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Formulation in terms of hyperspherical harmonics of the effect of breakup in deuteron-nucleus collisions

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The overall two-nucleon–nucleus wave function is decomposed into two parts ψ_1 and ψ_2. The coordinates of the latter are the hyperspherical ones, suitable for the description of breakup, while the former, which contains both the scattered deuteron channel and additional breakup components, is given in terms of the conventional coordinates $\vec{r}$ and $\vec{R}$. A basis set of functions in ψ_1 are the (continuum) eigenstates of H_{np} and the basis set for ψ_2 are the discrete hyperspherical harmonics. Coupled equations for the coefficients of each basis set are derived. The inhomogeneous source terms which couple ψ_1 to ψ_2 are evaluated for a numerical example, and the coupling terms within ψ_2 space are discussed in a separate publication. It appears feasible to calculate the effect of ψ_2 upon ψ_1 and vice versa in an iterative sequence, but the calculation within ψ_2 space requires the noniterative solution of coupled equations. Such a calculation appears possible with the computer power currently available, but has not been carried out.

[NUCLEAR REACTIONS Formulation of three body scattering in terms of hyperspherical harmonics. Application to deuteron-nucleus scattering.]

I. INTRODUCTION

The scattering of a composite projectile on a target invariably leads to a three body situation, in view of the possibility of breakup of the projectile during the collision. A certain class of methods of treating breakup in the description in deuteron-nucleus scattering, to be called "k by k" methods,[1-5] expands the breakup space into a continuum of relative neutron-proton momentum states and the corresponding coupled equations are then rendered manageable by a more or less sophisticated discretization procedure. However, it was found[2,4,5] that the breakup continuum is quite complex, in that several relative n-p orbital angular momenta (l) could be involved, and breakup energies up to possibly 40 MeV or more (depending on the energy of the incident deuteron) could be excited.[2-4] As a result, it would seem desirable to find a treatment of the breakup space which does not suffer from the discretization errors which are a yet not well understood.

The purpose of the present paper is to investigate the suitability of the use of hyperspherical harmonics[6] for such an alternate description. The hyperspherical coordinate space appears to be a suitable choice to describe the breakup process since these coordinates provide a natural way of introducing a complete orthogonal system of basis functions which depend on a *discrete* set of quantum numbers. Furthermore, they can be chosen in such a way that the angular momentum quantum numbers l and L are the same as the ones which occur in the k by k description, thus facilitating comparison between the two methods.

According to Fabre de la Ripelle,[6] who gives extensive references, the hyperspherical (HS) coordinates were used in physical problems since 1914 (E. Borel) and the HS harmonics (HSH) have been extensively studied since 1926 (Appel and Kampe). Since 1952 (Morpurgo) HSH were employed in the analysis of three nucleon bound state systems,[6] and in 1959 Delves[7] formulated three body scattering problems by means of HSH. Nevertheless, not much is as yet known regarding the practical usefulness of the HSH for numerically calculating the scattering of three body systems.[8-11] These functions have also been used, mainly by Levinger and collaborators, in the description of the photodisintegration of ^{3}He.[12]

One of the difficulties in using HSH for the description of three body scattering processes is that they do not provide a well convergent basis set for representing the two cluster channels in which two of the three particles are bound to each other. For this reason separate parts of the wave functions are introduced to describe these channels.[8-11]

In our treatment the method of separating the wave function into its various cluster pieces bears a close similarity to the Faddeev approach,[13] but is less general in that the stripping channels are suppressed in order to permit comparison with the k by k method, in which they are likewise suppressed. The main emphasis of the present study is to examine the feasibility of solving the resulting coupled channel equations, as well as

paying close attention to the number of angular momenta involved in the breakup description. It is known that this number increases as the c.m. energy increases.[4,10] The work is presented in two papers. In the present one the coupled equations are derived and the inhomogeneous coupling terms between two cluster pieces and the three body pieces of the wave function components are numerically evaluated. In a companion paper,[14] hereafter referred to as II, the coupling terms within the three-body space are expanded in terms of hyperspherical harmonics and numerically evaluated for a special case. However, the actual solution of the coupled equation is relegated to a future study.

Other three body methods also exist which describe in configuration space the scattering of a composite particle in the presence of breakup.[15] However, at the present stage of the development it is not yet possible to draw detailed comparisons between these various methods.

II. THEORY

Following an earlier practice,[1-5] the internal degrees of freedom of the target nucleus are suppressed, and the nucleon-nucleus interaction is described by a phenomenological complex optical model potential. The sum of the neutron-nucleus and proton nucleus optical potentials at a nucleon energy half the incident deuteron energy E_D is denoted as V_N:

$$V_N(\vec{r},\vec{R}) = V_{n\text{-}A}(r_n) + V_{p\text{-}A}(r_p) \ . \tag{2.1}$$

The neutron (proton) radius vector measured from the center of the nucleus is $\vec{r}_n(\vec{r}_p)$, the nucleus is considered infinitely heavy, and the coordinates $\vec{r}$ and $\vec{R}$ are given as

$$\vec{r} = \vec{r}_n - \vec{r}_p, \quad \vec{R} = (\vec{r}_n + \vec{r}_p)/2. \tag{2.2}$$

The overall wave function describing the neutron and the proton, in the combined bound deuteron and breakup states, is ψ and obeys the equation

$$(T_R + H_{np} + V_N - E)\psi = 0. \tag{2.3}$$

Here

$$H_{np} = T_r + v_{np}(r) \ , \tag{2.4a}$$

$$T_R = -(\hbar^2/4m)\nabla_R^2, \quad T_r = -(\hbar^2/m)\nabla_r^2 , \tag{2.4b}$$

v_{np} is the nucleon-nucleon interaction in free space, E is the total energy, and m is the nucleon mass.

