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Inversion of triton moments

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We use the formalism of hyperspherical harmonics to calculate several moments for the triton photoeffect, for a Volkov spin-independent potential. First, we improve the accuracy of Maleki's calculations of the moments σ_2 and σ_3 by including more terms in the hyperspherical expansion. We also calculate moments σ_0 and σ_1 for a Serber mixture. We find reasonable agreement between our moments found by sum rules and those found from the cross sections calculated by Fang *et al.* and Levinger-Fitzgibbon. We then develop a technique of inversion of a finite number of moments by making the assumption that the cross section can be written as a sum of several Laguerre polynomials multiplied by a decreasing exponential. We test our inversion technique successfully on several model potentials. We then modify it and apply it to the five moments (σ_{-1} to σ_3) for a force without exchange, and find fair agreement with Fang's values of the cross section. Finally, we apply the inversion technique to our three moments (σ_{-1}, σ_0, and σ_1) for a Serber mixture, and find reasonable agreement with Gorbunov's measurements of the ^{3}He photoeffect.

[NUCLEAR REACTIONS Triton photoeffects, hyperspherical harmonics, moments of photoeffect, inversion of moments.]

I. INTRODUCTION

The trinucleon photoeffect for electric dipole transitions has been calculated using different formalisms and potentials by various authors. We use the calculational technique of hyperspherical harmonics (h.h.) and a spin-independent Volkov potential for two exchange mixtures: a completely ordinary force, and a Serber mixture. These potentials are known to be inaccurate and therefore detailed agreement with experiment[1] is not expected. Rather, our goal is to obtain consistent results with the formalism to gain confidence in applying it to more realistic and difficult problems. For this purpose, the results obtained herein can be compared with those of Myers et al.[2] (MFL) and Fang et al.[3] (FLF) for a Wigner mixture and Levinger-Fitzgibbon[4] (LF) for a Serber mixture.

The cross sections $\sigma(E_\gamma)$ for the triton photoeffect, as calculated by MFL, FLF and LF use ground state wave functions $|i\rangle$, and final state wave functions $|f\rangle$ for a Volkov potential. A characteristic problem in photoeffect calculations is finding good continuum wave functions, $|f\rangle$. A comparison of the energy-weighted moments of $\sigma(E_\gamma)$ calculated from two different methods provides a means of testing the accuracy of the final state wave functions. The moments can be calculated numerically from the curves found by MFL, FLF and LF, or alternatively using only the triton ground state wave function $|i\rangle$ and nuclear sum rules. Another use for the sum rule moments is to invert them to find the cross section $\sigma(E_\gamma)$ independent of any calculation of $|f\rangle$. This latter method of finding $\sigma(E_\gamma)$ may even prove superior to the standard method.

The aforementioned work[2,3,4] on the triton is characterized by the use of h.h. The construction of these functions involves eliminating the motion of the center of mass, enabling the three-body system wave function to be written in terms of two vectors $\vec{\xi}$ and $\vec{\eta}$. These two vectors depend on differences of the coordinates, and can be further transformed to one "hyperradius" r, and five angle variables: $\Omega(\theta_1,\phi_1,\theta_2,\phi_2,\Phi)$. Simonov shows[5] that finding an explicit form for the h.h. reduces to formulating a set of basis functions which realize a representation of the group of rotations in ordinary three-dimensional space, and a group of permutations of the three particles. An individual h.h. is given by $H^{m_1,m_2}_{L,\ell_1,\ell_2}(\Omega)$, while $y^{(\nu)}(\Omega)$ denotes a particular linear combination of the h.h.'s giving a definite total orbital angular momentum L_{tot}. The superscript ν refers to the type of permutation symmetry. Note that the subscript L, which Fabre calls[6] the "grand orbital," is not the same as L_{tot}. For more properties of h.h. we refer the reader to MFL whose notation we follow.

The organization of this work is as follows: Section II gives derivations for the triton's sum rule moments σ_{-1} thru σ_3 for a Wigner mixture. Here we improve the calculations of Maleki and Levinger[7] (ML) by using a second term in the h.h. expansions for the wave function and potential. Section III presents calculations of σ_0 and σ_1 for a Serber mixture using only the first term in the h.h. expansions. We compare these results with expressions given by Levinger-Bethe[8] (LB), Verde,[9] Leonardi-Lipparini[10] (LL), and O'Connell-

Prats[11] (OP). Section IV outlines a general method of inverting moments using Laguerre polynomials similar to Langhoff's earlier work[12] with moment inversion for atomic and molecular photoeffects. We also test some model cross section curves for convergence. In Section V we apply the moment inversion technique to the triton moments and discuss the results.

