u_0|V|u_1\rangle y_2, \quad |u_2\rangle = |u_2\rangle = u_0\rangle + |\chi_2\rangle y_2 , \text{ etc.}$$

For convenience, we define the following quantities:

$$\alpha = \langle u_0|V|u_0\rangle , \quad \beta = \langle u_0|u_0|VG_0V|u_0\rangle ,$$
$$\gamma = \langle u_0|VG_0VG_0V|u_0\rangle , \text{ etc.}$$

Then it is simple to show first that for $n=1$

$$|\tilde{\omega}_1\rangle = -|\chi_1\rangle/\alpha ,$$

$$|\tilde{u}_1\rangle = |u_1\rangle, \quad \tilde{t}_1 = t_1 .$$

Next, for $n=2$, we have

$$|\tilde{\omega}_2\rangle = |\chi_2\rangle \Big/ \left(-\frac{\alpha^2}{\alpha-\beta}\right)$$

and thus

$$\tilde{t}_2 = -\frac{\alpha^3}{\alpha^2 - \alpha\beta - \alpha\gamma + \beta^2} ,$$

$$y_2 = -\frac{\alpha(\alpha-\beta)}{(\alpha-\beta)^2 + \alpha(\beta-\gamma)} ,$$

so that

$$t_2 = -\left(\alpha + \frac{\alpha\beta}{\alpha-\beta}\right) y_2 = \tilde{t}_2$$

and

$$|\tilde{u}_2\rangle = |u_0\rangle + |\tilde{\omega}_2\rangle \tilde{t}_2 = |u_0\rangle + |\chi_2\rangle y_2 = |u_2\rangle ,$$

and so on for the higher order iterations. Therefore, (2.13) with the choice $\varphi = u_0$ and $\omega = G_0 V u / t$ generates an iteration series which is identical to the SA iteration series defined by (2.7)–(2.9). Our numerical study of course is in agreement with this.

For the more general case in which $\varphi \neq u_0$ but still $\omega = G_0 V u / C$ and $C = -\langle \varphi | V | u \rangle$, we again have $K = 0$. However, $u = u_0 + \omega C$ of (2.16) generates the iteration series which is different from that obtained by V_s of (2.13).

[1]S. Weinberg, Phys. Rev. 130, 776 (1963); 131, 440 (1963); 133, B232 (1964).
[2]T. Sasakawa, Prog. Theor. Phys. (Kyoto) Suppl. 27, 1 (1963); Nucl. Phys. A186, 417 (1972); 203, 496 (1973).
[3]N. Austern, Phys. Rev. 188, 1595 (1969).
[4](a) R. J. R. Soper, Phys. Rev. C 10, 1282 (1974); (b) P. J. Griffin and R. D. Koshel, Ann. Phys. (N.Y.) 122, 271 (1979).
[5]J. A. Retter and Y. Hahn, Phys. Rev. C 23, 454 (1981).
[6]F. Coester, Phys. Rev. C 3, 525 (1971).
[7]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).
[8]See for example, Y. Hahn, Phys. Rev. A 9, 2014 (1974); Y. Hahn and D. W. Rule, Phys. Rev. C 12, 1607 (1975).
[9]Y. Hahn, Phys. Rev. 139, B212 (1965), Sec. V and Appendix C; 169, 794 (1968), Sec. III C.
[10]Y. Hahn (unpublished).