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Breakup coupling potentials for deuteron-nucleus collisions in the space of hyperspherical harmonics

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Formulas are derived for expressing the sum of neutron-nucleus and proton-nucleus optical potentials in terms of a series of hyperspherical harmonic functions and Legendre polynomials. Matrix elements of this expansion between two hyperspherical harmonics are obtained analytically and given in terms of triple sums of finite number of terms. The results are evaluated numerically for the breakup space associated with 21.6 MeV deuterons scattering on the nucleus of nickel. Conclusions are drawn regarding the feasibility of solving the corresponding coupled equations which describe breakup space in the hyperspherical basis.

[NUCLEAR REACTIONS Mathematical techniques for expanding three-body wave functions in terms of hyperspherical harmonics. Numerical examples.]

I. INTRODUCTION

The purpose of the present paper is to provide the mathematical tools needed to implement a formulation of breakup effects[1] which utilizes hyperspherical harmonics (HSH).[2] A description of the formulas derived in this study is given below.

The HSH functions $\phi_\beta(\underline{\theta})$ depend on five angular coordinates $\underline{\theta}$ and the subscript β represents sets of five discrete quantum numbers, which will be detailed further on. The remaining coordinate has dimensions of length and is denoted as y.

The piece of the three-body wave function, ψ_2, which contains only breakup components, i.e., no two cluster pieces, is expanded in terms of the HSH's,

$$\psi_2(y,\underline{\theta}) = y^{-5/2}\sum_\beta g_\beta(y)\phi_\beta(\underline{\theta}) \ , \tag{1.1}$$

and the coefficients $g_\beta(y)$ are unknown functions which are to be determined by solving a set of coupled equations. The latter are derived in another study[1] where the separation of the total wave functions ψ into the components ψ_1 and ψ_2 is described. The coupled equations for the g_β's contain two types of terms: (a) The potentials $V_{\beta\beta'}(y)$ which couple breakup terms g_β to other breakup terms $g_{\beta'}$ in the form $\sum_{\beta'} V_{\beta\beta'}(y)g_{\beta'}(y)$ and (b) the inhomogeneous (source) terms $Z_\beta(y)$ which couple the g_β's to the remaining part ψ_1 of the total wave function ψ. The part ψ_1 contains the incident and scattered deuteron components, and is written in terms of the coordinates $\vec{r}$ and $\vec{R}$, which are more natural for the description of the internal deuteron wave functions ($\vec{r}$) and the motion of the c.m. of the deuteron relative to the nucleus ($\vec{R}$). The internal coordinates of the nucleons in the nucleus are not explicitly included.

The main objective of the present paper is to provide convenient expressions for calculating the coupling potentials $V_{\beta\beta'}(y)$ in terms of the sum of the nucleon-nucleus optical potentials

$$V_N = V_{n-A}(r_n) + V_{p-A}(r_p) \ , \tag{1.2}$$

and also to give a numerical example.

The $V_{\beta\beta'}(y)$'s are the matrix elements

$$V_{\beta\beta'}(y) = \int \phi^*_\beta(\underline{\theta})V_N\phi_{\beta'}(\underline{\theta})\,d\tau_{\underline{\theta}} \ , \tag{1.3}$$

and in order to evaluate them one proceeds in two steps. The first consists of an expansion of V_N in terms of the coordinates y and $\underline{\theta}$, using for the θ space the HSH functions,

$$V_N(r_n, r_p) = \sum_\lambda V_{\bar{\lambda}}(y)\Phi_{\bar{\lambda}}(\underline{\theta}) \ , \tag{1.4}$$

and subsequently one needs the angular integration matrix elements

$$M^{\bar{\lambda}}_{\beta,\beta'} = \int \phi^*_\beta(\underline{\theta})\Phi_{\bar{\lambda}}(\theta)\phi_{\beta'}(\underline{\theta})d\tau_{\underline{\theta}} \tag{1.5}$$

An integral expression for $V_{\bar{\lambda}}(y)$ is given, and an expression for the M's are given in terms of expansions into triple sums which have to be carried out numerically. The expression for M contains 6-j symbols since 4 of the 5 angular variables are of the usual geometric type, and the overall result has the nature of a Clebsch-Gordan coefficient

since it contains triangular conditions also in the 5th quantum number.

Formulas for evaluating matrix elements of this type have also been given by Raynal and Revai,[3] in terms of transformations of HSH from one set of HS coordinates, for which evaluation of the matrix elements are straightforward, to another set of HS coordinates for which the matrix elements are actually needed. If harmonic oscillator functions in the coordinate y are also involved, then transformations which are similar to Moshinsky brackets have also been given.[4] However, the formulas given in the present paper for $V_{\beta\beta'}$ appear to be simpler and better suited for the purpose of studying deuteron-nucleus scattering.

Numerical examples are given for the $V_\chi(y)$ as well as for the M's which refer to the case of 21.6 MeV deuteron, incident on the nucleus of nickel. From these results the numerical effort required to solve the coupled equations in the g_β's becomes more transparent: i.e., one sees how many coupling terms in β' surrounding a given value of β have to be kept when actually solving the coupled equations.

The solution of the coupled equations is, however, beyond the scope of the present work. All we intend here is merely to understand the nature of the $V_{\beta\beta'}(y)$'s for the case of deuteron-complex nucleus scattering.

II. THEORY

Continuum-continuum transitions are caused in this formalism by the neutron-proton interaction $v_{np}(r)$, as well as by the interactions of the neutron and of the proton with the target nucleus, which—following earlier calculations[5]—is taken as an optical potential at a nucleon energy half the incident deuteron energy E_b:

$$V_N(\vec{r},\vec{R}) = V_{n-A}(r_n) + V_{p-A}(r_p) \ . \tag{2.1}$$

The neutron and proton radius vectors $\vec{r}_n$ and $\vec{r}_p$ are measured from the center of mass of the nucleus, which is considered infinitely heavy, and the coordinate vectors $\vec{r}$ and $\vec{R}$ are given as

$$\vec{r} = \vec{r}_n + \vec{r}_p, \quad \vec{R} = (\vec{r}_n + \vec{r}_p)/2 \ . \tag{2.2}$$

The polar angles of the directions $\hat{r}$ and $\hat{R}$ are denoted by θ_r, ϕ_r and θ_R, ϕ_R, respectively. The hyperspherical coordinates y and x are related to r and R by

$$r = 2^{1/2} y \sin x, \quad R = 2^{-1/2} y \cos x \ . \tag{2.3}$$

The five angles x, θ_r, ϕ_r, θ_R, and ϕ_R are the hyperspherical angles, which are the arguments of the hyperspherical harmonic $\phi_{\lambda l m_l M_2}(x, \theta_r, \phi_r, \theta_R, \phi_R)$. Then l, m_l are the angular momentum quantum numbers for the relative motion of neutron and proton, L, M_L the same for the relative motion of the neutron-proton center of mass with respect to the nucleus. λ comes in through the variable χ and has integer or half integer values with the lower limit λ_0:

$$\lambda_0 = (l+L)/2, \quad \lambda = \lambda_0, \ \lambda_0 + 1, \ldots \ . \tag{2.4}$$

λ is related to the eigenvalues of K^2, the "grand" angular momentum squared which is obtained, when transforming the total kinetic energy to the hyperspherical coordinates[6]:

$$T_r + T_R = -(\hbar^2/2m)\left[d^2/dy^2 + (5/y)d/dy - K^2/y^2\right] . \tag{2.5}$$

K^2 is the operator containing the five hyperspherical angles and has the eigenfunctions $\phi_{\lambda l L m_l M_L}$:

$$K^2 \phi_{\lambda l L m_l M_L} = 2\lambda(2\lambda+4)\phi_{\lambda l L m_l M_L} \ . \tag{2.6}$$