II. SUM RULE CALCULATIONS: WIGNER FORCE

The triton photoeffect is treated by assuming the photon is polarized along the z axis and interacts with the proton which has coordinates $\vec{r}_3$. The cross section for electric dipole transitions from initial state $|i\rangle$ to final state $|f\rangle$ in the triton is[13]

$$\sigma(E_\gamma) = (4\pi^2/\hbar c)E_\gamma|\langle i|D|f\rangle|^2\rho_f \qquad (2.1)$$

where ρ_f is the density of final states and E_γ is the photon energy. The dipole operator D is given by[2]

$$D = e(z_3 - Z) = e\eta_z/\sqrt{3} \qquad (2.2)$$

with Z being a component of the center of mass of the triton and η_z a component of one of the Jacobi coordinates according to the notation of MFL.

The moments σ_p of the photoeffect cross section $\sigma(E_\gamma)$ are defined by

$$\sigma_p \equiv \int_0^\infty E_\gamma^p\sigma(E_\gamma)dE_\gamma\ , \qquad (2.3)$$

or using Eq. (2.1) and changing from the continuum to a discrete system for $|f\rangle$ yields

$$\sigma_p = (4\pi^2/\hbar c)\sum_f E_\gamma^{p+1}|\langle i|D|f\rangle|^2 . \qquad (2.4)$$

We use this formula to calculate the sum rule moments σ_{-1} thru σ_3 for a Volkov potential with no exchange in this section, and σ_{-1} thru σ_1 for a Volkov potential with Serber exchange in Section III.

MFL formulated expressions for σ_{-1}, σ_0 and σ_1 without exchange. For p = -1, the quantum mechanical closure relation is applied to Eq. (2.4) to give

$$\sigma_{-1} = (4\pi^2/\hbar c)\langle i|D^2|i\rangle = (4\pi^2/3)\alpha\langle i|\eta_z^2|i\rangle\ , \qquad (2.5)$$

where α is the fine structure constant. Calculation of σ_0 and σ_1 involves using the Heisenberg relations

$$E_\gamma\langle i|D|f\rangle = -\langle i|[H,D]|f\rangle,\quad E_\gamma\langle i|D|f\rangle^* = \langle f|[H,D]|i\rangle, \qquad (2.6)$$

where we used the fact that D is a Hermitian operator. The general expressions for σ_0 and σ_1 are derived by MFL

$$\sigma_0 = (4\pi^2/\hbar c)\sum_f E_\gamma[\langle i|D|f\rangle\langle i|D|f\rangle^*] = (2\pi^2/\hbar c)\langle i|[D,[H,D]]|i\rangle\ , \qquad (2.7)$$

and

$$\sigma_1 = -(4\pi^2/\hbar c)\langle i|([H,D])^2|i\rangle\ . \qquad (2.8)$$

The commutator bracket [H,D] is evaluated using $(P_\xi^2 + P_\eta^2)/M$ for the kinetic energy operator[5] and by assuming the potential energy term without exchange commutes with D. The results of numerical evaluation of these expectation values are given in Table I.

We now improve the calculations of σ_2 and σ_3 of ML by using a second term in the h.h. expansions of the wave function and potential. For p = 2, we can write Eq. (2.4) as

$$\sigma_2 = (4\pi^2/\hbar c)\sum_f E_\gamma^3|\langle i|D|f\rangle|^2 = (2\pi^2/\hbar c)\sum_f\{[E_\gamma^2\langle i|D|f\rangle][E_\gamma\langle i|D|f\rangle^*] + [E_\gamma\langle i|D|f\rangle][E_\gamma^2\langle i|D|f\rangle^*]\}. \qquad (2.9)$$

The Heisenberg relation is used to show

$$E_\gamma\langle i|D|f\rangle = -A\langle i|\frac{\partial}{\partial\eta_z}|f\rangle,\quad E_\gamma^2\langle i|D|f\rangle = A\langle i|\partial V/\partial\eta_z|f\rangle, \qquad (2.10)$$

where $A = 2e\hbar^2/\sqrt{3}M$. Similar relations hold for the complex conjugate matrix elements. Using (2.10) and closure we find

$$\sigma_2 = (8\pi^2/3)\alpha(\hbar^4/M^2)\langle i|\partial^2V/\partial\eta_z^2|i\rangle. \qquad (2.11)$$

TABLE I. Moments for triton photoeffect.

	No exchange		Serber exchange mixture	
	From cross section[a]	Sum rule	From cross section[e]	Sum rule[d]
σ_{-1} (mb)	2.80	2.87[b]	2.87	2.87
σ_0 (MeV mb)	41.0	39.8[b]	58.6	66.3
σ_1 (MeV^2 mb)	690	613[c], 611[d]	1566	1950
σ_2 (MeV^3 mb)	1.43×10^4	1.19×10^4[c], 1.38×10^4[d]		
σ_3 (MeV^4 mb)	6.03×10^5	5.62×10^5[c], 6.06×10^5[d]		

[a] See FLF, Ref. 3.
[b] See MFL, Ref. 2.
[c] See ML, Ref. 7; note that here we correct an error of a factor of 2 in the ML value of σ_3.
[d] This paper.
[e] See LF, Ref. 4.