The hyperspherical coordinates will now be defined.[6,16] They are given in terms of five angles θ_r, ϕ_r, θ_R, ϕ_R, and x and a variable y which has dimensions of a length. The first four of the five angles are the polar angles of $\vec{r}$ and $\vec{R}$, respectively, and x and y are related to r and R through the relation

$$r/2 = 2^{-1/2} y \sin x, \quad R = 2^{-1/2} y \cos x \ , \tag{2.5a}$$

$$x = \tan^{-1}(r/2R), \quad y = (r^2/2 + 2R^2)^{1/2} \ . \tag{2.5b}$$

The hyperspherical harmonics $\phi_\beta(x, \theta_r, \phi_r, \theta_R \phi_R)$ are eigenfunctions of K^2, the "grand" angular momentum squared,[6] which is the generalization to hyperspherical space of the orbital angular momentum

$$K^2 \phi_\beta = 2\lambda(2\lambda + 4)\phi_\beta \ . \tag{2.6}$$

The quantum number λ is particular to HS space. Its values are integer or half integer, increasing in integer steps n from its lowest value λ_0,

$$\lambda_0 = (l + L)/2 \ , \tag{2.7a}$$

$$\lambda = \lambda_0, \ \lambda_0 + 1, \ \lambda_0 + 2, \ \ldots, \ \lambda_0 + n, \ \ldots . \tag{2.7b}$$

In view of the identity,[16] which is a consequence of Eq. (2.6),

$$(T_R + T_r) y^{-5/2} \phi_\beta = y^{-5/2}(-\hbar^2/2m) \times [d^2/dy^2 - \Lambda(\Lambda+1)/y^2]\phi_\beta \ , \tag{2.8}$$

where

$$\Lambda = 2\lambda + \tfrac{3}{2} = l + L + 2n + \tfrac{3}{2} \ , \tag{2.9}$$

one sees that the presence of K^2 gives rise to a centrifugal potantial $\Lambda(\Lambda+1)/y^2$, which in turn is responsible for the convergence in the expansion in the basis of HSH's. Delves[7] denotes Λ by l and calls it the "channel order," since it determines the threshold properties of each channel. The dependence of the HSH on the angles θ_r, ϕ_r, θ_R, and ϕ_R is given in terms of the usual spherical harmonics $Y_{lm}(\hat{r})$ and $Y_{LM}(\hat{R})$, which are vector-coupled into bipolar spherical harmonics $\mathcal{Y}_{(lL)JM}(\hat{r},\hat{R})$. One obtains[14]

$$\phi_\beta(x,\hat{r},\hat{R}) \equiv \phi_\alpha^{(J)}(x,\hat{r},\hat{R}) = F_\alpha(x)\mathcal{Y}_{(lL)JM_J}(\hat{r},\hat{R}) , \tag{2.10a}$$

$$\alpha \equiv \lambda, l, L, \tag{2.10b}$$

where the $F_\alpha(x)$ is given in terms of polynomials in $\sin^2 x$ in Eq. (2.8) of II.

The overall wave function is now separated into two components

$$\psi = \psi_1(\vec{r},\vec{R}) + \psi_2(y,x,\hat{r},\hat{R}) \ , \tag{2.11}$$

and ψ_1 and ψ_2 are defined in such a way that ψ_2 does not contain bound deuteron components so as to avoid poor convergence of the expansion in HSH. One method of accomplishing this is by requiring that the ψ_1 and ψ_2 obey[17]

$$(T_R + T_r + v_{np} + \langle V_N\rangle - E)\psi_1 = -v_{np}\psi_2 \ , \tag{2.12}$$

$$(T_R + T_r + V_N - E)\psi_2 = -[V_N - \langle V_N\rangle]\psi_1 \ . \tag{2.13}$$

Here $\langle V_N\rangle$ is the Watanabe or folding potential

$$\langle V_N(R)\rangle = \int |\phi_b(r)|^2 V_N(\vec{r},\vec{R}) d^3r \ . \tag{2.14}$$

In Appendix A the connection between the Faddeev equations and Eqs. (2.10) and (2.11) is established. The proof that ψ_2 cannot acquire bound state components, provided that they are not already present initially, is given in Appendix B. The main reason is that v_{np} is absent from Eq. (2.13). The boundary condition on ψ_2 is that it contain only outgoing components in the y coordinate, while ψ_1 contains a deuteron channel with ingoing and outgoing waves in the coordinate R. Breakup components are also present in ψ_1. Stripping channels should, in principle, also be present in ψ_2; however, in the present study the boundary conditions imposed on ψ_2 suppress these channels. In view of this deficiency, the wave function ψ obtained by the solution of Eqs. (2.10) and (2.11) does not provide a complete description of the physical scattering problem; however, the stripping channels can be incorporated at a later stage.

An appealing alternate decomposition of ψ is given by

$$\psi = \chi'(\vec{R})\phi_b(r) + \psi_2'(y,x,\hat{r},\hat{R}) \ , \tag{2.15}$$

where χ' and ψ_2' are defined through the equations

$$[T_R + \langle V_N\rangle - E_D]\chi' = -\int d^3r\, \phi_b U \psi_2' \ , \tag{2.16}$$

$$[T_r + T_R + V_N + v_{np} - E]\psi_2' - \phi_b \int d^3r\, \phi_b U \psi_2' = -U\chi'(\vec{R})\phi_b(r) \ , \tag{2.17}$$

and where

$$U(\vec{r},\vec{R}) = V_N(\vec{r},\vec{R}) - \langle V_N\rangle \ . \tag{2.18}$$

The boundary conditions are that χ' has ingoing as well as outgoing waves in the R coordinate, while ψ_2' has only outgoing waves in the y coordinate. Again the stripping channels are suppressed. The advantage of Eqs. (2.16) and (2.17) is that the whole breakup amplitude is contained in ψ_2', while in Eqs. (2.12) and (2.13) some breakup is also contained in ψ_2. It can further be shown (Appendix C) that when the system of Eqs. (2.16) and (2.17) are solved in an iterative fashion, then if initially no deuteron components are contained in ψ_2', in principle, the iterations will not introduce such components into ψ_2'. However, due to the occurrence of numerical inaccuracies, a bound deutron component may inadvertently appear in ψ_2' in the process of solution of Eqs. (2.16) and (2.17) in view of the presence of v_{np} in Eq. (2.17). As a result, the convergence of the expansion of ψ_2' into HSH's could be jeopardized. For this reason Eqs. (2.12) and (2.13) are preferred.

The present discussion addresses itself to the inclusion of breakup space in the solution of Eq. (2.3). Physical questions which arise due to the Pauli antisymmetrization of the incident nucleons with the nucleons in the nucleus are ignored here. Such effects would lead, for example, to a density dependence of v_{np}.[18] Furthermore, the energy dependence of the nucleon optical potentials contained V_N of Eq. (2.1) is also ignored. Since the nucleons in the broken-up state can share their energies in various ways, this latter approximation could be troublesome for large incident deuteron energies.