The last term in Eq. (2.5) gives rise to a centrifugal barrier of the form $2\lambda(2\lambda+4)/y^2$ in the hyperspherical space, which for potentials of finite range in y provides the desired cutoff in λ in an expansion of the wave function in hyperspherical harmonics. The latter can be written as

$$\phi_{\lambda l L m_l M_L} = F_{\lambda l L}(x) Y_{l m_l}(\theta_r \varphi_r) Y_{L M_L}(\theta_R \varphi_R) \ . \tag{2.7}$$

Here[2-4,6]

$$F_{\lambda l L}(x) = N_{\lambda l L} (\sin x)^l (\cos x)^L \, {}_2F_1\left[(l+L)/2 - \lambda, \ (l+L)/2 + \lambda + 2; \ l + \tfrac{3}{2}; \ \sin^2 x\right], \tag{2.8}$$

where ${}_2F_1$ are the usual hypergeometric functions. The $N_{\lambda l L}$ are normalization constants such that

$$\int_0^{\pi/2} F_{\lambda l L} F_{\lambda' l L} \sin^2 x \cos^2 x \, dx = \delta_{\lambda\lambda'} \ . \tag{2.9}$$

Some of the functions $F_{\lambda l L}$ are illustrated in Fig. 1.

The general ansatz for the wave function discussed in Ref. 1 consists of two parts, the first, ψ_1, describing—asymptotically—the elastic channel, the second, ψ_2, the breakup channel:

$$\psi_2(y, x, \hat{r}, \hat{R}) = y^{-5/2} \sum_{J,\alpha} g_\alpha^{(J)}(y) \phi_\alpha^{(J)}(x, \hat{r}, \hat{R}) \ , \tag{2.10}$$

where $\alpha = (\lambda, l, L)$, J is the total orbital angular

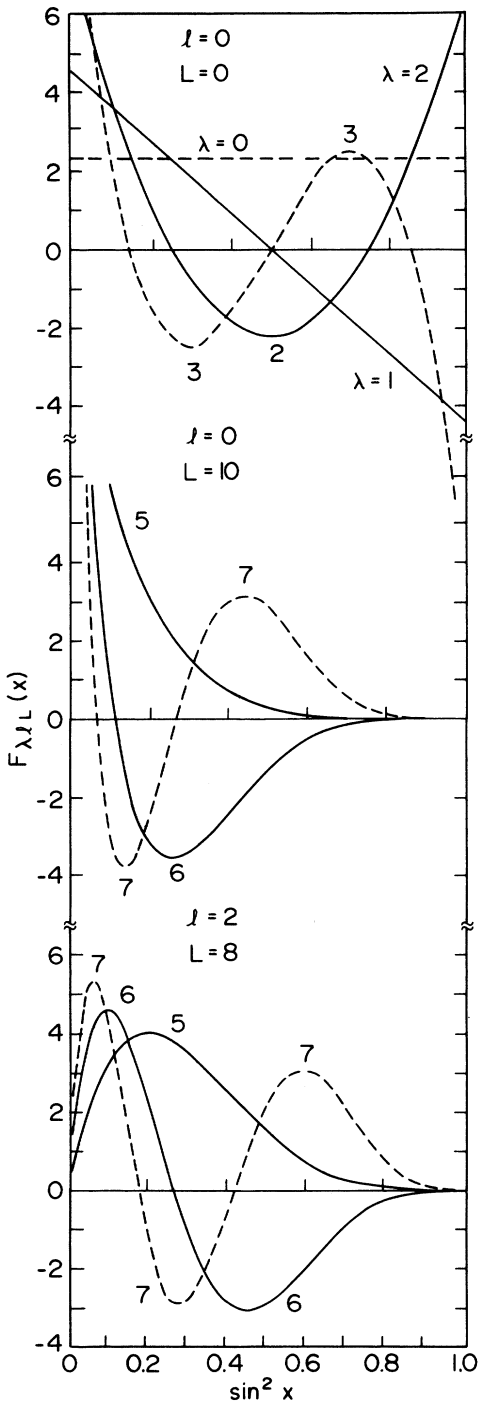

FIG. 1. The hyperspherical harmonics $F_{\lambda lL}(x)$ defined Eq. (2.8) as a function of $\sin^2 x$. The values of l and L are indicated on top of each panel. The values of λ are written next to each curve.

momentum, and

$$\phi_\alpha^{(J)} = \sum_{m_l, M_L} \langle JM_J | \, l L m_l M_L \rangle \, \phi_{\lambda l L m_l M_L} . \quad (2.11)$$

The bracket symbols are the usual Clebsch-Gordan coefficients.[7] M_J is suppressed from the set of quantum numbers α, since it is not changed in the interaction.

From the Schrödinger equation one obtains two sets of equations[1]; the set which is of interest here is

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dy^2} - \frac{\Lambda(\Lambda+1)}{y^2}\right] - E\right\} g_\alpha^{(J)} + \sum_{\alpha'} \langle \phi_\alpha V_N \phi_{\alpha'} \rangle = -Z_\alpha^{(J)}(y) , \quad (2.12)$$

where $\Lambda = 2\lambda + \frac{3}{2}$.

The term $Z_\alpha^{(J)}$ in Eq. (2.12) is the source term for the breakup channel, that means the coupling of ψ_2 to ψ_1. The continuum-continuum coupling terms, that is, the matrix elements $\langle \phi_\alpha V_N \phi_{\alpha'} \rangle$, are the quantities to be discussed in this paper. In the asymptotic region they decrease as y^{-3} and hence, at large values of y, the asymptotic behavior of g is that of an outgoing cylindrical Bessel function:

$$g_\alpha^{(J)}(y) \approx S_\alpha^{(J)} (\pi K y/2)^{1/2} H^{(1)}_{2\lambda+2}(Ky)/2i , \quad (2.13)$$

where $K = (mE/\hbar^2)^{1/2}$ and where S_α^J is the value of the S-matrix element for deuteron-breakup transitions. For small values of y one has

$$g^{(J)}_{\lambda lL}(y) \sim M_{\lambda lL} y^{2\lambda+5/2} , \quad (2.14)$$

where $M_{\lambda lL}$ is a normalization factor.

III. THE BREAKUP TO BREAKUP COUPLING TERMS $\langle \phi_\alpha V_N \phi_{\alpha'} \rangle$

These matrix elements are quadruple integrals, whose evaluation is facilitated by expanding $V_N(\vec{r}, \vec{R})$ in hyperspherical harmonics

$$V_N(\vec{r}, \vec{R}) = \sum_{\bar{\lambda}\bar{l}} V_{\bar{\lambda}\bar{l}}(y) F_{\bar{\lambda}\bar{l}\bar{l}}(x) (2\bar{l}+1)^{1/2} P_{\bar{l}}(\cos\delta) , \quad (3.1)$$

where δ is the angle between $\hat{r}$ and $\hat{R}$.