The derivation of σ_3 is a simple application of Eqs. (2.4) and (2.10) which gives

$$\sigma_3 = (16\pi^2/3)\alpha(\hbar^4/M^2)\langle i|(\partial V/\partial\eta_z)^2|i\rangle. \quad (2.12)$$

Neglecting spin-dependent forces the triton wave function $|i\rangle$ is assumed symmetric under spatial permutations of any pair of particles. Simonov selects a linear combination of the h.h. denoted by $y_L^{(o)}(\Omega)$ and having total angular momentum zero, even parity, and complete symmetry (o) for particle exchange. Even parity excludes odd L values and Simonov shows[5] that terms with L = 2 are also missing. The coefficient of a given angular function is $r^{-5/2}u_L(r)$ so the triton ground state wave function is written

$$r^{5/2}\langle r,\Omega|i\rangle = u_o(r)y_o^{(o)}(\Omega) + u_4(r)y_4^{(o)}(\Omega) + \dots . \quad (2.13)$$

Substituting this expression into Schrödinger's equation yields an infinite set of coupled differential equations for $u_L(r)$, which are truncated at some L_{max}. The numerical values of $u_o(r)$ and $u_4(r)$ were provided in a private communication by Beiner and Fabre. (Also see Beiner and Fabre[14] and Ballot et al.[15])

The spin-independent Volkov potential for a two-body interaction is given by[2]

$$V(r_{ij}) = 144.86\ \exp[-(r_{ij}/0.82)^2] - 83.40\ \exp[-(r_{ij}/1.60)^2]. \quad 2.14)$$

This is expanded in h.h. yielding[2]

$$\pi^{-3/2}V(r,\Omega) = 3V_o(r)y_o^{(o)}(\Omega) + 3^{3/2}V_4(r)y_4^{(o)}(\Omega) + \dots , \quad (2.15)$$

where the radial "hypermultipoles" $V_L(r)$ are defined by Fabre and Levinger[6]

$$V_{2k}(r) = 289.72\ \exp(-x)I_{k+1}(x)/x - 166.80\ \exp(-x')I_{k+1}(x')/x' \quad (2.16)$$

with

$$x = \tfrac{1}{2}(r/0.82)^2; \quad x' = \tfrac{1}{2}(r/1.60)^2.$$

The $I_n(x)$ are modified Bessel functions.

The expressions for σ_2 and σ_3 can now be evaluated using Eqs. (2.13) and (2.15) for the potential $V(r,\Omega)$. The spherically symmetric S ground state of the triton gives us the identity

$$\langle i|\partial^2V/\partial\eta_z^2|i\rangle = \frac{1}{6}\langle i|\nabla^2V|i\rangle.$$

Then Eq. (2.11) becomes

$$\sigma_2 = (4\pi^2/9)\alpha(\hbar^4/M^2)\langle i|\nabla^2V|i\rangle. \quad (2.17)$$

For two terms in the h.h. expansions of V and $|i\rangle$, the expectation value (2.17) is given by

$$\langle i|\nabla^2V|i\rangle = 3[\langle u_o|\nabla^2V_o|u_o\rangle + \langle u_4|\nabla^2V_o|u_4\rangle + 2\sqrt{3}\langle u_o|\nabla^2V_4|u_4\rangle], \quad (2.18)$$

where orthonormality of the $y_L^{(o)}(\Omega)$'s is used. Numerical evaluation of the radial integrals in (2.18) gives for a final result

$$\sigma_2 = 1.38 \times 10^4\ \text{MeV}^3\ \text{mb}. \quad (2.19)$$

This is about a 14% correction to the ML calculation, which truncates at one term in the h.h. expansion, and is seen to be in good agreement with the results of FLF (Table I).