III. THE COUPLED EQUATION

The functions ψ_1 and ψ_2 are expanded in a set of eigenstates of H_{np} and of K^2 respectively, as follows. The bound and continuum eigenstates of H_{np} are $\phi(b,r)$ and $\phi_l(k,r)$, respectively. They obey

$$H_{np}\phi(b,r) = \epsilon_b \phi(b,r) \ , \tag{3.1a}$$

$$H_{np}\phi_l(k,\vec{r}) = \epsilon_k \phi_l(k,\vec{r}) \ , \tag{3.1b}$$

where $\epsilon_k = (k^2/m)$, H_{np} is defined in Eq. (2.4), and k and l are the linear momentum and the angular moment of the neutron-proton relative motion, respectively. In the absence of spin, the value of l for the bound state is set equal to zero. The boundary condition for the continuum states $\phi_l(k,\vec{r})$ can be chosen to be standing waves.[19] Denoting the radial parts of $\phi(b,r)$ and $\phi_l(k,r)$ as $u(b,r)$ and $u_l(k,r)$, respectively,[2]

$$\phi_{lm_l}(k,r) = (1/r)u_l(k,r)Y_{lm_l}(\hat{r}) \ , \qquad \phi(b,r) = (1/r)u(b,r)/(4\pi)^{1/2} \ , \tag{3.2}$$

one obtains for the expansion of ψ_1

$$\psi_1(\vec{r},\vec{R}) = \frac{1}{R}\frac{1}{r}\sum\!\!\!\!\!\!\int f_{lL}^{(J)}(k,R)u_l(k,r)dk]\mathcal{Y}_{(lL)JM_J}(\hat{r},\hat{R}) \ . \tag{3.3}$$

The symbol $\sum\!\!\!\!\!\!\int$ denotes that the bound state component $u(b,r)\, f_{0J}^{(J)}(b,R)$ is included in the square bracket. The $\mathcal{Y}$'s are the usual bipolar spherical harmonics.

The expansion of ψ_2 in terms of HSH $\phi_\alpha^{(J)}$ is

$$\psi_2(y,x,\hat{r},\hat{R}) = y^{-5/2}\sum_{J,\alpha} g_\alpha^{(J)}(y)\phi_\alpha^{(J)}(x,\hat{r},\hat{R}) \ . \tag{3.4}$$

Upon inserting Eq. (3.3) into Eq. (2.12), making

use of the orthonormality of the $u(k,r)$'s, multiplying the left hand side of Eq. (2.12) by $Rru_l^*(k,r)\mathcal{Y}^*_{(l_L)_J}(\hat{r},\hat{R})$, and integrating over $d\Omega_r d\Omega_R dr$, one obtains the first set of the coupled equations

$$\left\{-\hbar^2/4m\left[\frac{d^2}{dR^2}-\frac{J(J+1)}{R^2}\right]+\langle V_N\rangle-E_k\right\}f_{lL}^{(J)}(k,R)$$
$$=-X_{lL}^{(J)}(k,R)\ , \quad (3.5)$$

where the inhomogeneous term is given by

$$X_{lL}^{(J)}=R\sum_\lambda\int_0^\infty ru_l(r,k)v_{np}(r)y^{-5/2}g_{\lambda lL}^{(J)}(y)F_{\lambda lL}(x)dr\ . \quad (3.6)$$

A similar equation for the bound state component of ψ_1 is obtained when k is replaced by b, l is set equal to zero, and L is replaced by J.

The assumption that v_{np} is not dependent upon the direction of $\vec{r}$, together with the orthonormality of the bipolar spherical harmonics, was used for the derivation of the above equations. For the evaluation of the integral in Eq. (3.6), y and x are assumed to be given in terms of r and R according to Eq. (2.5b), with the value of R kept fixed.

The $l=0$ components of $X_{lL}^{(J)}$ are the largest, and they should also have the longest range in R, as will be discussed in Sec. V. It is shown there that each λ component of X, for $l=0$, decreases like

$$R^{-3/2}g_{\lambda 0J}^{(J)}(\sqrt{2}R) \quad (3.7)$$

for R large compared to the range of v_{np}. The functions g are oscillatory functions of their argument with constant amplitude. However, even if cancellations between the various λ contributions to X do not decrease the range of X, an R dependence of the type of Eq. (3.7) should not cause undue mathematical difficulties if the experience in handling Coulomb integrals,[20] also of long range, is made use of. The coupled equations for the g_α's are obtained by inserting the expansion Eq. (3.4) into Eq. (2.13), making use of Eq. (2.8), multiplying the result on the left by $y^{5/2}\phi_\alpha^{(J)}$, integrating over $d\Omega_r d\Omega_R$, and $d\tau_x$, and making use of the orthonormality of the ϕ's. The result is

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dy^2}-\frac{\Lambda(\Lambda+1)}{y^2}\right]-E\right\}g_{\lambda lL}^{(J)}(y)+\sum_{\lambda' l' L'}\langle\phi_{\lambda lL}V_N\phi_{\lambda' l' L'}\rangle g_{\lambda' l' L'}^{(J)}(y)=-Z_{\lambda lL}^{(J)}(y)\ , \quad (3.8)$$

where

$$Z_{\lambda lL}^{(J)}(y)=y^{5/2}\int\phi_{\lambda lL}^{(J)}[V_N-\langle V_N\rangle]\psi_1 d\Omega_r d\Omega_R d\tau_x\ . \quad (3.9)$$

The second line in the above equation contains the breakup coupling terms and the inhomogeneous third term couples ψ_2 to ψ_1. In the asymptotic region of y, the two last terms in Eq. (3.8) decrease as y^{-3} and $y^{-5/2}$, respectively, and hence, for large distances y where these terms can be ignored, $g_{\lambda lL}^J$ is Ky times a combination of regular and irregular spherical Bessel functions of half integer index $2\lambda+\frac{3}{2}$ and argument Ky. Asymptotically only the outgoing irregular spherical Bessel function is permitted and one obtains

$$g_\alpha^{(J)}(y)\approx S_\alpha^{(J)}(\pi Ky/2)^{1/2}H_{2\lambda+2}^{(1)}(Ky)/2i\ , \quad (3.10)$$

where

$$K=(2mE/\hbar^2)^{1/2} \quad (3.11)$$

and S is the asymptotic normalization constant. Near the origin, $\Lambda(\Lambda+1)/y^2$ dominates and one obtains

$$g_\alpha^{(J)}(R)\simeq M_\alpha^{(J)}y^{2\lambda+5/2}\ , \quad (3.12)$$

where $M_\alpha^{(J)}$ is a normalization constant. The overall wave function for the two outgoing particles is obtained when $g_\alpha^{(J)}$, given by Eq. (3.12), is inserted into Eq. (3.4) and the summation is carried over J and α. Owing to an identity for Hankel functions, one can transform this wave function from the coordinates y, x back into the coordinates r and R, or r_n and r_p (Ref. 7), and thereby get information about the distribution in energy and angle of the two outgoing particles. Since here we are interested in the effect of breakup on the elastic channel wave function, this transformation will not be pursued.