In what follows, equality of the neutron-nucleus and proton-nucleus optical potentials is assumed, deferring the treatment of the Coulomb potential to a later stage. Then one has with $\epsilon = 2 \sin x \cos x \cos\delta$

$$V_N(\vec{r}, \vec{R}) = V[y(1+\epsilon)^{1/2}] + V[y(1-\epsilon)^{1/2}/2] . \quad (3.2)$$

As (3, 2) is invariant under the replacement of δ by $(\pi - \delta)$ only even values of $\bar{l}$ contribute to (3.1). A similar argument shows that $\bar{\lambda}$ must be even. Both results follow, of course, from the general derivation of $V_{\bar{\lambda}\bar{l}}$, for which the result is (Appendix A)

$$V_{\bar{\lambda}\bar{l}}(y)=K_{\bar{\lambda}\bar{l}}\,V_{\bar{\lambda}}(y);\quad \bar{\lambda}\text{ and }\bar{l}\text{ even},\tag{3.3}$$

with

$$V_{\bar{\lambda}}=\pi^{-1/2}(\bar{\lambda}+1)^{-1}\int_0^{\pi}\sin\alpha\,\sin[(\bar{\lambda}+1)\alpha]\times V_{n-A}(y\sin\tfrac{1}{2}\alpha)\,d\alpha\ ,\tag{3.4}$$

and

$$K_{\bar{\lambda}\bar{l}}=(-)^{(\bar{\lambda}-\bar{l})/2}\,2^{\bar{l}}(2\bar{l}+1)^{1/2}[(\bar{\lambda}+\bar{l})/2]!\,/[(\bar{\lambda}-\bar{l})/2]!\times[(\bar{\lambda}+1)(\bar{\lambda}-\bar{l})!\,/(\bar{\lambda}+\bar{l}+1)!]^{1/2}\ .\tag{3.5}$$

Inserting Eq. (3.3) into Eq. (3.1) and carrying out the sum over $\bar{l}$ one obtains an expression of the type given by Eq. (1.4).

For large values of y, $V_{\bar{\lambda}}$ decreases as y^{-3}, like

$$V_{\bar{\lambda}}(y)\approx y^{-3}8\pi^{-1/2}\int_0^{\infty}dr_n\,r_n^{\,2}V_{n-A}(r_n)\ .\tag{3.6}$$

The fact that for large values of y there remain regions in phase space where V_N does not vanish is not in contradiction with the result that $\langle\phi_\alpha V_N\phi_{\alpha'}\rangle$ goes to zero as y^{-3}. In fact, the size of these phase space regions shrinks like $(a/y)^3$.

If V_N is not to vanish as y goes to infinity, then $\vec{r}/2$ and $\vec{R}$ must be such that r_n or r_p remain small, within the region of the range a of the nucleon-nucleus optical potential. This means that $\vec{r}/2$ and $\vec{R}$ must have approximately the same magnitude, so that they can nearly cancel in the expression $\vec{r}_n=\vec{R}+\vec{r}/2$ or $\vec{r}_p=\vec{R}-\vec{r}/2$. In this case x is close to $\pi/4$ and the angle δ between the directions of $\vec{R}$ and $\vec{r}$ has to lie in a narrow region around zero or π. Under these conditions it can be shown that the allowed regions of δ and x decrease like $(a/y)^2$ and a/y, respectively. Hence the integration over the angular phase space decreases like $(a/y)^3$ as $y\to\infty$, in agreement with the result of Eq. (3.6).

A list of values of $K_{\bar{\lambda}\bar{l}}$ is given in Table I. For a given value of $\bar{\lambda}$ the values of $\bar{l}$ run from 0 to $\bar{\lambda}$. The largest values of K occur for $\bar{l}=\bar{\lambda}$. The potentials $V_{\bar{\lambda}}(y)$ are shown in Fig. 1 for a particular example which corresponds approximately to the target of Ni. The nucleon-nucleus optical potential has the Woods-Saxon form

$$V_{n-A}(r_n)=V_0/(1+e)+i\,4W_0e'/(1+e')^2\ ,\quad e=\exp[(r_n-R)/a]\ ,\quad e'=\exp[(r_n-R')/a']\ ,\quad V_0=-50\text{ MeV},\quad W_0=-15\text{ MeV},\quad R=R'=5\text{ fm},\quad a=a'=0.65\text{ fm}\ .\tag{3.7}$$

Results for $V_\lambda(y)$ are illustrated in Fig. 2. The potentials $V_{\bar{\lambda}}(y)$ decrease in magnitude with increasing $\bar{\lambda}$, they all have the same asymptotic form (3.6); the number of oscillations for small values of y increases with $\bar{\lambda}$.

With the exception of $\bar{\lambda}=0$, the potentials $V_{\bar{\lambda}}$ provide the coupling potentials between different continuum states. The potential $V_0(y)$ is a diagon-

TABLE I. Values of $K_{\bar{\lambda}\bar{l}}$ defined in Eq. (3.5).

$\bar{\lambda}$ \ $\bar{l}$	0	2	4	6	8	10
0	1					
2	−1	+2.828				
4	1	−2.390	4.276			
6	−1	+2.309	−3.411	+5.571		
8	1	−2.278	3.218	−4.315	6.770	
10	−1	+2.265	−3.138	+4.008	−5.151	+7.901

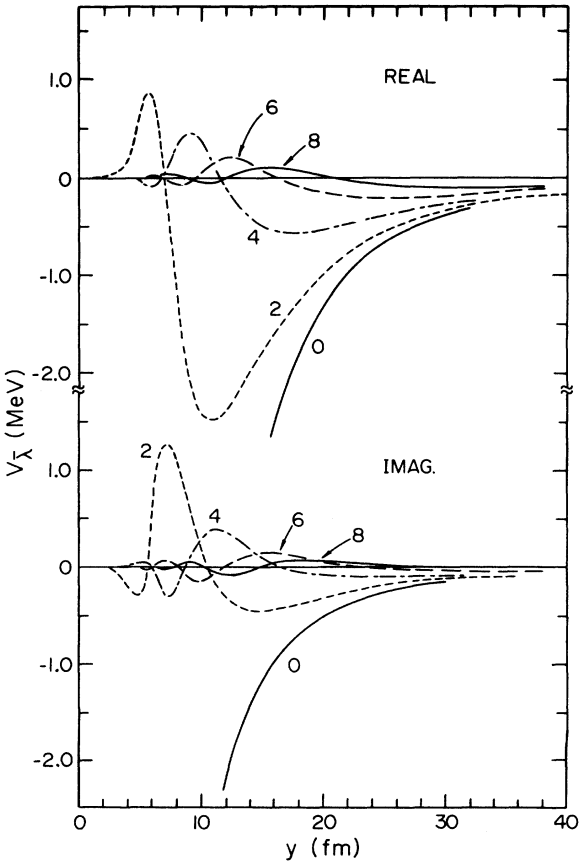

FIG. 2. The potentials $V_{\bar{\lambda}}(y)$, which are defined in Eqs. (3.3). The values of $\bar{\lambda}$ are indicated next to each curve. According to Eq. (1.4), the $V_{\bar{\lambda}}$'s are the HSH components of the sum of the proton-nucleus and the neutron-nucleus optical potentials, V_N defined in Eq. (1.2). The parameters for the optical potentials are given in Eq. (3.7). The functions Φ in Eq. (1.4) can be obtained in terms of Eqs. (3.1) and (3.3).

al distorting potential which is the same in all channels.