The calculation of σ_3 From Eq. (2.12) is considerably more complicated. First, we write

$$\partial V(r,\Omega)/\partial\eta_z = 3\Big[(dV_o/dr)(\partial r/\partial\eta_z) + \sqrt{3\pi^3}\ (dV_4/dr)(\partial r/\partial\eta_z)y_4^{(o)}(\Omega) + \sqrt{3\pi^3}\ (V_4)\ \partial y_4^{(o)}(\Omega)/\partial\eta_z\Big]. \quad (2.20)$$

Then we find[16]

$$\partial y_4^{(o)}(\Omega)/\partial\eta_z = (2\sqrt{2}/r)\Big[y_3^{(+)}(\Omega) - y_1^{(+)}(\Omega)\Big] \quad (2.21)$$

and substitute in (2.20) to show

$$\partial V(r,\Omega)/\partial\eta_z = f(r)y_1^{(+)}(\Omega) + g(r)y_3^{(+)}(\Omega), \quad (2.22)$$

where

$$f(r) = 3\sqrt{\pi^3/6}\ (dV_o/dr) + 6\sqrt{6\pi^3}(V_4/r)$$

$$g(r) = 3\sqrt{\pi^3/2}\ (dV_4/dr) + 6\sqrt{6\pi^3}(V_4/r).$$

Now it can be shown[16] that

$$\begin{aligned}\langle i|[\partial V(r,\Omega)/\partial\eta_z]^2|i\rangle &\simeq 3/2\langle u_o|(dV_o/dr)^2|u_o\rangle \\ &+ 9/2\langle u_o|(dV_4/dr)^2|u_o\rangle \\ &- 36\langle u_o|(dV_o/dr)(V_4/r)|u_o\rangle \\ &+ 36\sqrt{3}\ \langle u_o|(dV_4/dr)(V_4/r)|u_o\rangle \\ &+ 6\sqrt{3}\ \langle u_o|(dV_o/dr)(dV_4/dr)|u_4\rangle \\ &+ 72\langle u_o|(dV_o/dr)(V_4/r)|u_4\rangle \\ &- 72\sqrt{3}\langle u_o|(dV_4/dr)(V_4/r)|u_4\rangle, \end{aligned} \quad (2.23)$$

where terms with $[u_4(r)]^2$ and $[V_4(r)]^2$ are neglected. Inserting (2.23) into (2.12) gives the numerical result

$$\sigma_3 = 6.06 \times 10^5\ \text{MeV}^4\ \text{mb}. \quad (2.24)$$

The second term in the h.h. expansions gives about 7% of the new moment's value when compared to the corrected ML calculation.We note that (2.24) is in excellent agreement with the FLF result (Table I).

III. SUM RULE CALCULATIONS: SERBER MIXTURE

Now consider a mixture of Wigner force (fraction 1-x) with two-body Majorana exchange force, fraction x. In our numerical work below we use a Serber mixture with x = ½.

Equation (2.5) shows that the moment σ_{-1} depends only on the ground state expectation value of the squared dipole moment and is therefore independent of x. On the other hand, the Majorana force contributes an extra term to the commutator [H,D] in

Eq. (2.7) for the integrated cross section. The resulting increase in σ_0 has been studied for over forty years with general agreement amongst various workers: Siegert,[17] LB,[8] Verde,[9] LL,[10] OP.[11] We sketch the derivation here as parts are used in the calculation of the moment σ_1.

We use Eq. (2.2) for the dipole moment D to show

$$[H,D] = [T,D] + ex[\{V(r_{12})P_{12} + V(r_{13})P_{13} + V(r_{23})P_{23}\},z_3]$$
$$= [T,D] + ex\{V(r_{13})(z_1-z_3)P_{13} + V(r_{23})(z_2-z_3)P_{23}\}. \quad (3.1)$$

In writing Eq. (3.1) we have noted that P_{ij} commutes with the center of mass coordinate and that P_{12} commutes with z_3. The double commutator [D,[T,D]] gives us the well known Thomas-Reiche-Kuhn result used in Eq. (2.7). Addition of the commutator [D,[V,D]] gives the integrated cross section

$$\sigma_0 = (4\pi^2/3)\alpha(\hbar^2/M) - 2\pi^2\alpha\, x \times \langle i|\{(z_1-z_3)^2V(r_{13})P_{13} + (z_2-z_3)^2V(r_{23})P_{23}\}|i\rangle. \quad (3.2)$$

The totally symmetric ground state allows us to write

$$P_{ij}|i\rangle = |i\rangle . \quad (3.3)$$

We approximate the two-body potential by one term in the h.h. expansion (2.15),i.e.,

$$V(r_{ij}) \simeq V_0(r) . \quad (3.4)$$

Changing from particle to h.h. coordinates, we write

$$(z_1-z_3)^2 + (z_2-z_3)^2 = (\tfrac{1}{2}\xi_z - \tfrac{1}{2}\sqrt{3}\,\eta_z)^2 + (-\tfrac{1}{2}\xi_z - \tfrac{1}{2}\sqrt{3}\,\eta_z)^2 = \tfrac{1}{2}\xi_z^{\,2} + (3/2)\eta_z^{\,2} . \quad (3.5)$$

We approximate the ground state wave function with the first term in the h.h. expansion of Eq. (2.13). Performing the angular integrations we can then write

$$\langle i|V_0(r)\xi_z^{\,2}|i\rangle = \langle i|V_0(r)\eta_z^{\,2}|i\rangle = (1/6)\langle u_0|V_0(r)r^2|u_0\rangle . \quad (3.6)$$