The set of Eqs. (3.5), (3.6), (3.8), and (3.9) represents the coupled equations to be solved. In Ref. 14 the coupling potentials $\langle\phi_\alpha V_N\phi_{\alpha'}\rangle$ are discussed and evaluated, and in the next two sections the inhomogeneous terms $Z_\alpha^{(J)}(y)$ and $X_{lL}^{(J)}(k,R)$ are described.

IV. THE INHOMOGENEOUS COUPLING TERMS $Z_\alpha^{(J)}(y)$

By utilizing the expansion (3.3) for ψ_1, and by expanding $V_N-\langle V_N\rangle$ in a series of multipoles described below, one obtains for $Z_\alpha^{(J)}$, Eq. (3.9), the expression

$$Z^{(J)}_{\lambda l L}(y)=\sum\!\!\!\!\!\!\int Z^{(J)}_{\lambda l L}(k,y)dk \ , \tag{4.1}$$

where

$$Z^{(J)}_{\lambda l L}(k,y)=2y^{-3/2}\sum_{\bar{l}l'L'} C^{\bar{l}J}_{l L;l'L'}\int_0^{y/\sqrt{2}} R\,dR\,F_{\lambda l L}(x)v_{\bar{l}}(r\ R)u_{l'}(k,r)f^{(J)}_{l'L'}(k,R) \ . \tag{4.2}$$

In the integral the independent variables (y,x) are changed to (y,R) and the variables x and r are considered to be functions of R and y according to

$$\sin^2 x=1-2R^2/y^2;\quad r=(2y^2-4R^2)^{1/2} \ , \tag{4.3}$$

and $d\tau_x=\sin^2x\cos^2x\,dx$ is changed to $2rR^2dR/y^4$. The constants C are angular momentum coupling coefficients

$$C^{\bar{l}J}_{l L;l'L'}=\int \mathcal{Y}^*_{(l L)JM_J}(\hat{r},\hat{R})(2\bar{l}+1)^{1/2}P_{\bar{l}}(\cos\delta)\mathcal{Y}_{(l'L')JM_J}(\hat{r},\hat{R})d\Omega_r d\Omega_R \ , \tag{4.4}$$

where δ is the angle between $\hat{r}$ and $\hat{R}$, and where $P_{\bar{l}}$ are Legendre polynomials. Expressions for C in terms of $6-j$ and $3-j$ symbols are given in Eq. (3.10) of II. Further, $v_{\bar{l}}(r,R)$ is the multipole expansion coefficient of $V_N-\langle V_N\rangle$

$$V_N-\langle V_N\rangle=\sum_{\bar{l}} v_{\bar{l}}(r,R)(2\bar{l}+1)^{1/2}P_{\bar{l}}(\cos\delta) \ . \tag{4.5}$$

The functions $F_\alpha(x)$ are parts of the hyperspherical harmonics defined in Eq. (2.10). The function $v_{\bar{l}}[r(y),R]$ can be expressed in terms of the nucleon-nucleus optical potentials as

$$v_{\bar{l}}(y,R)=(2\bar{l}+1)^{1/2}(Rr)^{-1}\int_{Z\min}^{Z\max}[V_{n-A}(Z)+(-)^{\bar{l}}V_{p-A}(Z)]P_{\bar{l}}[(Z^2-R^2-r^2/4)/Rr]Z\,dZ-\delta_{\bar{l}0}\langle V_N(R)\rangle \tag{4.6}$$

with $Z_{\min}=|R-r/2|$, $Z_{\max}=R+r/2$, and r given by Eq. (4.3).

For large values of y, $Z^{(J)}_{\lambda l L}$ decreases at least as fast as $y^{-5/2}$

$$Z^{(J)}_{\lambda l L}(k,y)\propto y^{-5/2} \ , \tag{4.7a}$$

$$Z^{(J)}_{\lambda l L}(b,y)\propto y^{-5/2}\exp[-(\hbar^2\epsilon_b/m)^{1/2}y] \ . \tag{4.7b}$$

This can be seen from Eq. (4.2) by noting that for large values of y, $V_{\bar{l}}(r,R)$ is large only when $r/2$ is approximately equal to R, to within a distance of the order of the diameter of the target nucleus, in which case $r/2\sim R\sim y/2$. For these distances it follows from Eq. (4.6) that $V_{\bar{l}}\propto(Rr)^{-1}\propto y^{-2}$, and the result (4.7a) follows from (4.2). For the bound component, $Z^J_{\lambda 0 J}(b,y)$, $u_b(r)\propto\exp(-\alpha y)$ and (4.7b) follows.

In order to obtain some insight into the λ, l, L, and y dependence of $Z^{(J)}_\alpha(y)$, the expression (4.2) was evaluated for a numerical example. The bound channel case was taken, $k\to b$, l' was set equal to zero, and the Hulthén form for $u_b(r)$ was used

$$u(b,r)=[(2\gamma)^{-1}+(2\beta)^{-1}-2(\gamma+\beta)^{-1}]^{-1/2}\times[\exp(-\gamma r)-\exp(-\beta r)] \ , \tag{4.8a}$$

$$\gamma=0.231\ \text{fm}^{-1},\quad \beta=1.199\ \text{fm}^{-1} \ . \tag{4.8b}$$

The nucleon-nucleus optical potentials were taken of the Woods-Saxon form $V=V_0/(1+e)+i4W_0e'/(1+e')^2$, where $e=\exp(r-R)/a$ and where

$$\begin{aligned} &V_{0n}=-52.016\ \text{MeV},\quad R_n=4.529\ \text{fm},\quad a_n=0.75\ \text{fm},\\ &W_{0n}=-9.886\ \text{MeV},\quad R_n'=4.877\ \text{fm},\quad a_n'=0.58\ \text{fm},\\ &V_{0p}=-54.248\ \text{MeV},\quad R_p=4.529\ \text{fm},\quad a_p=0.75\ \text{fm},\\ &W_{0p}=-9.514\ \text{MeV},\quad R_p'=5.110\ \text{fm},\quad a_p'=0.534\ \text{fm}. \end{aligned} \tag{4.9}$$

In addition, a small volume Woods-Saxon potential with $W_0=-0.816$ MeV, $R=4.877$, and $a=0.58$ was present in the neutron channel. The above potential corresponds to the Greenlees parameters[21] for the 10 MeV nucleons incident on Ni^{58}. The Watanabe potantial $\langle V\rangle$ was calculated from Eq. (2.14) using Eq. (4.8) for u_b and the deuteron radial wave functions $f_{(0J)J}(b,r)$ were calculated solving a homogeneous Schrödinger equation distorted by the above-mentioned Watanabe potential to which a Coulomb potential was added. This Coulomb potential is due to a uniform charge distribution of radius $R=4$ fm. The incident deuteron energy was 21.6 MeV.

