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# Possible bound states of repulsive potentials

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The Dirac equation predicts the existence of deeply bound states in the field of a repulsive potential. We discuss some of the implications of these states, especially with regard to the possible existence of the "charged vacuum." We also suggest an experiment to detect the $V^{++}$—a bound state of a proton and a positron.

[NUCLEAR REACTIONS Relativistic wave equations. Bound states. Predict bound state of $p$–$e^+$. $T_{1/2} \sim 10^{-8}$ sec. Suggest experiment to detect in bending magnetic field.]

## I. INTRODUCTION

Ever since the inception of the Dirac equation, the physical (and metaphysical) interpretation of the negative energy solutions has been recurrently troublesome. Dirac's description of the free electron as accompanied by a vacuum consisting of a "Fermi sea" of filled negative energy states was vindicated by the discovery of the positron and by the phenomena of pair creation and annihilation. This had the effect of turning a single-particle theory into a field theory and led, ultimately, to the successful development of quantum-electrodynamics as the second-quantized version of the Dirac equation (even though in the development the concept of the negative energy sea was pretty much abandoned).

Despite these successes of the Dirac theory, the introduction of bound states has led to conceptual difficulties. These difficulties could be overcome, at least formally,[1] if the bound states could be effectively decoupled from the Dirac sea and so, for example, radiative corrections to bound states (Lamb shift) could be calculated to high precision.

More recently, the advent of nuclear experiments involving collisions between heavy ions has focused interest on the Dirac equation for nuclear charges $Z > 137$, including a series of solutions for cut-off Coulomb potentials (summarized in a couple of review articles)[2,3] which exhibit the ground state's dropping to negative energies with increasing $Z$, and extrapolated eventually to "dive" below $-m$. A ground state depressed below the top of the Fermi sea energetically favors spontaneous positron emission. Clearly, the approximation that the bound states are decoupled from the vacuum no longer holds, and so there has been a major theoretical effort to treat quantum electrodynamics in the presence of strong fields[3,4] as well as considerable experimental effort devoted to observing the "charged vacuum."[5]

But even for weak fields, where ordinary Dirac theory and quantum electrodynamics are on a much sounder theoretical foundation, situations exist in which the bound states are not decoupled from the vacuum. This occurs because of the following interesting, and basically trivial, observation: *If the Dirac equation describing a particle of charge e (in an external field $A_\mu$) has a bound state with energy E, then the equation describing a particle of charge -e has a bound state with energy -E.* Specifically, particles can be bound by repulsive forces; for example, the Dirac theory predicts bound states of the proton-positron system.

The above observation is not new; it is mentioned in Ref. 2 and previously was pointed out by Messiah.[6] Nonetheless, the ramifications of this fact have not been fully explored. For example, Messiah simply states that the repulsive bound states have never been observed, while in Ref. 2 these states are regarded as artifacts arising from charge symmetrization (as in quantum electrodynamics, for example, when one normal-orders the current operator).[7] However, we should point out that such repulsive bound states were originally discovered, for Klein-Gordon particles, some forty years ago,[8] and taken very seriously indeed. (The same authors looked for such states of the Dirac equation, but due to a mathematical slipup failed to discover them, although they were the first to predict the existence of the charged vacuum. We explain, in a later section, how these states were missed.)

In Sec. II of this paper, we present the elementary mathematical description of these states and go on, in Sec. III, to propose a possible experiment for their detection. This experiment should determine whether the point of view of Ref. 2 or that of Refs. 6 and 8 is correct. Section IV is devoted to a discussion of the implications of the existence of such states to the charged vacuum and to some other areas of particle physics.

## II. MATHEMATICAL DESCRIPTION

*For convenience only*, we shall consider the stationary single-particle Dirac equation. The same argument carries through for the full time-dependent equation and, indeed, for the Schwinger and Bethe-Salpeter equations,[9] so that it is in no way restricted to single-particle theory. Writing then

$$(\vec{\alpha}\cdot\vec{p}+\beta m-E)\psi=0\,, \tag{1}$$

we observe that there exist two positive-energy plane-wave solutions $u^{\pm}(\vec{p})$ and two corresponding negative-energy solutions $v^{\pm}(\vec{p})$. In fact, the negative-energy solutions can be constructed by application of the antiunitary charge-conjugation operator $K_c$ defined by

$$K_c f=\gamma_2 f^*\,, \tag{2}$$

where the asterisk denotes complex conjugation, and $\gamma_2=\beta\alpha_2$. Then one easily verifies from the explicit forms[10] of $u^{\pm}$ and $v^{\pm}$ that

$$K_c u^{\pm}(\vec{p})=v^{\mp}(-\vec{p})\,. \tag{3}$$

The same operator $K_c$ transforms Eq. (1) into

$$(\vec{\alpha}\cdot\vec{p}+\beta m+E)\psi_c=0,\quad \psi_c\equiv K_c\psi\,, \tag{4}$$

exhibiting explicitly the relationship and interdependence among positive and negative energy free-particle states. This is extremely well known and is the theoretical basis for Dirac's "hole theory."