Now we use Eqs. (3.2) thru (3.6) to write

$$\sigma_0 = (4\pi^2/3)\alpha(\hbar^2/M) - (2\pi^2/3)\alpha\, x \times \langle u_0|V_0(r)r^2|u_0\rangle = 39.8 + 26.5 = 66.3 \text{ MeV mb} . \quad (3.7)$$

In our numerical evaluation we use the Volkov potential with Serber force ($x=\tfrac{1}{2}$) and the Beiner-Fabre[14] radial function $u_0(r)$. The result given in Table I is 13% higher than that found from the LF cross sections for a Serber mixture. They[4] four closer agreement using the next term, $V_2(r)$, in the h.h. expansion of $V(r,\Omega)$; but here we confine ourselves to the lowest term in the h.h. expansion for both σ_0 and σ_1.

We calculate σ_1 by substituting Eq. (3.1) into the sum rule (2.8). We note that LB[8] and Verde[9] each obtain three terms: (i) independent of x from $[T,D]^2$, (ii) proportional to x from [T,D][V,D] + [V,D][T,D], and (iii) proportional to x^2 from $[V,D]^2$. However, LL[10] and OP[11] do not find a term proportional to x. (We do not agree with the OP argument that such a term should be absent due to certain "reality properties.") We also note that OP and LL disagree on the term proportional to x^2 and that (in a private communication) Prats and Lehman quote still another expression for the term proportional to x^2. Our result below is in agreement with the private communication from Prats-Lehman.

We express the commutator [T,D] in hyperspherical coordinates:

$$[T,D] = -(2e\hbar^2)/(\sqrt{3}\,M)(\partial/\partial\eta_z). \quad (3.8)$$

The term of order x (denoted by superscript x) is

$$\sigma_1^{\,x} = (4\pi^2/\hbar c)(-2e\hbar^2/\sqrt{3}\,M)(ex) \times \langle i|\partial/\partial\eta_z\{V(r_{13})(z_1-z_3)P_{13} + V(r_{23})(z_2-z_3)P_{23}\} + \{V(r_{13})(z_1-z_3)P_{13} + V(r_{23})(z_2-z_3)P_{23}\}\partial/\partial\eta_z|i\rangle . \quad (3.9)$$

We evaluate (3.9) using Eqs. (3.3), (3.4), and the following relations:

$$\partial V_0/\partial\eta_z = (\eta_z/r)dV_0/dr , \quad (3.10)$$
$$P_{13}[\partial\psi_i/\partial\eta_z] = (\tfrac{1}{2}\sqrt{3}\,\xi_z - \tfrac{1}{2}\eta_z)r^{-1}d\psi_i/dr.$$
$$P_{23}[\partial\psi_i/\partial\eta_z] = -(\tfrac{1}{2}\sqrt{3}\,\xi_z + \tfrac{1}{2}\eta_z)r^{-1}d\psi_i/dr.$$

After some algebra we find

$$\sigma_1^{\,x} = -(4\pi^2/3)\alpha(\hbar^2/M)x \times \langle u_0|rdV_0/dr + 6V_0|u_0\rangle = 920 \text{ MeV}^2/\text{mb}. \quad (3.11)$$

The term of order x^2 (denoted by superscript xx) is

$$\sigma_1^{\,xx} = -4\pi^2\alpha x^2\langle i|\{V(r_{13})(z_1-z_3)P_{13} + V(r_{23})(z_2-z_3)P_{23}\}^2|i\rangle \quad (3.12)$$

We again use (3.3) and (3.4) to show

$$\sigma_1^{\,xx} = 4\pi^2\alpha x^2|i\, V_0^{\,2}(r)[(z_1-z_3)^2 + (z_2-z_3)^2 + (z_1-z_2)^2]|i\rangle = 2\pi^2\alpha x^2\langle u_0|V_0^{\,2}(r)r^2|u_0\rangle = 420 \text{ MeV}^2\text{-mb}. \quad (3.13)$$

We combine Eqs. (3.11) and (3.13) with the term involving the squared kinetic energy commutator[7] to give the value for σ_1 in Table I. We note a disagreement of 24.5% with the LF value.

IV. MOMENT INVERSION WITH LAGUERRE POLYNOMIALS

Moment inversion entails finding an expression for a function from its moments.

We outline a technique using Laguerre polynomials to determine the photoeffect cross section from the triton moments.