The coupling potentials $\langle \phi_\alpha V_N \phi_{\alpha'} \rangle$ are obtained by inserting the expansion (3.1) for V_N and (2.11) for ϕ_α

$$\langle \phi_\alpha V_N \phi_{\alpha'} \rangle = \sum_{\bar{\lambda}} M^{\bar{\lambda} J}_{\lambda l L, \lambda' l' L'} V_{\bar{\lambda}}(y) , \tag{3.8}$$

where

$$M^{\bar{\lambda} J}_{\lambda l L, \lambda' l' L'} = \sum_{\bar{l}=0}^{\bar{\lambda}} C^{\bar{l} J}_{l L, l' L'} F^{\bar{\lambda}\bar{l}}_{\lambda l L, \lambda' l' L'} K_{\bar{\lambda}\bar{l}} \tag{3.9}$$

is a matrix element of the type given by Eq. (1.5), where

$$C^{\bar{l} J}_{l L, l' L'} = (-)^{J+\bar{l}} \begin{Bmatrix} l & l' & \bar{l} \\ L' & L & J \end{Bmatrix} \begin{pmatrix} \bar{l} & L & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{l} & l & l' \\ 0 & 0 & 0 \end{pmatrix} \times \{(2\bar{l}+1)(2l+1)(2l'+1)(2L'+1)(2L+1)\}^{1/2} , \tag{3.10}$$

and where

$$F^{\bar{\lambda}\bar{l}}_{\lambda l L, \lambda' l' L'} = \int F^{*}_{\lambda l L}(x) F_{\bar{\lambda}\bar{l}}(x) F_{\lambda' l' L'}(x) d\tau_x , \tag{3.11}$$

with

$$d\tau_x = \sin^2 x \cos^2 x \, dx .$$

The integration in (3.11) can be carried out and leads to a triple sum which can be evaluated by computer. The quantum numbers λ, λ', and $\bar{\lambda}$ satisfy the triangular condition (Appendix B). Furthermore, $2\lambda + 2\lambda' + 2\bar{\lambda}$ has to be even. This follows as

$$\lambda = (l+L)/2, \ (l+L)/2+1 \cdots$$

$$\lambda' = (l'+L')/2, \ (l'+L')/2+1 \cdots$$

$\bar{\lambda}$ even, $L+L'$ even, $l+l'$ even.

The latter two conditions follow from Eq. (3.10).

The diagonal distorting potential [term $\bar{\lambda}=0$ in the sum (3.8)]

$$V(y) = 4\pi^{-1/2} V_0(y) \tag{3.12}$$

is illustrated in Fig. 3 for the numerical examples given by Eqs. (3.7). The dashed curves illustrate the distorting potentials which are encountered in the formulation of the discretized "k by k" coupled equations for approximately the same nucleon-nucleus optical potential as that given by Eq. (3.7), in which case R is identified with $y/\sqrt{2}$. One sees that $V(y)$ is larger, both in magnitude and in range, compared to the k by k potentials.

As a result of Eqs. (3.8)–(3.11), Eq. (2.12) can

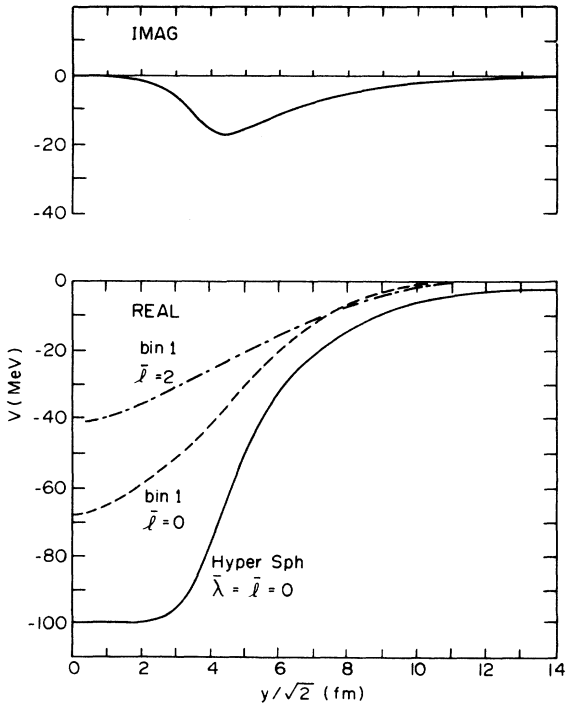

FIG. 3. The distorting potential $V(y)$, defined in Eq. (3.12). The real part of $V(y)$ is compared with the central distorting potentials which appear in the k by k method described in the text. Momentum bin 1 corresponds to a breakup energy interval from 0 to 10 MeV. The x axis represents $y/\sqrt{2}$, since R equals $y/\sqrt{2}$ when $r \ll R$, and the k by k potentials are defined in terms of R.

be cast in the form

$$\left\{ -\frac{\hbar^2}{2m}\left[\frac{d^2}{dy^2} - \frac{\Lambda(\Lambda+1)}{y^2}\right] + V(y) - E \right\} g^{(J)}_{\alpha}(y) + \sum_{\alpha'} V^{(J)}_{\alpha\alpha'}(y) g^{(J)}_{\alpha'}(y) = -Z^{(J)}_{\alpha}(y) , \tag{3.13}$$

where

$$V^{(J)}_{\alpha\alpha'}(y) = \sum_{\bar{\lambda}=2,4,\ldots} M^{\bar{\lambda} J}_{\lambda l L, \lambda' l' L'} V_{\bar{\lambda}}(y) . \tag{3.14}$$

Equation (3.13) can be brought to a more elegant form by the introduction of the column vectors $\hat{g}^{J}_{\lambda}$:

$$\hat{g}^{(J)}_{\lambda} = \begin{bmatrix} g^{(J)}_{\lambda l_1 L_1} \\ g^{(J)}_{\lambda l_2 L_2} \\ \cdot \\ \cdot \\ \cdot \\ g^{(J)}_{\lambda l_n L_n} \end{bmatrix} . \tag{3.15}$$

The entries l_i, L_i are all the combinations which

belong to the same values of λ and J. The matrix of the elements $M^{\bar{\lambda}J}_{\lambda l L,\,\lambda' l' L'}$ is called $\hat{M}^{\bar{\lambda}J}_{\lambda\lambda'}$ and the column vector of the inhomogeneous term is $\hat{Z}^{J}_{\lambda}(y)$. Then Eq. (3.13) can be written in the succinct form

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dy^2}-\frac{\Lambda(\Lambda+1)}{y^2}\right]+V(y)-E\right\}\hat{g}^{(J)}_{\lambda}(y)$$

$$+\sum_{\lambda'}\hat{\hat{V}}^{(J)}_{\lambda\lambda'}(y)\,\hat{g}^{(J)}_{\lambda'}(y)+\hat{Z}^{(J)}_{\lambda}(y)=0\ , \qquad (3.16)$$

where

$$\hat{\hat{V}}^{(J)}_{\lambda\lambda'}(y)=\sum_{\bar{\lambda}=2,4,\ldots}\hat{M}^{\bar{\lambda}J}_{\lambda\lambda'}\,V_{\bar{\lambda}}(y)\ . \qquad (3.17)$$

The main usefulness of Eq. (3.16) is to bring out the central role of the quantum numbers λ. The Green's functions $G_{\lambda}(E,y,y')$ which resolve the square bracket of Eq. (3.16) are the same for all elements of the vector $\hat{g}_{\lambda}$.

Whether Eq. (3.16) can be solved by iteration in a series in G_{λ} depends on the magnitude of the coupling potentials $\hat{\hat{V}}^{J}_{\lambda\lambda'}$ as will now be examined. The values of $V_{\bar{\lambda}}(y)$ shown in Fig. 2 are sufficiently small so that the viability of performing an iteration depends on the size of the coefficients $M^{\bar{\lambda}J}_{\lambda\lambda'}$. A numerical example of the elements of $M^{\bar{\lambda}J}_{\lambda\lambda'}$ is given in Appendix C for the case $J=4$, for values of λ and λ' of 2, 3, and 4, and for values of $\bar{\lambda}$ of 2 and 4. For $\bar{\lambda}=2$ the values of the elements of $M^{\bar{\lambda}J}_{\lambda\lambda'}$ in this example are not larger than 4.1 (which occurs for $\lambda=2$, $\lambda'=3$). The corresponding maximum value of $M^{2,4}_{\lambda\lambda'}\cdot V_2(y)$ is 10.25 MeV. This magnitude is similar in size to the one found for the coupling between continuum states in the k by k formulation, which was too large for an iteration to converge. As a result, the iterative solution of Eq. (3.16) is not recommended. On the other hand, the contribution from the potentials $V_{\bar{\lambda}}(y)$ with $\bar{\lambda}>2$ as well as the contributions to $\hat{\hat{V}}$ from large values of y are likely to be amenable to a treatment by perturbation.