The resulting function $Z^{(J)}_\alpha(b,y)$ is complex, the

real and imaginary parts are oscillating functions of y. The absolute value of the $Z_\alpha^{(J)}(b,y)$, but not including the factor $C^{lJ}_{lL;0J}$,

$$|Z^{(J)}_{\lambda lL}(b,y)/C^{lJ}_{lL;0J}| \tag{4.10}$$

is plotted as a function of $y/\sqrt{2}$ in Figs. 1 and 2 for $J=3$ and 10, respectively. The oscillation results from the fact that the real and imaginary parts of $Z_\alpha^{(J)}$ are reversing sign in between lobes. For $l=0$ the curves for the smallest value of λ (1.5 for $J=3$ and $\lambda=5$ or 6 for $J=10$) have the largest value of Z/C. When λ increases by 4 or 5 units, the corresponding value of Z/C is reduced by a factor of 2 or more. The maximum value of $|Z/C|$ is about 3.5 MeV for $J=3$ and less than 1 MeV for $J=10$. Such a magnitude is comparable to, if not less than, the value of the coupling potential between the bound and continuum components in the k by k method.[2-4]

The results for $l=2$ are shown in Figs. 1(b) and 2(b). Of the several combinations of λ and L only the ones which give the largest values of $|Z/C|$ are displayed. Comparison between Figs. 1(a) and 2(b) shows that the $l=2$ values of $|Z/C|$ is comparable to the $l=0$ value, but is displaced to larger distances in y. For $J=10$ the $l=2$ value of $|Z/C|$ is even larger than for $l=0$. The $l=4$ results for $|Z/C|$ are smaller than the $l=2$ results, but the decrease is less pronounced the larger the J value.

Again there is some resemblance to the k by k results,[2-4] where it was found that the $l=2$ breakup continua are excited as strongly (if not more so) than the $l=0$ breakup continua and the $l=4$ breakup played a negligible role.

In summary, the bound to continuum coupling terms are of the order of a few MeV, and in the numerical case illustrated here the values of $l=0$ and 2 play the largest role. Both features are similar to what was found in the k by k method. Further, the λ dependence of $Z_\alpha^{(J)}(y)$ appears to be determined by $n=\lambda-(l+L)/2$, the number of nodes in the hyperspherical function $F_{\lambda lL}(x)$, $|Z|$ decreasing by a factor of 2 or 3 as n increases by about 6 units. A similar n dependence is found[14] for the breakup to breakup transition potentials $\langle\phi_\alpha V_N \phi_{\alpha'}\rangle$, but the magnitudes of these potentials are much larger, of the order of 10 or 20 MeV.

V. THE COUPLING TERM $X^J_{lL}(R)$

This term, which is defined in Eq. (3.6), couples the breakup function ψ_2 to the function ψ_1. For values of R large compared to the range of V_{np}, it will now be shown that $X^{(J)}_{lL}$ can be approximated by

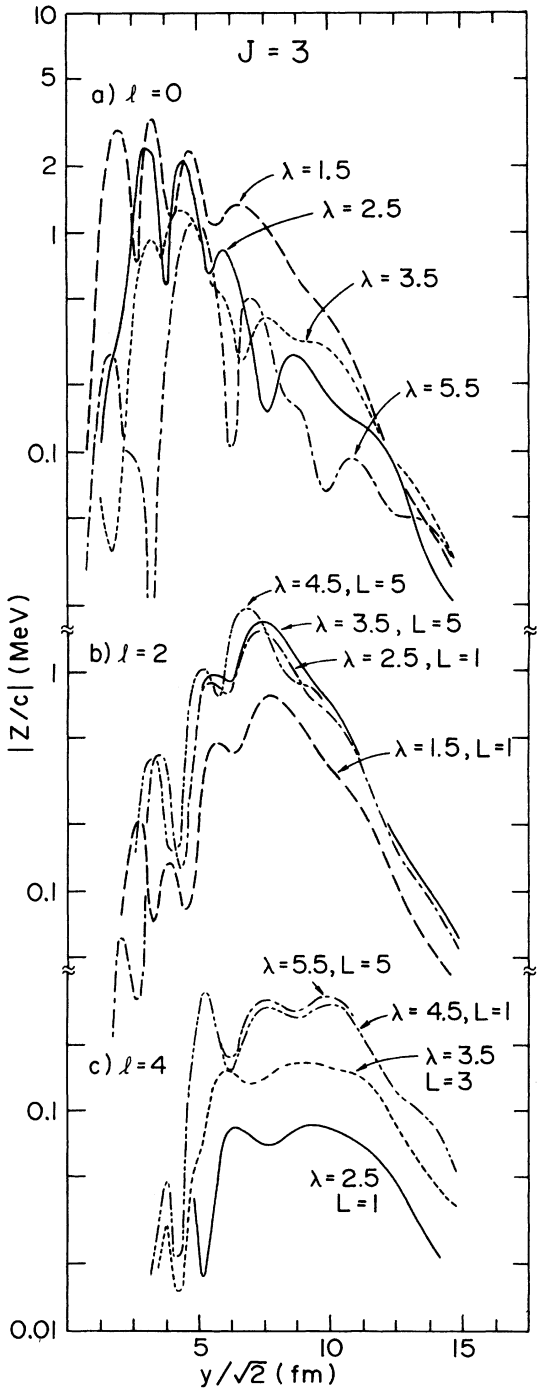

FIG. 1. The absolute value of the inhomogeneous term $Z^{(J)}_{\lambda lL}(y)$ which couples the deuteron component of ψ_1 to the breakup components of ψ_2 for a total angular momentum $J=3$. The term is defined in Eqs. (3.9) and (4.2). The bound state component $f^{(J)}_{nL}(b,R)$ is used in Eq. (4.2). In this case $l'=0, L'=J$, and $\bar{l}=l$, hence the sum in Eq. (4.2) contains only one term. The corresponding value of C is divided out, and only $|Z/C|$ is plotted. Panels a, b, and c illustrate the $l=0$, 2, and 4 results; the values of λ and L are indicated next to the curves. The real and imaginary parts of Z change sign from one lobe of $|Z/C|$ to the next one.