The general shape of the experimental triton photoeffect curve is characterized by the threshold energy at 8.48 MeV (for 3-body breakup), a pronounced peak around 14-20 MeV, followed by a steadily decreasing tail at higher energies. Also, ML find that all moments are finite. These features prompted the assumption that the cross section curve $\sigma(x)$ can be written in the form

$$\sigma(x) = F(x)\exp(-x), \quad (4.1)$$

where F(x) and its derivatives are continuous. We discuss the relation between the dimensionless variable x and the photon energy E_γ in the next section.

In analogy to Eq. (2.3), the moments of $\sigma(x)$ are

$$\sigma_p = \int_0^\infty \sigma(x)x^p dx = \int_0^\infty \exp(-x)F(x)x^p dx$$

$$p = 0, 1, 2, \dots . \quad (4.2)$$

The existence of these integrals allows us to expand F(x) in a series of Laguerre polynomials $L_n(x)$:

$$F(x) = \sum_{n=0}^{\infty} \lambda_n L_n(x) \quad n = 0,1,2,\dots . \quad (4.3)$$

An explicit formula for the Laguerre polynomials is

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k x^k/k! \quad (4.4)$$

and the coefficients λ_n are found to be

$$\lambda_n = \int_0^\infty \exp(-x)\, L_n(x)F(x)dx$$

$$= \int_0^\infty \sigma(x)L_n(x)dx , \quad (4.5)$$

from the orthonormality relation

$$\int_0^\infty \exp(-x)L_n(x)L_m(x)dx = \delta_{nm}. \quad (4.6)$$

The moment inversion is now accomplished by expressing the coefficients λ_n in terms of the moments σ_p. We do this by inserting (4.4) into (4.5) leading to

$$\lambda_n = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \sigma_k/k! \quad (4.7)$$

Hence, the cross section curve in terms of the moments is given by[18]

$$\sigma(x) = \exp(-x) \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\frac{(-1)^k}{k!}\sigma_k L_n(x). \quad (4.8)$$

We now test the inversion formula (4.8) against some model curves before applying it to the triton moments. The convergence of five terms in the Laguerre series is of particular interest since Eq. (4.7) shows that the five triton moments can generate five coefficients of expansion. It might be noted that since there are no negative powers of x in $L_n(x)$, the inversion problem necessitates defining only positive moments. A method for utilizing the triton moment σ_{-1} is developed in the next section.

The first model examined is $\sigma^{(1)}(x) = \sqrt{x}\exp(-x)$ where $F(x) = \sqrt{x}$ and exact convergence for a finite number of terms in the Laguerre series expansion cannot be attained. The coefficients are found from Eq. (4.5) to be

$$\lambda_n = \sum_{k=0}^{n} \binom{n}{k}\frac{(-1)^k}{k!}\Gamma(k + \tfrac{3}{2}) . \quad (4.9)$$

Figure 1 displays a comparison of the model curve $\sigma^{(1)}(x)$ with five terms in the series. The convergence is seen to be quite good except in the region of x = 0. This behavior seems characteristic of our inversion formula since it recurs in all models.

We choose the second model, $\sigma^{(2)}(x) = x\exp(-cx)$, to examine the consequences of the constant c in the exponential. For values of $c > \frac{1}{2}$, exp(-cx) is a valid weighting function for the Laguerre polynomials[19] (with respect to orthonormality) and the analysis can be carried through. The coefficients λ_n now contain a c dependence as follows from Eq. (4.5)

$$\lambda_n(c) = \sum_{k=0}^{n} \binom{n}{k}\frac{(-1)^k(k+1)}{c^{k+2}};\ c > \tfrac{1}{2}. \quad (4.10)$$

We have plotted five terms for c = 0.70 versus the model in Fig. 2. In general, we find convergence becoming better as c is increased.[16]

We choose the last model, $\sigma^{(3)}(x) = x^5\exp(-x)$ to demonstrate that five terms in the Laguerre series expansion will not always be enough to produce reasonable convergence. In this case, $F(x) = x^5$ and exact convergence is attainable with six terms. Estimating $\sigma^{(3)}(x)$ by five terms is equivalent to saying

$$\sigma^{(3)}(x) \approx e^{-x}[x^5 + 120L_5(x)] \quad (4.11)$$

which is a poor approximation.

Our models indicate that convergence should generally be good for five terms in the Laguerre series expansion. The case of

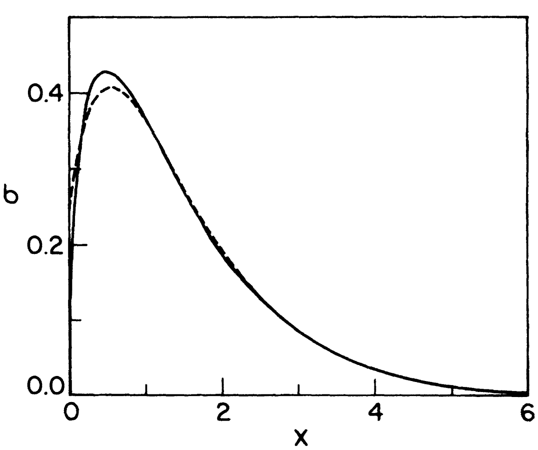

FIG. 1. The solid curve shows model 1; the dashed curve shows an inversion using five moments, see Eqs. (4.8) and (4.9).