If we now add a potential $V$ to Eq. (1), so that

$$(\vec{\alpha}\cdot\vec{p}+\beta m+eV-E)\psi=0\,, \tag{5}$$

then we find that $\psi_c$ obeys

$$(\vec{\alpha}\cdot\vec{p}+\beta m-eV+E)\psi_c=0\,. \tag{6}$$

[If $V$ is a central potential, so that angular momentum states make sense, then Eq. (3) also carries over.] Comparing Eqs. (5) and (6) leads directly to our observation in Sec. I, namely that corresponding to an eigenvalue $+E$ in an attractive potential $eV$ there also exists an eigenvalue $-E$ in the repulsive potential $-eV$. These repulsive states are very deeply bound. If $V$ represents the Coulomb potential of a proton, then the bound states of the electron fall between $m$ and $m-13.6$ eV, while the bound states of the positron-proton system lie between $-m+13.6$ eV and $-m$ and are, except for spin idiosyncrasies, the exact "mirror images" about $E=0$ of the bound electron states.

As we have already mentioned, these negative energy states are described in Ref. 2 as somehow arising from a symmetric description of the Dirac vacuum, as occurs in quantum electrodynamics when the current operator is normal ordered. In this view, these states are "sucked up" out of the vacuum by the Coulomb potential of the proton, and thus are ordinarily filled. A vacancy in such a state would correspond to a bound electron, and thus they represent an alternate, charge conjugate, but otherwise entirely equivalent description of the bound Dirac electron.

The alternate view, expressed implicitly in Refs. 6 and 8, is that these are actual physical states. The fact that they have not been observed is due to the fact that the ground state ($E=-m$) is degenerate with the vacuum. A system prepared in such a state would decay on an atomic transition time scale ($\sim 10^{-8}$ sec) to a state basically indistinguishable from the vacuum. This point of view requires some physical explanation of how repulsive forces can bind, and the answer is given in Ref. 6, namely that a particle with negative total energy has, by virtue of $E=mc^2$, a negative mass, and hence obeys $F=-ma$ (anti-Newton's second law). The reader should not be misdirected by the semantics of the last comment into seeking a connection between the present discussion and old cosmological speculations as to a negative gravitational mass for antimatter. We have accepted the evidence for the principle of equivalence[11] and have consistently ascribed a positive *rest mass* to the positron. Negative energy states arise because the (negative) binding energy exceeds the (positive) rest mass. The negative-energy bound states discussed in this paper are for real particles with positive rest mass.

Which of these points of view is correct? In the next section we describe an experiment to detect a bound state of the $p$-$e^+$ system (dubbed hereafter the $V^{++}$) which might answer the question definitively. If the second point of view outlined above is indeed correct, i.e., that the $V^{++}$ can actually be detected, we feel that the charged vacuum concept is in serious trouble. This is discussed in detail in Sec. IV.

Before going on, we return to the question of why Schiff *et al.*[8] conclude (erroneously) that the Dirac equation had no repulsive bound states. The explanation is somewhat subtle. If Eqs. (5) and (6) are separated in spherical coordinates, the identical radial equation and, *ipso facto*, the identical equation for the eigenvalue $E$ are obtained. That is, the signs of the potential and of the energy are already built into the radial equations, and two solution sets $(E, V)$ and $(-E, -V)$ are obtained simultaneously. This can be seen by looking at the transcendental equation describing the eigenvalues in a square-well potential of depth $V_0$, range $R_0$ (the case studied in Ref. 8)[12]:

$$\left(\frac{E+V_0+m}{E+V_0-m}\right)^{1/2}\left[\frac{1}{R_0}\left(\frac{1}{E+V_0+m}\right)-\left(\frac{1}{E+m}\right)^{1/2}\right]$$

$$\times\tan\{R_0[(E+V_0)^2]^{1/2}-m^2\}=1\,.$$

The procedure, followed in Ref. 8, of attempting to solve this under $V \to -V$ is simply incorrect. (The corresponding equation for the Coulomb potential[10]

$$\{(m^2-E^2)[(j+\tfrac{1}{2})^2-\mu^2]\}^{1/2}=E\mu$$

is obviously invariant under $\mu$, $E \to -\mu$, $-E$, which perhaps explains why Messiah discovered these states.)