The largest matrix element in $M^{4,4}_{\lambda\lambda'}$ has the value of 5.3 and occurs for $\lambda=2$, $\lambda'=4$. The corresponding maximum value of $M^{44}_{\lambda\lambda'}\cdot V_4(y)$ is 3.18 MeV, which is sufficiently small for a distorted Green's function iteration to converge.

The coupling between the various λ values decreases as the difference between λ and λ' increases. This fact is not evident from inspection of the matrix elements $\hat{M}^{\bar{\lambda}}_{\lambda\lambda'}$ given in Appendix C because this example does not have enough values of λ and λ'. However, examination of the coefficients $F^{\bar{\lambda}\bar{l}}_{\alpha\alpha'}$ shows these quantities to depend most strongly on the difference between the number of nodes n

$$n=\lambda-(l+L)/2 \qquad (3.18)$$

of each $F_{\lambda lL}$. The coefficient $F^{\bar{\lambda}\bar{l}}_{\alpha\alpha'}$, Eq. (3.11), is conveniently expressed as

$$F^{\bar{\lambda}\bar{l}}_{\alpha\alpha'}=\langle\,\lambda l L\,|\,\bar{\lambda}\bar{l}\,|\,\lambda' l' L'\,\rangle\ . \qquad (3.19)$$

For example, when $l=l'=2$, $L=L'=6$, $\bar{l}=0$, $\lambda=4$, and $\bar{\lambda}$ as well as λ' are variable, then the plot of $F^{\bar{\lambda}\bar{l}}_{\alpha\alpha'}$ as a function of $n'-\bar{n}$ shows that points for different values of $\bar{\lambda}$ and λ' all lie roughly on the same universal curve. This curve is illustrated in Fig. 4.

This discussion shows that the n' has to differ approximately by 6 units from $\bar{n}$ before the values of $F_{\alpha\alpha'}$ decreases by a factor of 3 or 4. Inspection of other cases reveals a similar type of mean free path in λ space of approximately 6.

In conclusion, the various (l,L) components of the continuum functions $\hat{g}^{(J)}_{\lambda}$ are coupled strongly to each other, and so are the various λ components. As a result, the flux "injected" into an (α,J) component of the ψ_2 breakup space via the coupling to the bound space [through the inhomogeneous term $Z^{(J)}_{\alpha}$ in Eq. (3.13)] is expected to propagate strongly to other (α',J) values. The distance of propagation is governed by the change in the parameter $\lambda-(l+L)/2$, which could change by as much as 6 units.

IV. SUMMARY AND CONCLUSIONS

The main content of the present work is to provide the mathematical background needed for the

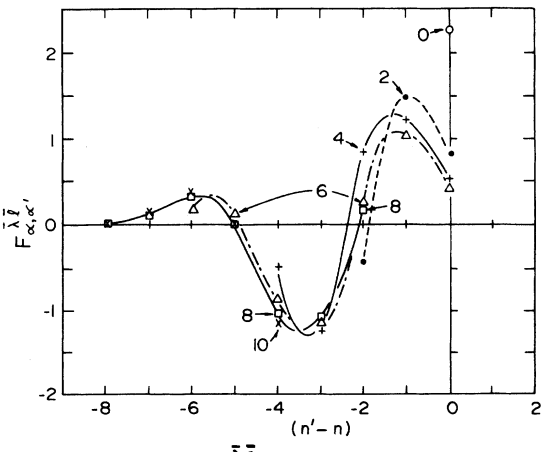

FIG. 4. The quantity $F^{\bar{\lambda}\bar{l}}_{\lambda lL,\,\lambda' l' L'}$ plotted as a function of the difference between the number of nodes $n'-\bar{n}$. Here F represents the integral of the product of three hyperspherical harmonics $F_{\lambda lL}(x)$, as is defined in Eq. (3.11), and the corresponding number of nodes is given by Eq. (3.18), and $\bar{n}=\bar{\lambda}-\bar{l}$. The F's contain the λ and λ' dependence of the coefficients $M^{\bar{\lambda}J}_{\lambda lL,\,\lambda' l' L'}$ according to Eq. (3.9). The latter, which are tabulated in Appendix D, are needed in the calculation of the overlap integrals $\langle\phi_{\alpha}V_N\phi_{\alpha'}\rangle$, Eq. (3.8).

formulation of deuteron breakup in terms of hyperspherical coordinates. The sum of the proton-nucleus and neutron-nucleus optical potentials is expanded in a basis of hyperspherical harmonics functions as a first step for the calculations of the matrix element of this potential between initial and final HSH functions. The next step for the calculation of this matrix element then involves the integral of the product of three HSH functions. This integral is expressed in terms of a triple sum involving factorials and gamma functions in a form suitable for numerical evaluation. The above chain of steps differs from a method developed by Raynal and co-workers.[3,1] The present method suggests that the coupled equations between the breakup functions $g^{(J)}_{\lambda lL}(y)$ be written in a form of coupled matrix equations which underlines the dominant role played by the HS quantum number λ. The matrices in this formulation act on column vectors $\hat{g}^{(J)}_{\lambda}(y)$ which are formed by the functions $g^{(J)}_{\lambda lL}(y)$ for all the combinations of l and L which belong both to the given value of λ and of the total angular momentum J.

Numerical evaluation for the case of 21.6 MeV deuterons incident on a nucleus of nickel shows that the two steps needed in the calculation of the matrix elements discussed above can be carried out without much loss of accuracy. The size of the coupling potentials (10 to 20 MeV) is so large that iterative solution of the coupled equations between the functions $g^{(J)}_{\lambda lL}(y)$ is not recommended. By contrast, the coupling between deuteron space (ψ_1) and breakup space (ψ_2) evaluated in Ref. 1 appears to be sufficiently weak, of the order of a few MeV, so that iterations for going from ψ_1 to ψ_2 and back promise to converge.

The strong coupling between the function $g^{(J)}_{\lambda lL}(y)$ shows that many values of l (the relative neutron-proton orbital momentum values for the breakup states) are expected to be required in the description of breakup space, regardless of the formulation in terms of which breakup is expressed.

TABLE II. Values for $N_{\lambda lL}$.

$N_{\lambda lL}$, $l=0$

n [a]	$L=0$	$L=1$	$L=2$	$L=5$	$L=10$	$L=19$
$n=0$	2.257	3.191	4.037	6.286	9.531	14.604
1	4.513	5.528	6.474	9.058	12.880	18.945
2	6.770	7.818	8.809	11.564	15.699	22.338
3	9.027	10.092	11.112	13.972	18.316	25.352
4	11.284	12.361	13.398	16.332	20.828	28.163
5	13.541	14.425	15.675	18.665	23.277	30.849
6	15.797	16.888	17.947	20.980	25.685	33.451
7	18.054	19.149	20.216	23.283	28.065	35.994
8	20.311	21.409	22.483	25.577	30.424	38.492
9	22.568	23.669	24.747	27.865	32.767	40.957
10	24.824	25.928	27.010	30.148	35.097	43.394

$N_{\lambda lL}$, $l=2$

n [a]	$L=0$	$L=5$	$L=10$	$L=19$
$n=0$	4.037	27.54	66.40	169.64
1	15.105	67.03	146.52	350.47
2	37.000	126.21	256.37	577.22
3	73.336	208.83	396.79	853.81
4	127.725	318.51	574.42	1183.64
5	203.779	458.88	791.84	1570.15
6	305.108	633.55	1052.66	2016.79
7	435.324	846.13	1360.47	2527.07
8	598.037	1100.24	1718.87	3104.50
9	796.858	1399.48	2131.47	3752.65
10	1035.397	1747.47	2601.87	4475.06

$N_{\lambda lL}$, $l=4$

n [a]	$L=0$	$L=2$	$L=10$	$L=19$
$n=0$	5.572	21.53	259.2	1047.0
1	30.176	85.35	760.5	2818.0
2	101.774	238.37	1685.3	5789.9
3	266.51	548.93	3227.21	10368.5
4	593.44	1112.83	5624.63	17022.3
5	1179.15	2057.99	9166.45	26289.8
6	2152.30	3548.98	14196.5	38785.0
7	3678.25	5791.65	21118.05	55201.7
8	5963.62	9037.69	30398.80	76319.0
9	9260.87	13589.23	42575.02	103304.0
10	13872.93	19803.46	58256.42	136219.0

[a] Defined in Eq. (A2).