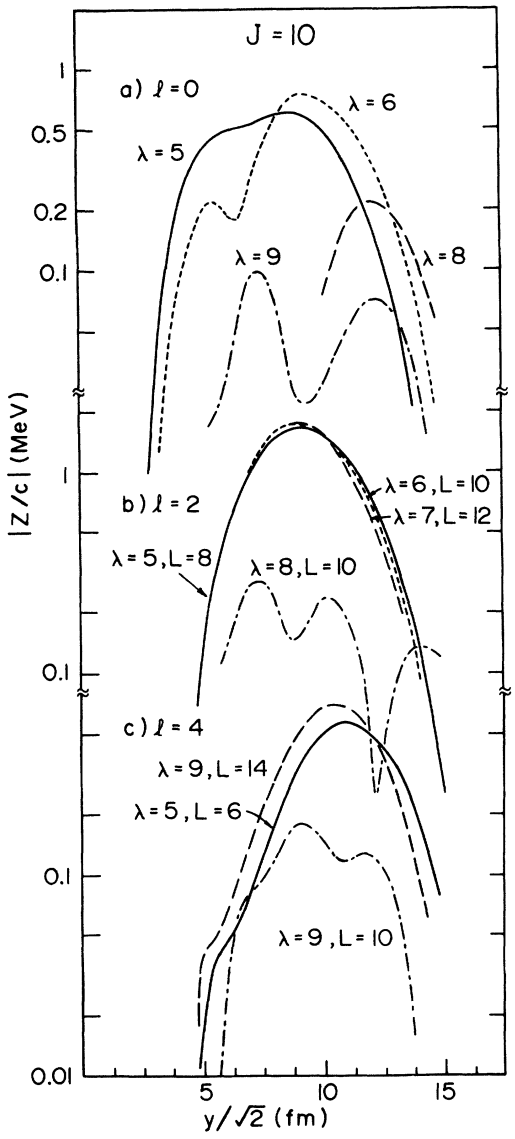

FIG. 2. Same as Fig. 1 for $J=10$.

$$X_{lL}^{(J)}(k,R) \approx 2^{1/4}(2R)^{-3/2-l} D_l^{(l)}(k) \mathcal{Y}_{lL}^{(J)}(R) \ , \quad (5.1)$$

where

$$\mathcal{Y}_{lL}^{(J)}(R) = \sum_{\lambda} N_{\lambda lL} g_{\lambda lL}^{(J)}(y = \sqrt{2}\,R) \quad (5.2)$$

and

$$D_l^{(k)}(k) = \int_0^{\infty} r^{n+1} u_l(k,r) v_{np}(r) dr \ . \quad (5.3)$$

The N_α's are normalization constants which occur in the definition of the hyperspherical harmonics, Eq. (2.8) of II, the functions u_l describe the neutron-proton relative motion, and are defined in Eq. (3.2), and the corresponding integrals $D_l^{(n)}(k)$ are also defined and tabulated in Ref. 22. The derivation of Eq. (5.1) is based on the fact that when $R \gg r$, the angle x is small, of the order $(r/2R)$, and y equals $2^{1/2}R$ to within terms of order $(r/2R)^2$, according to Eq. (2.5b). As a result of the smallness of x, the function $F_\alpha(x)$, given in Eq. (2.8) of II, can be approximated by $(r/2R)^l N_\alpha$ and the result in Eq. (5.1) follows.

For R very small compared to the range of V_{np} one obtains

$$X_{lL}^{(J)}(k,R) \sim \sum_{\lambda} N_{\lambda lL} M_{\lambda lL} (2)^{(L-l)/2} R^{L+1} \times \int_0^{\infty} r^{l+1} u_l^*(k,r) v_{np}(r) (r^2/2 + 2R^2)^n \times {}_2F_1[a,b;c;r^2/(r^2+4R^2)]\, dr \ . \quad (5.4)$$

In the above, n is the number of nodes in $F_{\lambda lL}(x)$, M is the normalization of $g_{\lambda lL}(y)$ near the origin, Eq. (3.12) and a, b, and c are defined in Eq. (2.8) of II. The integral in Eq. (4.13) is of the order of $D_l^{(l+2n)}(k)\ {}_2F_1(a,b,c;1)$ and one sees that, for small values of R, X goes to zero like R^{L+1}:

$$X_{lL}^{(J)}(k,R) \sim 2^{(L-l)/2} R^{L+1} \sum_{\lambda} F_{\lambda lL}(x = \pi/2) M_{\lambda lL} D_l^{(l+2n)}(k) . \quad (5.5)$$

The integral over k of $f(k,R)\ u(k,r)$ is required in the calculation of ψ_1, according to Eq. (3.3). This procedure can be simplified if the f's and u's are averaged over momentum bins by means of a discretization procedure described in Refs. 2 and 22. Each momentum bin is denoted by s ($s=1,2,\ldots$) and it has a width Δk. The corresponding functions $f(k,R)$ and $u(k,r)$ averaged over their respective bins and multiplied by $(\Delta k)^{1/2}$ are denoted by $f(s,R)$ and $u(s,r)$. As a result $F_{lL}^{J}(s,R)$ is obtained from Eq. (3.5) if $X_{lL}^{(J)}(k,R)$ is replaced by $X_{lL}^{J}(s,R)$:

$$X_{lL}^{J}(s,R) = (\Delta k)^{1/2} [X_{lL}^{(J)}(k,R)]_{\text{Aver. over bin } s} \ , \quad (5.6)$$

and E_k is replaced by a bin averaged value E_s. Likewise, one can obtain $X(s,R)$ from Eq. (3.6) if $u(r,k)$ is replaced by $u(r,s)$. Also, if in Eq. (5.1), $D_l^{(x)}$ is replaced by[22]

$$D_l^{(n)}(s) = (\Delta k)^{1/2} [D_l^{(u)}(k)]_{\text{Aver. over bin } s} \ , \quad (5.7)$$

one obtains

$$X_{lL}^{(J)}(s,R) \sim 2^{1/4}(2R)^{-3/2-l} D_l^{(l)}(s) Y_{lL}^{(J)}(R) \ . \quad (5.8)$$

A rough estimate of $X_{lL}^{(J)}(s,R)$ based on the above equation will now be described for $s=b$, 1, and 2. Bins 1 and 2 correspond to breakup energies ϵ_k, which range from 0–10 MeV and 10–40 MeV, respectively. The value of $D_0^{(0)}(s)$ for $s=b$, 1, and 2 is 36, 20, and 25 MeV fm$^{3/2}$, respectively, according to the table of $D_l^{(n)}(k)$ in the appendix

of Ref. 22, while $D_2^{(2)}(s)$ for $s=1$ and 2 is approximately equal to 5 to 25 MeV fm$^{7/2}$, respectively. In view of the extra factor of R^{-2} in Eq. (5.8) for $l=2$ as compared to $l=0$, the $X_{iL}^{(J)}$ values for $l=2$ are expected to be smaller than those for $l=0$ for bins 1 and 2. However, the values of $D_2^{(2)}$ rapidly increase with k (and hence with s) and hence the $l=2$ components of ψ_1 may become important for the higher bins.