## III. A PROPOSED EXPERIMENT

If a proton is bombarded with photons of energy ~16 eV, a negative energy positron can be excited from the continuum into the top bound state described in Sec. II. The cross section for this process can be calculated by standard methods[13]; the result is, except for the phase-space factor of $\frac{1}{2}\pi$, identical with the cross section for photoionization of a neutral hydrogen atom, or about $10^6$ b. As we have mentioned earlier, this state, the $V^{++}$, must be detected in a time of $10^{-8}$ sec. (In view of the better availability of high-intensity light sources at lower frequencies, it might be advantageous to target the "mirror-$L$" or "mirror-$M$" state instead of the "mirror-$K$.")

The basic experimental setup is shown in Fig. 1. A beam of nonrelativistic protons passes through a region in which a photon flux can be turned on and off into a magnetic field ~1$T$. The cyclotron frequency $Be/m$ in a field of 1$T$ is about $10^8$ rad/sec, so that the $V^{++}$ will travel about 1 rad before decaying; furthermore, the radius of curvature of the $V^{++}$, $mv/Be$, is one half that of the proton, and spatial separation would occur as depicted in Fig. 1. The dimension of the separation is ~0.05 m for 500 keV protons or 0.5 m for 50 MeV protons. The object of the experiment would be to detect the appearance and disappearance of counts in the detector "$A$" as the light source is turned on and off.

Although we have described the production of the $V^{++}$ in terms of excitation of vacuum positrons,

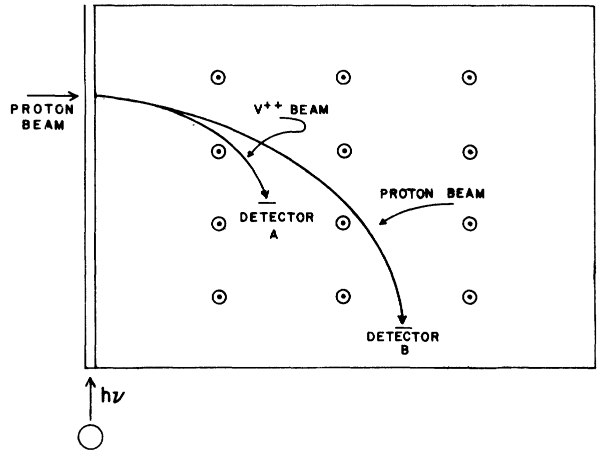


FIG. 1. Schematic of experiment designed to detect the $V^{++}$. The distance between detectors $A$ and $B$ is ~0.5 m for 500 kev, and ~0.5 m for 50 Mev protons. ⊙ represents a magnetic field of 1T directed out of the paper.

it is clear that a more field-theoretic description is possible, not requiring a description in terms of "holes." Relevant Feynman diagrams are shown in Fig. 2.

## IV. EXISTENCE OF THE CHARGED VACUUM AND OTHER SPECULATIONS

The construction of the charged vacuum is based on a numerical solution of the Dirac equation for a cut-off Coulomb potential. A typical plot of $E_0$ (ground state energy) vs $\mu(=\alpha Z)$ is shown as the solid line in Fig. 3, where the point nucleus (pure Coulomb) solution is also indicated (dotted) for comparison. As we see, the point-nucleus ground state approaches $E_0=0$ at $\mu=1$ with infinite slope. This fact is cited in Ref. 2 as evidence that the ground state energy dives into the negative-energy continuum for $\mu=1$ and, *mutatis mutandis*, that the ground state energy for the cut-off potential also dives into the continuum, giving rise to the phenomenon of spontaneous positron emission, or the charged vacuum. (Numerical results[3] indicate that the ground state

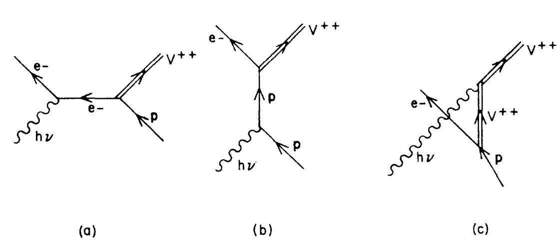


FIG. 2. Feynman diagrams for production of $V^{++}$.

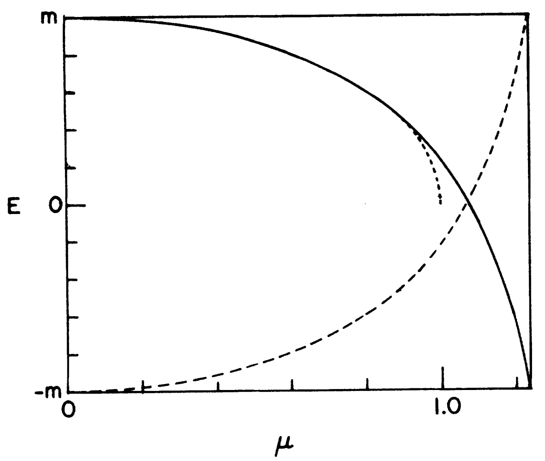


FIG. 3. Energy of the 1s level as a function of $\mu=\alpha Z$ for the finite-size nucleus model (solid line) from Ref. 11. The Sommerfeld point-nucleus eigenvalues are dotted. The dashed line represents the conjugate positron states.

is lowered as the cut-off radius is decreased, as would be expected.)