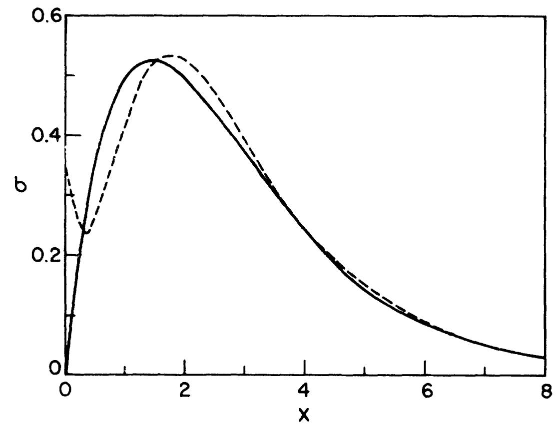

FIG. 2. The solid curve shows model 2; the dashed curve shows an inversion using five moments, see Eqs. (4.8) and (4.10).

$\sigma^{(3)}(x) = x^5\exp(-x)$ shows that further terms would likely be needed to converge on a sharply peaked curve, since higher powers of x are involved.

V. MOMENT INVERSION FOR THE TRITON

To apply the moment inversion technique just developed to the triton moments, σ_{-1} through σ_3, we must first modify Eq. (4.8). Recall that (4.8) was derived for non-negative moments; therefore, we outline a method for utilizing σ_{-1}. Then we consider the functional dependence of the cross section on the photon energy.

For the first purpose, we define a new function $\tau(x)$ by

$$\tau(x) \equiv \sigma(x)/x\ , \qquad (5.1)$$

where $\sigma(x)$ is the cross section curve discussed previously. The power moments of $\tau(x)$ are defined, in analogy to Eq. (4.2), to be

$$\tau_p \equiv \int_0^\infty \tau(x)x^p dx = \int_0^\infty \sigma(x)x^{p-1}dx \qquad (5.2)$$

for p = 0, 1, 2, ... Comparison of Eqs. (4.2) and (5.2) identifies the moments σ_{-1} through σ_3 as the first five moments τ_0 through τ_4 of the function $\tau(x)$. Following the procedure developed in Section IV, we write $\tau(x)$ in terms of its moments

$$\tau(x) = \exp(-x) \sum_{n=0}^{\infty} \lambda_n L_n(x)\ . \qquad (5.3)$$

where the coefficients are given by

$$\lambda_n = \int_0^\infty \tau(x)L_n(x)dx = \sum_{k=0}^{n} \binom{n}{k}\frac{(-1)^k}{k!}\tau_k . \qquad (5.4)$$

If $\tau(x)$ is calculated in this manner, we can determine $\sigma(x)$ from Eq. (5.1).

We now introduce a parameter having the units of energy and denoted by D when considering the functional dependence of the cross section on E_γ. This is because the arguments of $L_n(x)$ and the exponential function must be dimensionless. The previous equations for $\sigma(x)$ and $\tau(x)$ are now considered with the substitution $x = E_\gamma/D$. We also facilitate the computations by translating the axis to locate the origin at the threshold energy for three-body breakup (8.48 MeV), denoted by B. This essentially ignores the small contribution of the reaction $^3H + \gamma \to d + n$ from the threshold energy for two-body breakup to 8.48 MeV. If E is defined as the energy above threshold, such that $E_\gamma = E + B$, we have the following relations

$$\sigma(E_\gamma/D) = \begin{cases} 0 & : E_\gamma \le B \\ \sigma(E/D) & : E_\gamma \ge B \end{cases}$$
$$\tau(E_\gamma/D) = \begin{cases} 0 & : E_\gamma \le B \\ \tau(E/D) & : E_\gamma \ge B\ . \end{cases} \qquad (5.5)$$

Now we write Eq. (5.2) for the moments as

$$\tau_p = D^{-(p+1)}\int_0^\infty E_\gamma{}^p \tau(E/D)dE \qquad (5.6)$$

and the coefficients become

$$\Lambda_n = D^{-1}\int_0^\infty \tau(E/D)L_n(E/D)dE\ . \qquad (5.7)$$

Using Eqs.(5.1) through (5.7), our inversion formula for the photoeffect cross section in terms of the five triton moments is

$$\sigma(E/D) \simeq (E+B)D^{-1}\exp(-E/D) \sum_{n=0}^{4} \Lambda_n L_n(E/D)\ . \qquad (5.8)$$