In what follows we will restrict ourselves to the case $l=0$. The values of N_α and g_α in Eq. (5.2) need to be known before the magnitude of X_α can be estimated. The value of $g_\alpha(y)$ is obtained from the solution of Eq. (3.8). Since $Z_\alpha(y)$ has a magnitude similar to what is found in the k by k method, and since in that method the breakup functions were found[2,23] to have an amplitude not larger than 0.3, we will assume for $g_\alpha(y)$ a magnitude of 0.3. The values of N_α are tabulated in Appendix A of II. For $l=0$, and assuming the number of nodes n is not larger than 5, one finds that N_α ranges from 2 to 31 as L increases from 0 to 19. Choosing an average value of 10 for N_α, and assuming that in the sum over λ in the second line of Eq. (5.7) only one term contributes, one obtains for $X_{0J}^{(J)}(b,R)$ at $R=5$ fm the rough estimate

$$X_{0J}^{(J)}(b,R)\sim 2^{1/4}\times(10\,\text{fm})^{-3/2}\times(36\ \text{MeV fm}^{3/2})\times 10\times 0.3 \sim 4\ \text{MeV} . \quad (5.9)$$

This value is quite reasonable, in that it is comparable to, but somewhat larger than, the coupling term found in the k by k method between the continuum and the bound channel wave functions, and hence the effect of breakup on the elastic deuteron scattering should be of the same order of magnitude in the two methods of calculation. According to the discussion above, the value of $D_0^{(0)}(s)$ for bins 1 and 2 is comparable in magnitude to that for the bound value, $D_0^{(0)}(b)$. As a result, $X_{0J}^{(J)}(s,R)$ for $s=1$ and 2 is comparable in magnitude to the value of $X_{0J}^{J}(b,R)$, and hence the amount of $l=0$ breakup amplitude contained in ψ_1 is comparable to what it is in the k by k method, and non-negligible amounts of $l=0$ breakup should exist both in parts ψ_1 and ψ_2 of the total wave function.

However, it should be kept in mind that the above estimates depend strongly on the degree of cancellation which occurs in the sum over λ in Eqs. (5.2). A more accurate estimate of the effect of the breakup wave ψ_2 on ψ_1 requires detailed numerical calculations.

VI. SUMMARY AND CONCLUSIONS

The method described in this investigation consists in using simultaneously two sets of coordinates: (r,R) and (y,x). The former is suitable for the deuteron space ψ_1, the latter is a hyperspherical coordinate system suitable for the breakup space ψ_2. The expansion basis functions in either space are not orthogonal to each other, hence inhomogeneous terms appear in the coupled differential equations for the functions in each space. The magnitude of these inhomogeneous terms appears to be small enough so that the propagation of the wave function from one space to the other can be treated iteratively. By contrast, the coupling among the various $g_{\lambda lL}^{(J)}(y)$ components of ψ_2 is too strong[14] to be amenable to perturbative treatment. This coupling involves not only different λ values, but also different sets of l and L values belonging to the same J value. As a result, one of the conclusions of the present study is that the coupling between different l values of breakup components is expected to be strong also in different representations of breakup space, such as the k by k representation, or that used by Laverne *et al.*[15] Even if many l values are excited in breakup space during a deuteron-nucleus collision, it could nevertheless turn out that the effect of breakup on the bound deuteron part of the overall wave function does not depend sensitively on such complexity and that the inclusion of a few l values might suffice. This point, as well as a numerical study of the coupled equations described in the present approach using hyperspherical harmonics, awaits further investigation.

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APPENDIX A: CONNECTION WITH THE FADDEEV FORMULATION

The three coupled Faddeev equations are usually written as

$$[H_0 - E + V_i(x_i)]\psi^{(i)} + V_i(x_i)\sum_{j\neq i}\psi^{(j)} = 0, \quad i = 1,2,3 . \tag{A1}$$

Here $H_0 = T_R + T_r$ represents only the kinetic operators, x_i is the radius vector between particles j and k, and $\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)}$ is the total wave function. Each $\psi^{(i)}$ contains the cluster component corresponding to the bound state solution for each respective square bracket in Eq. (A1) as well as three body breakup components. If one identifies the particles 1, 2, and 3 with the nucleus, the proton, and the neutron, respectively, then $V_1 = v_{np}(r)$, $V_2 = V_{n\text{-}A}(r_n)$, and $V_3 = V_{p\text{-}A}(r_p)$, and the three equations (A1) become

$$(H_0 - E + v_{np})\psi^{(1)} + v_{np}[\psi^{(2)} + \psi^{(3)}] = 0 , \tag{A2}$$

$$(H_0 - E + V_{n\text{-}A})\psi^{(2)} + V_{n\text{-}A}[\psi^{(1)} + \psi^{(3)}] = 0 , \tag{A3}$$

$$(H_0 - E + V_{p\text{-}A})\psi^{(3)} + V_{p\text{-}A}[\psi^{(1)} + \psi^{(3)}] = 0 . \tag{A3}$$

If one sums Eqs. (A3) and (A4) and defines

$$\psi^{(2)} + \psi^{(3)} = \psi_2 , \tag{A5}$$

$$\psi^{(1)} = \psi_1 , \tag{A6}$$

and if one makes use of the definition, Eq. (2.1), of V_N, one obtains the exact result

$$(H_0 - E + v_{np})\psi_1 + v_{np}\psi_2 = 0 , \tag{A7}$$

$$(H_0 - E + V_N)\psi_2 + V_N\psi_1 = 0 . \tag{A8}$$

Finally, if one adds to Eq. (A7) the term $\langle V_N\rangle\psi_1$ and subtracts it again from Eq. (A8), one obtains the "distorted" Faddeev equations (2.12) and (2.13) written in the text. The main difference from the usual Faddeev treatment consists in ignoring the stripping (two cluster) components present in both $\psi^{(2)}$ and $\psi^{(3)}$. These components could arise if in Eqs. (A3) and (A4) the nucleon optical potentials $V_{n\text{-}A}$ and $V_{p\text{-}A}$ were real and admitted bound states. Because they are not, $\psi^{(2)}$ and $\psi^{(3)}$ has only breakup (three body) outgoing components, and hence their sum, Eq. (A5), does also.