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APPENDIX A

Values of the normalization constant $N_{\lambda lL}$ defined in Eq. (2.8):

$$N_{\lambda lL}=2(\lambda+1)^{1/2}/\Gamma(l+\tfrac{3}{2}) \times\left[\frac{\Gamma(n+l+\tfrac{3}{2})(L+l+n+1)!}{\Gamma(n+L+\tfrac{3}{2})\,n!}\right]^{1/2}, \quad \text{(A1)}$$

$$n=\lambda-(l+L)/2\,. \tag{A2}$$

Specific values for $N_{\lambda lL}$ are given in Table II.

APPENDIX B: THE POTENTIALS $V_{\bar{\lambda}\bar{l}}(y)$

Expanding the potential $V[y(1-\epsilon)^{1/2}/2]$ in Eq. (3.2), abbreviated as V below, in the form (3.1), one obtains for $V_{\bar{\lambda}\bar{l}}(y)$:

$$V_{\bar{\lambda}\bar{l}}(y)=(2\bar{l}+1)^{1/2}2^{-1}N_{\bar{l}\bar{l}\bar{\lambda}}\int_0^{\pi}d\delta\,\sin\delta P_{\bar{l}}(\cos\delta)\int_0^{\pi/2}d\tau_\chi(\sin\chi\cos\chi)^{\bar{l}}{}_2F_1(\bar{l}-\bar{\lambda},\bar{l}+\bar{\lambda}+2;\bar{l}+\tfrac{3}{2};\sin^2\chi)$$
$$\times V[(y/2)(1-\sin2\chi\cos\delta)^{1/2}]\,. \tag{B1}$$

The following transformation will lead to a simplification (reduction to a single integral in the end):

$$\cos2\chi=\sin\alpha\sin\beta,\quad \sin2\chi\cos\delta=\cos\alpha,\quad \sin2\chi\sin\delta=\sin\alpha\cos\beta\,. \tag{B2}$$

$V_{\bar{\lambda}\bar{l}}$ is now

$$V_{\bar{\lambda}\bar{l}}=(2\bar{l}+1)^{1/2}(16)^{-1}N_{\bar{l}\bar{l}\bar{\lambda}}\int_0^{\pi}d\alpha\int_{-\pi/2}^{+\pi/2}d\beta\,\sin^2\alpha\cos\beta$$
$$\times P_{\bar{l}}[\cos\alpha(1-\sin^2\alpha\sin^2\beta)^{-1/2}]2^{-\bar{l}}(1-\sin^2\alpha\sin^2\beta)^{\bar{l}/2}$$
$$\times{}_2F_1[\bar{l}-\bar{\lambda},\bar{l}+\bar{\lambda}+2;\bar{l}+\tfrac{3}{2};(1-\sin\alpha\sin\beta)/2]\,. \tag{B3}$$

The following discussion shows how to reduce (B3) to a single integral: The kinetic energy operator for three particles for total orbital angular momentum 0 is

$$T_r+T_R=-\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial y^2}+\frac{5}{y}\frac{\partial}{\partial y}+\frac{1}{y^2}\left[\frac{\partial^2}{\partial\chi^2}+4\frac{\cos2\chi}{\sin2\chi}\frac{\partial}{\partial\chi}+\frac{4}{\sin^2\!2\chi}\left(\frac{\delta^2}{\partial\delta^2}+\frac{\cos\delta}{\sin\delta}\frac{\partial}{\partial\delta}\right)\right]\right\}. \tag{B4}$$

The eigenfunctions of the χ,δ dependent operator in (B4) are the functions $F_{\bar{\lambda}\bar{l}\bar{l}}(\chi)\cdot P_{\bar{l}}(\cos\delta)$ employed in the expansion of V. Performing the transformation (B2) one has

$$T_r+T_R=-\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial y^2}+\frac{5}{y}\frac{\partial}{\partial y}+\frac{1}{y^2}\left[\frac{\partial^2}{\partial\alpha^2}+2\frac{\cos\alpha}{\sin\alpha}\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\left(\frac{\partial^2}{\partial\beta^2}-\frac{\sin\beta}{\cos\beta}\frac{\partial}{\partial\beta}\right)\right]\right\}. \tag{B5}$$

The α,β dependent operator has the eigenfunctions

$$P_l(\sin\beta)N_{l\bar{l}\bar{\lambda}}2^{-l}(\sin\alpha)^l{}_2F_1[l-\bar{\lambda},l+\bar{\lambda}+2;l+\tfrac{3}{2};\sin^2(\alpha/2)]\,. \tag{B6}$$

The functions $F_{\bar{\lambda}\bar{l}\bar{l}}(\chi)P_l(\cos\delta)$ are linear combinations of (B6) with the same $\bar{\lambda}$:

$$N_{\bar{l}\bar{l}\bar{\lambda}}P_{\bar{l}}(\cos\delta)(\sin\chi\cos\chi)^{\bar{l}}{}_2F_1(\bar{l}-\bar{\lambda},\bar{l}+\bar{\lambda}+2;\bar{l}+\tfrac{3}{2};\sin^2x)$$
$$=\sum_{l'}a_{l'}N_{l'\bar{l}\bar{\lambda}}P_{l'}(\sin\beta)2^{-l'}(\sin\alpha)^{l'}{}_2F_1(l'-\bar{\lambda},l'+\bar{\lambda}+2;l'+\tfrac{3}{2};\sin^2(\alpha/2))\,. \tag{B7}$$

Only the term $l'=0$ in (B7) will contribute to the integral (B3). So only a_0 is needed, which is found from (B7) by setting $\alpha=0$ ($\sin2\chi=1$, $\cos\delta=1$). One finds

$$a_0=(N_{\bar{l}\bar{l}\bar{\lambda}}/N_{0\bar{l}\bar{\lambda}})\pi^{1/2}2^{-\bar{l}}\Gamma(\bar{l}+\tfrac{3}{2})\{\Gamma[(\bar{l}-\bar{\lambda})/2+\tfrac{1}{2}]\Gamma[(\bar{l}+\bar{\lambda})/2+\tfrac{3}{2}]\}^{-1} \tag{B8}$$

because[8]

$${}_2F_1(\bar{l}-\bar{\lambda},\bar{l}+\bar{\lambda}+2;\bar{l}+\tfrac{3}{2};\tfrac{1}{2})=\pi^{1/2}\Gamma(\bar{l}+\tfrac{3}{2})\{\Gamma[(\bar{l}-\bar{\lambda})/2+\tfrac{1}{2}]\Gamma(\bar{l}+\bar{\lambda})/2+\tfrac{3}{2}]\}^{-1}\,. \tag{B9}$$