The coefficient $(E+B)D^{-1}$ arises from the initial substitution $x = E_\gamma/D$ in Eq. (5.1), while the product of the exponential function and the Laguerre series represents the expansion of the function $\tau(E/D)$. The moment inversion is accomplished by expressing the coefficients Λ_n in terms of the moments τ_p. Eqs. (5.6) and (5.7) show that for n=0

$$\Lambda_0 = D^{-1}\int_0^\infty \tau(E/D)dE = \tau_0, \qquad (5.9)$$

since $L_0(E/D) = 1$. We calculate higher coefficients by substituting $E = E_\gamma - B$ in the argument of the Laguerre polynomial; for example, with n=1,

$$\begin{aligned}\Lambda_1 &= D^{-1}\int_0^\infty \tau(E/D)L_1[(E_\gamma-B)/D]dE \\ &= D^{-1}\int_0^\infty \tau(E/D)[1-(E_\gamma-B)/D]dE \\ &= (B+D)\tau_0/D-\tau_1\ . \end{aligned} \qquad (5.10)$$

We use expressions[16] for Λ_0 through Λ_4.

We now apply the inversion formula (5.8) to the five triton moments for a Volkov potential with no exchange, which were developed in Section II. The value D = 7.3 MeV is chosen for our parameter since it gives best agreement with the known threshold behavior, $\sigma(E=0)=0$. Figure 3 compares our result with the calculation of FLF. We feel that the discrepancy occurs for reasons indicated by our third model of the previous section; i.e., more terms are needed to converge toward such a sharply peaked curve. Additional moments would probably also remedy the negative portion of the inverted curve arising at higher energies.

The moments of Section III are calculated for a Volkov potential with Serber exchange

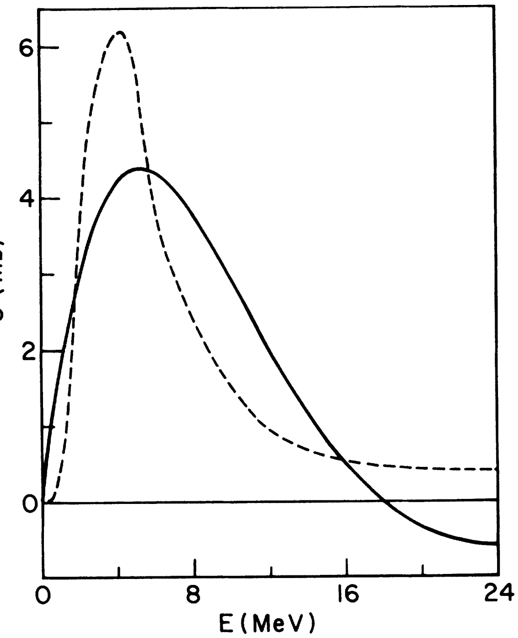

FIG. 3. Cross sections for triton photoeffect in mb vs energy above threshold in MeV. Fang's curve, dashed, is from Ref. 3, for a Volkov potential with no exchange. The solid curve uses the five triton moments from Table I, in Eq. (5.8).

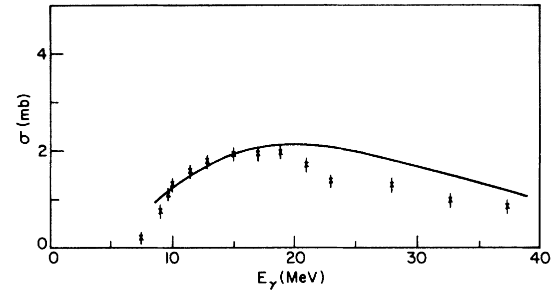

FIG. 4. Triton photoeffect cross section in mb vs photon energy in MeV. The solid curve uses Eq. (5.8) with three moments from Table I for a Volkov potential with Serber exchange. The crosses with error bars are Gorbunov's measurements for ^{3}He, Ref. 1.

neglecting spin. We invert these moments, even though we have only three ($\sigma_{-1}, \sigma_0, \sigma_1$), and obtain very promising results. The value D = 8.8 MeV was chosen to minimize $\sigma(E=0)$. The inverted cross section versus Gorbunov's experimental data[1] is displayed in Fig. 4. We extended the calculation out to E_γ = 300 MeV and found that the inverted curve remains positive (barely) although it does fall below the experimental data. We feel our results are almost too close to experiment considering we use only three moments calculated with one term in the h.h. expansions, and completely neglect spin-dependent forces.

Future work in this area entails finding σ_2 and σ_3 for the potential of Section III, and extending moment calculations to a second term in the h.h. expansions of V and $|i\rangle$. We could also use a more accurate nuclear potential. Another direction, having less priority in our opinion, is to utilize an inversion technique other than that developed in Sections IV and V.

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