APPENDIX B: DOES ψ_2 HAVE DEUTERON COMPONENTS?

Let us assume that not only ψ_1 but also ψ_2 has a bound state component. If they are denoted as $\phi_b(r)X_1(\vec{R})$ and $\phi_b(r)X_2(\vec{R})$ respectively, and if $\bar{\psi}_1$ and $\bar{\psi}_2$ denote the purely breakup components of ψ_1 and ψ_2, then one has

$$\psi_1 = \phi_b X_1 + \bar{\psi}_1 , \tag{B1}$$

$$\psi_2 = \phi_b X_2 + \bar{\psi}_2 . \tag{B2}$$

If r is kept finite and R goes to infinity, then asymptotically $\bar{\psi}_1$ and $\bar{\psi}_2$ vanish[19] as $R^{-5/2}$, and X_1 and X_2 vanish as R^{-1}. If one inserts Eqs. (B1) and (B2) into Eqs. (2.12) and (2.14), still keeping r finite and ignoring terms of order $R^{-5/2}$, and making use of Eq. (3.1), $(T_r + v_{np})\phi_b = \epsilon_b\phi_b$, one obtains

$$[T_R + \langle V_N\rangle - E_D]\phi_b X_1 + v_{np}\phi_b X_2 = O(R^{-5/2}) , \tag{B3}$$

$$[T_R + V_N - v_{np} - E_D]\phi_b X_2 + [V_N - \langle V_N\rangle]\phi_b X_1 = O(R^{-5/2}) , \tag{B4}$$

where $E_D = E - \epsilon_b$ is the physical deuteron kinetic energy. Multiplying Eq. (B4) by $\phi_b(r)$ and integrating over d^3r, the second square bracket cancels exactly and one obtains

$$[T_R + \langle V_N\rangle - \langle v_{np}\rangle - E_D]X_2(R) = O(R^{-5/2}) . \tag{B5}$$

Here $\langle v_{np}\rangle$ is a negative constant defined as

$$\langle v_{np}\rangle = \int \phi_b v_{np}\phi_b d^3r .$$

Likewise, one obtains from Eq. (B3)

$$[T_R + \langle V_N\rangle - E_D]X_1 + \langle v_{np}\rangle X_2 = O(R^{-5/2}) . \tag{B6}$$

Thus, according to Eq. (B5), if X_2 exists it behaves asymptotically like $\exp(-iK'_D R)/R$, where K'_D is an unphysical wave number

$$K'_D = [(2m/\hbar^2)(E_D + \langle v_{np}\rangle)]^{1/2} . \tag{B7}$$

According to Eq. (B6), X_1 must also have such an unphysical component, in order for it to cancel $\langle v_{np}\rangle X_2$ to order $R^{-5/2}$. However, if the calculation of ψ_1 and ψ_2 is performed by successive iterations of Eqs. (2.12) and (2.13), starting with $\psi_1^{(0)} = \phi_b X_1^{(0)}$, $\psi_2^{(0)} = 0$, where $X_1^{(0)}$ is a physical distorted deuteron optical wave function, then $X_1^{(0)}$ does not have unphysical components to start with. The boundary conditions on ψ_2 do not introduce such unphysical components into $\psi_2^{(1)}$ (only breakup components) because the Green's function in Eq. (2.13) does not have bound deuteron components. Hence, $X_1^{(2)}$ does not acquire unphysical components, and so forth.

APPENDIX C: DOES ψ'_2 HAVE DEUTERON COMPONENTS?

The basic coupled equations are now Eqs. (2.16) and (2.17). If ψ'_2 had a bound deuteron component, given by

$$\phi_b(r)x'_2(\vec{R}) = \phi_b \int \phi_b(r)\psi'_2(\vec{r},\vec{R})d^3r , \tag{C1}$$

then it would have to obey the equation

$$[T_R + \langle V_N \rangle - E_D] x_2'(\vec{R}) = 0 . \quad \text{(C2)}$$

This can be seen by multiplying Eq. (2.17) by $\phi_b(r)$, integrating over d^3r, and remembering Eq. (3.1). One obtains

$$(T_R - E_D) \int \phi_b \psi_2' d^3r + \int \phi_b V_N \psi_2' d^3r - \int \phi_b V \psi_2' d^3r = 0 . \quad \text{(C3)}$$

Making use of Eq. (2.18) for U, a cancellation occurs between the two last terms in Eq. (C3), and Eq. (C2) follows rigorously. Had U been replaced by λU in Eqs. (2.16) and (2.17), then ψ given by Eq. (2.15) would still obey the overall Schrödinger Eq. (2.3), but the cancellation mentioned above would no longer take place. Since Eq. (C2) is a homogeneous equation, the solution x_2' must have both ingoing and outgoing components. However, in view of the boundary conditions imposed on ψ_2', x_2' should have only outgoing components. Hence x_2' must vanish. Even if ψ_2, in the process of solving Eq. (2.17), inadvertently did acquire a small bound component, that component would not affect x' via the inhomogeneous term in Eq. (2.16), because $\int \phi_b U \phi_b d^3r \equiv 0$.

A possible difficulty with Eq. (2.17) is that the term in square brackets contains v_{np}. Hence, the Green's function which resolves the square bracket contains (bound) deuteron components. If the function ψ_2' is expanded in hyperspherical harmonics, this expansion may converge poorly as it tries to represent the presence of a deuteron component. Due to inaccuracies in the numerical calculation, the cancellation mentioned after Eq. (C3) may not take place and the amount of deuteron components may increase with each iteration. This danger does not appear to be present in the set of Eqs. (2.12) and (2.13), which is also simpler in that no integrals occur, but the disadvantage is the slow decrease with R of the inhomogeneous term in Eq. (2.12), explained near Eq. (3.7), as well as the fact that ψ_1 contains breakup components.

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†On leave of absence at the University of Connecticut during the academic year 1978–1979.

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