The following replacements are now performed in (B7) and (B8):

$$N_{\bar{l}\bar{l}\bar{\lambda}}=[2(\bar{\lambda}+1)^{1/2}/\Gamma(\bar{l}+\tfrac{3}{2})][(\bar{\lambda}+\bar{l}+1)!/(\bar{\lambda}-\bar{l})!]^{1/2},$$
$${}_2F_1(-\bar{\lambda},\bar{\lambda}+2;\tfrac{3}{2};\sin^2\alpha/2)=\sin[(\bar{\lambda}+1)\alpha]/(\bar{\lambda}+1)\sin\alpha,$$
$$\Gamma(n+\tfrac{1}{2})=(2n)!\pi^{1/2}/(2^{2n}n!), \tag{B10}$$
$$\Gamma(-n+\tfrac{1}{2})=n!2^{2n}(-)^n\pi^{1/2}/(2n)!\,.$$

With (B10) one obtains

$$V^{-}_{\bar{\lambda}\bar{l}}(y)=[(1+(-)^{\bar{\lambda}-\bar{l}})/2]\pi^{-1/2}(-)^{(\bar{\lambda}-\bar{l})/2}2^{\bar{l}-1}[(2\bar{l}+1)/(\bar{\lambda}+1)]^{1/2}[(\bar{\lambda}-\bar{l})!/(\bar{\lambda}+\bar{l}+1)!]^{1/2}$$

$$\times\{[(\bar{\lambda}+\bar{l})/2]![(\bar{\lambda}-\bar{l})/2]!\}\int_0^{\pi}d\alpha\sin\alpha\sin[(\bar{\lambda}+1)\alpha]V(y\sin(\alpha/2))\,. \tag{B11}$$

For $V^{+}_{\bar{\lambda}\bar{l}}(y)$ one obtains the same expression with $V[y\sin(\alpha/2)]$ replaced by $V[y\cos(\alpha/2)]$. As a consequence one obtains for $V_{\bar{\lambda}\bar{l}}(y)$ of Eq. (3.1)

$$V_{\bar{\lambda}\bar{l}}(y)=V^{-}_{\bar{\lambda}\bar{l}}(y)[1+(-)^{\bar{\lambda}}][1+(-)^{\bar{l}}]/2\,. \tag{B12}$$

APPENDIX C: INTEGRALS OVER THREE FUNCTIONS $F_{\lambda lL}(\chi)$

The integrals in question are the ones in Eq. (3.11) which, apart from normalization factors, are

$$I=\int_0^{\pi/2}d\tau_\chi(\sin\chi)^{l+l'+\bar{l}}(\cos\chi)^{L+L'+\bar{l}}{}_2F_1[(l+L)/2-\lambda,(l+L)/2+\lambda+2;l+\tfrac{3}{2};\sin^2\chi]$$

$$\times{}_2F_1[(l'+L')/2-\lambda',(l'+L')/2+\lambda'+2;l'+\tfrac{3}{2};\sin^2\chi\,{}_2F_1(\bar{l}-\bar{\lambda},\bar{l}+\bar{\lambda}+2;\bar{l}+\tfrac{3}{2};\sin^2\chi)\,. \tag{C1}$$

These $l,l',\bar{l}$ are connected by the triangular condition, as are $L,L',\bar{l}$ [see Eq. (3.10)]. $l+l'+\bar{l}$ and $L+L'+\bar{l}$ are even. With $\sin^2\chi=(x+1)/2$ and[9]

$$(-)^n\binom{n+\alpha}{n}{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;(x+1)/2)=[(-)^n/2^n n!](x+1)^{-\alpha}(1-x)^{-\beta}(d/dx)^n[(x+1)^{n+\alpha}(1-x)^{n+\beta}] \tag{C2}$$

one finds

$$I=2^{-(l+l'+L+L'+2\bar{l})/2-3}(2l+1)!!(2l'+1)!!(2\bar{l}+1)!!\,[(2\lambda-L+1)!!(2\lambda'-L'+1)!!(2\bar{\lambda}-\bar{l}+1)!!]^{-1}$$

$$\times\int_{-1}^{+1}dx(1+x)^{-1-(l+l'+\bar{l})/2}(1-x)^{-1-(L+L'+\bar{l})/2}\{(d/dx)^{\lambda-(l+L)/2}[(1+x)^{\lambda+(l-L+1)/2}(1-x)^{\lambda+(L-l+1)/2}\}$$

$$\times\{(d/dx)^{\lambda'-(l'+L')/2}[(1+x)^{\lambda'+(l'-L'+1)/2}(1-x)^{\lambda'+(L'-l'+1)/2}]\}\{(d/dx)^{\bar{\lambda}-\bar{l}}[(1+x)^{\bar{\lambda}+(1/2)}(1-x)^{\bar{\lambda}+(1/2)}]\}\,. \tag{C3}$$

The existence of the triangular condition for $\lambda,\lambda',\bar{\lambda}$ can be proved in the following way: One integrates by parts several times in such a way that the $(d/dx)^{\bar{\lambda}-\bar{l}}$ is removed. One obtains

$$\int_{-1}^{+1}dx(1+x)^{-1-(l+l'+\bar{l})/2}\ldots$$

$$=(-)^{\bar{\lambda}-\bar{l}}\int_{-1}^{+1}dx(1-x^2)^{\bar{\lambda}+1/2}(d/dx)^{\bar{\lambda}-\bar{l}}\{(1+x)^{-1-(l+l'+\bar{l})/2}(1-x)^{-1-(L+L'+\bar{l})/2}$$

$$\times(d/dx)^{\lambda-(l+L)/2}[(1+x)^{\lambda+(l-L+1)/2}(1-x)^{\lambda+(L-l+1)/2}]$$

$$\times(d/dx)^{\lambda'-(l'+L')/2}[(1+x)^{\lambda'+(l'-L'+1)/2}(1-x)^{\lambda'+(L'-l'+1)/2}]\} \tag{C4}$$

The expression in curly braces is a polynomial in x; the highest power of which is $\lambda+\lambda'-\bar{l}$. The $(\bar{\lambda}-\bar{l})$'th derivative of this polynomial is nonzero only when

$$\bar{\lambda}-\bar{l}\leq\lambda+\lambda'-\bar{l}\ \text{or}\ \bar{\lambda}\leq\lambda+l'\,. \tag{C5}$$

On account of the complete symmetry of (C3) in $\lambda,\lambda',\bar{\lambda}$ one also has

$$\lambda\leq\lambda'+\bar{\lambda},\quad\lambda'\leq\lambda+\bar{\lambda}, \tag{C6}$$

which proves the triangular condition.

TABLE III. Values of $M^{\bar{\lambda}J}_{\lambda lL,\lambda' l' L'}$.