But this interpretation of Fig. 3 is suspect on two grounds. The first is the issue of whether or not it is possible for the Dirac particle to be bound at all with $E<0$ (binding energy greater than $m$). Recall the Messiah arguments for the existence of the bound states of the proton-positron system, namely the anti-Newton second law. Now superpose on the bound electron states plotted in Fig. 3 the mirror image positron states predicted by Eq. (6) (dashed curve). As we see, at $Z=Z_0$ the energy of the bound positron becomes positive, so that the Messiah argument no longer applies. And yet the theory predicts these states. Clearly, the theory must be wrong, at least for high $Z$ (recall that the Dirac equation is a weak-field approximation in any event). If the theory breaks down for high $Z$, there is no reason to believe that the electron states go below $E=0$, either. Furthermore, the Messiah anti-Newton argument itself militates against the existence of a bound electron with negative total energy. Either argument compels us to the conclusion that any bound state in an attractive potential must have a positive (total) energy.[14] While this argument would be strengthened if the $V^{++}$ could actually be detected, it does not seem to depend on that in any essential fashion. Conclusion: The charged vacuum cannot exist.

Our second point has to do with the point Coulomb potential, for which the eigenvalues are given analytically by the Sommerfeld formula,[6,10] which breaks down when $\mu>j+\frac{1}{2}$. For the ground state, this is at $\mu=1$, where the ground state has energy $E=0$, the first excited state $E=2^{-1/2}m$, and as we have noted, $\partial E/\partial\mu$ becomes infinite for all $j=\frac{1}{2}$ states. We have solved the Dirac equation for the point Coulomb potential[15] with $\mu>1$, utilizing the method of self-adjoint extension of symmetric operators.[16] A one-parameter infinite family of self-adjoint extensions can be generated in this way, but we have proven[15] that for all these extensions there can be no eigenvalue with $E\leqslant -m$. If we stipulate that the eigenvalue be continuous at $\mu=1$, a unique value of the parameter is thereby selected (a distinguished extension). This solution is plotted in Fig. 4 for the ground state. Quite unlike the finite-nucleus calculations, the eigenvalues remain near $E=0$ with a trend to more positive values for increasing $Z$ (the curve was obtained by using the distinguished value of the extension parameter at $\mu=1$ for all higher $\mu$; if we allow this parameter to vary quite modestly with $\mu$, we can avoid there being any negative eigenvalues at all). The break in the curve is not surprising; the conditions under which the Kato-Rellich theorem[17] predicts analyticity of $E(\mu)$ are violated at $\mu=1$ when the domain is extended there (in order to restore self-adjointness). Nonetheless, the sharp contrast between the two curves is counterintuitive, and also suggests the charged vacuum may be a calculational ghost state. In addition to the possible breakdown of the Dirac equation itself, one must consider the possibility that the vacuum polarization effects have been drastically understated in Refs. 3 and 4.

It is tempting to speculate more widely on the consequences of the prospect that repulsive forces actually can be proved to have bound states. We note that these bound states would be very strongly bound; in particular the ratio of the binding energies of an electron and a positron in the field of a proton is about $10^5$. Can some such mechanism be responsible for the strong interaction? The $V^{++}$ system is short lived because its ground state is degenerate with the vacuum,

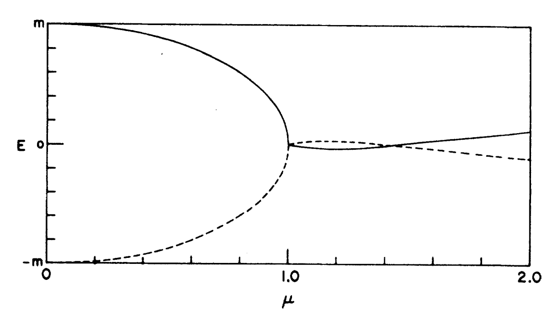


FIG. 4. Energy of the 1s level as a function of $\mu$ for the point nucleus (solid line) from Ref. 14 and conjugate positron states (dashed).

but a potential with a "mass gap," such as a square well, would bind particles with very long lifetimes, since the bound particle would have no place to go.

Another attractive speculative idea is that the force between quarks might, at least in some of its manifestations, be repulsive, leading not only to deeply bound states (confinement) but "antiscreening" (asymptotic freedom). If the experiment described in Sec. III is carried out and the $V^{**}$ is found to exist, then these ideas might be worth pursuing.

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