$\bar{\lambda}=2,\ J=4,\ \lambda=\lambda'=2$

	(0,4)	(2,2)	(4,0)
(0,4)	−1.451	2.369	
(2,2)	2.369	2.418	2.369
(4,0)		2.369	−1.451

$\bar{\lambda}=2,\ J=4,\ \lambda=2,\ \lambda'=3$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)
(0,4)	−2.328	1.489		−2.603	
(2,2)	−2.851	0	2.851	−3.188	−3.188
(4,0)		−1.489	−4.054		−2.603

$\bar{\lambda}=2,\ J=4,\ \lambda=2,\ \lambda'=4$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)	(2,6)	(4,4)
(0,4)	−0.981	0.311		−0.657		3.551	
(2,2)	0.962	−1.553	0.962	1.264	−1.264		1.749
(4,0)		0.311	−0.981		0.657		

$\bar{\lambda}=2,\ J=4,\ \lambda=3,\ \lambda'=3$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)
(0,4)	−0.681	0.403		2.435	
(2,2)	0.403	0.580	0.403	−0.902	0.902
(4,0)		0.403	−0.681		−2.435
(2,4)	2.435	−0.902		−0.910	0.774
(4,2)		0.902	−2.435	0.774	−0.910

$\bar{\lambda}=2,\ J=4,\ \lambda=3,\ \lambda'=4$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)	(2,6)	(4,4)
(0,4)	−1.757	1.534		−1.793		−2.543	
(2,2)	−2.473	0	2.473	−2.275	−2.275		0
(4,0)		−1.534	1.757		−1.793		
(2,4)	−0.079	0.539		0.244	+0.558	−1.990	−2.572
(4,2)		−0.539	−0.079	−0.558	−0.244		−2.572

$\bar{\lambda}=2,\ J=4,\ \lambda=4,\ \lambda'=4$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)	(2,6)	(4,4)
(0,4)	−0.946	−0.517		2.478		−0.701	
(2,2)	−0.517	−0.207	−0.517	−1.262	1.262		−0.493
(4,0)		−0.517	−0.946		−2.478		
(2,4)	2.478	−1.262		−1.702	0.276	−0.179	1.158
(4,2)		1.262	−2.478	0.276	−1.702		−1.158
(2,6)	−0.701			−0.179		3.022	0.179
(4,4)		−0.493		1.158	−1.158	0.179	0.862

$\bar{\lambda}=4,\ J=4,\ \lambda=2,\ \lambda'=2$

	(0,4)	(2,2)	(4,0)
(0,4)	0.054	−0.263	0.860
(2,2)	−0.263	0.483	−0.263
(4,0)	0.860	−0.263	0.054

TABLE III. (*Continued*)

$\bar{\lambda}=4,\ J=4,\ \lambda=2,\ \lambda'=3$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)
(0,4)	0.635	−1.489	1.693	0.710	−1.893
(2,2)	−1.296	0	1.296	−1.255	−1.449
(4,0)	−1.693	1.489	−0.635	−1.893	0.710

$\bar{\lambda}=4,\ J=4,\ \lambda=2,\ \lambda'=4$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)	(2,6)	(4,4)
(0,4)	1.874	−2.447	1.428	2.174	−1.876	−1.614	2.596
(2,2)	0.437	−2.134	0.437	1.514	−1.514	5.272	3.444
(4,0)	1.438	−2.447	1.874	1.876	−2.174		2.596

$\bar{\lambda}=4,\ J=4,\ \lambda=3,\ \lambda'=3$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)
(0,4)	1.918	−0.916	−2.483	1.456	3.184
(2,2)	−0.916	−2.042	−0.916	0.409	−0.409
(4,0)	−2.483	−0.916	1.918	−3.184	−1.456
(2,4)	1.456	0.409	−3.184	3.064	2.375
(4,2)	3.184	−0.409	−1.456	2.375	3.064

$\bar{\lambda}=4,\ J=4,\ \lambda=3,\ \lambda'=4$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)	(2,6)	(4,4)
(0,4)	1.325	0.220	−0.793	0.273	1.539	−1.956	−3.736
(2,2)	0.951	0	−0.951	1.033	1.033	2.294	0
(4,0)	0.793	−0.219	−1.325	1.539	0.273		3.736
(2,4)	−0.695	2.204	−1.854	−0.836	1.712	−3.169	0.110
(4,2)	−1.854	2.204	−0.695	−1.712	0.836	−0.437	0.110

$\bar{\lambda}=4,\ J=4,\ \lambda=4,\ \lambda'=4$

	(0,4)	(2,2)	(4,0)	(2,4)	(4,2)	(2,6)	(4,4)
(0,4)	0.937	−0.775	−1.624	−0.304	1.381	0.990	1.286
(2,2)	−0.775	−1.212	−0.775	−0.057	0.057	−1.126	−1.951
(4,0)	−1.624	−0.775	0.937	−1.381	0.304		1.286
(2,4)	−0.304	−0.057	−1.381	2.446	0.066	1.234	0.443
(4,2)	1.381	0.057	0.304	3.123	2.446	−0.416	−0.443
(2,6)	0.990	−1.126		1.234	−0.417	0.129	1.679
(4,4)	1.286	−1.951	1.286	0.443	−0.443	1.679	−0.878

When carrying out the derivatives in (C3) one gets a triple sum for the integrand. Apart from the binomial coefficients originating from the derivatives one has the x-dependent factors $(1+x)^{p+1/2}(1-x)^{q+1/2}$, with p and q integers, for which the integrals are known:

$$\int_{-1}^{+1} dx(1+x)^{p+1/2}(1-x)^{q+1/2}=\pi(2p+1)!!(2q+1)!!/(p+q+2)! \tag{C7}$$

Denoting

$$x=(l+L)/2,\quad x'=(l'+L')/2\,, \tag{C8}$$

and remembering that Γ denotes the usual gamma function, one obtains the final result

$$F^{\bar{\lambda}\bar{l}}_{\lambda l L;\lambda' l' L'} = N_{\lambda l L} N_{\lambda' l' L'} N_{\bar{\lambda}\bar{l}} \pi \sum_{k_1=0}^{\lambda-x} \sum_{k_2=0}^{\lambda'-x'} \sum_{k_3=0}^{\bar{\lambda}-\bar{l}} (-)^{k_1+k_2+k_3} [k_1!\, k_2!\, k_3!]^{-1}$$

$$\times \frac{(\lambda-x)!}{(\lambda-x-k_1)!} \frac{(\lambda'-x')!}{(\lambda'-x'-k_2)!} \frac{(\bar{\lambda}-\bar{l})!}{(\bar{\lambda}-\bar{l}-k_3)!}$$

$$\times \frac{(\lambda+x+k_1+1)!}{(\lambda+x+1)!} \frac{(\lambda'+x'+k_2+1)!}{(\lambda'+x'+1)!} \frac{(\bar{\lambda}+\bar{l}+k_3+1)!}{(\bar{\lambda}+\bar{l}+1)!}$$

$$\times \frac{\Gamma(l+\frac{3}{2})}{\Gamma(l+\frac{3}{2}+k_1)} \frac{\Gamma(l'+\frac{3}{2})}{\Gamma(l'+\frac{3}{2}+k_2)} \frac{\Gamma(\bar{l}+\frac{3}{2})}{\Gamma(\bar{l}+\frac{3}{2}+k_3)}$$

$$\times [2(k_1+k_2+k_3)+1+l+l'+\bar{l}]!\,(1+L+L'+\bar{l})!$$
$$\times \{[k_1+k_2+k_3+\tfrac{1}{2}(l+l'+\bar{l})]!\,[\tfrac{1}{2}(L+L'+\bar{l})]!$$
$$\times [k_1+k_2+k_3+2+\bar{l}+\tfrac{1}{2}(l'+L'+l+L)]!\}^{-1}$$
$$\times 2\exp[-2(k_1+k_2+k_3)-3-2\bar{l}-l-l'-L-L'] \qquad \text{(C9)}$$

when $\bar{\lambda}=0$, C in Eq. (3.10) reduces to $\delta_{ll'}\delta_{LL'}$, $K=1$, and $F=(N_{000}=4/\sqrt{\pi}=2.257)\times\delta_{\lambda\lambda'}$.

APPENDIX D: VALUES OF $M^{\bar{\lambda}J}_{\lambda l L,\lambda' l' L'}$

Table III gives the values of $M^{\bar{\lambda}J}_{\lambda l L,\lambda' l' L'}$ which was defined in Eq. (3.9). The values of (l', L') are indicated in the top row, the values of (l, L) are indicated on left column.

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