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# Relativistic effects in the three-nucleon bound-state problem

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We calculate the binding energy of tritium using two relativistic versions of the Faddeev equations with separable potentials, and compare with the nonrelativistic results. We find that the relativistic effects increase the binding energy by less than 0.5 MeV. We do our calculations considering only $S$ waves for the two-body interactions, but take into account the tensor force for different values of the deuteron $D$-state probability.

[NUCLEAR STRUCTURE Tritium binding energy, relativistic effects.]

Relativistic effects in the three-nucleon bound-state problem are not well known although they are expected to be small since the binding energy of tritium is much smaller than its total mass. However, it is important to have exact calculations in which one can check how important these corrections really are.

The first estimates of relativistic effects in the three-nucleon problem[1] were performed within the framework of the standard Faddeev equations by simply considering corrections to the kinetic energy and two-body potentials up to order $(v/c)^2$ which lead to an increase in binding energy of 0.5 MeV. Since the corrections to the kinetic energy and to the two-body potentials are of opposite sign, it was found in later calculations[2,3] which neglected the kinetic energy piece that the relativistic effects due to the potential alone decreased the binding energy by 0.3 MeV.

Two calculations based in relativistic formulations of the three-body problem with local potentials have been performed[4,5] which, however, lead to quite different results. While Jackson and Tjon[4] found that the relativistic effects increase the binding energy by 0.25 MeV, Bawin and Lavine[5] on the other hand, using Vinogradov's relativistic formulation,[6] obtained an increase of 0.7 MeV.

Our calculations differ from those of Refs. 1–5, in that we perform the angular momentum decomposition using the full relativistic transformation that relates the momentum coordinates of the three-body system in the configuration $i$ to those of the configuration $j$. Also we use separable potentials and perform our calculations with several values of the deuteron $D$-state probability.

The nonrelativistic Faddeev equations for the bound-state problem are[7]

$$T_i = \sum_{j\neq i} t_i G_0 T_j , \quad i,j=1,2,3 \tag{1}$$

where $t_i$ is the off-energy-shell $T$ matrix of the pair $j$, $k$, and $G_0$ is the Green's function for three free particles,

$$G_0(E)=\frac{1}{E-p_i^2/2\eta_i - q_i^2/2\nu_i + i\epsilon}, \tag{2}$$

where $\vec{p}_i$ is the relative momentum between particles $j$ and $k$, and $\vec{q}_i$ that between particle $i$ and the center of mass of the pair $j$, $k$, while $\eta_i$ is the reduced mass of the pair $j$, $k$, and $\nu_i$ that between particle $i$ and the pair.[8] If the two-body $T$ matrix $t_i$ is separable with only the $S$ wave, that is,

$$\langle \vec{p}_i\vec{q}_i | t_i(E) | \vec{p}_i'\vec{q}_i'\rangle = \delta(\vec{q}_i - \vec{q}_i')\frac{1}{4\pi} g_i(p_i) \times \tau_i(E - q_i^2/2\nu_i) g_i(p_i'), \tag{3}$$

then the Faddeev equations (1) for the case of total angular momentum $J=0$ are

$$\langle q_i | T_i | \phi_0\rangle = \sum_{j\neq i}\int_0^\infty dq_j K_{ij}(q_i q_j)\langle q_j | T_j | \phi_0\rangle, \tag{4}$$

$$K_{ij}(q_i q_j) = \frac{q_j^2}{2}\tau_i(E - q_i^2/2\nu_i) \times \int_{-1}^{1} d\cos\Theta\, g_i(p_i)\frac{1}{E - p_i^2/2\eta_i - q_i^2/2\nu_i + i\epsilon} \times g_j(p_j), \tag{5}$$

where $\Theta$ is the angle between $\vec{q}_i$ and $\vec{q}_j$, and[8]

$$p_i^2 = \left(\vec{q}_j + \frac{\eta_i}{m_k}\vec{q}_i\right)^2 = q_j^2 + \frac{\eta_i^2}{m_k^2}q_i^2 + 2\frac{\eta_i}{m_k}q_i q_j\cos\Theta, \tag{6}$$

$$p_j^2 = \left(\vec{q}_i + \frac{\eta_j}{m_k}\vec{q}_j\right)^2 = q_i^2 + \frac{\eta_j^2}{m_k^2}q_j^2 + 2\frac{\eta_j}{m_k}q_i q_j\cos\Theta. \tag{7}$$

One can obtain the relativistic analog of the Faddeev equations[9-12] by summing all partial sets of diagrams in which only two particles interact while the third particle acts as spectator, which leads to equations identical in form to Eq. (1) but which depend on four-component relative momenta. In order to eliminate in a covariant way the fourth components of the relative momenta, one replaces the relativistic propagafors of the three particles by their delta function parts and evaluates a dispersion integral in the total energy of the system.[11] This leads to the relativistic Faddeev propagator

$$G_0(W_0)=\frac{1}{2\pi}\frac{1}{\omega_i\omega_j\omega_k}\frac{\omega_i+\omega_j+\omega_k}{W_0^2-(\omega_i+\omega_j+\omega_k)^2+i\epsilon}, \quad (8)$$

where $\omega_i=(k_i^2+m_i^2)^{1/2}$, and $W_0$ is the external total relativistic energy of the system. One can write the integral equations in terms of the relativistic relative momenta $\vec{p}_i$ and $\vec{q}_i$ by applying the Jacobian transformation[13]

$$\frac{d\vec{k}_i}{\omega_i}\frac{d\vec{k}_j}{\omega_j}\frac{d\vec{k}_k}{\omega_k}\delta(\vec{k}_i+\vec{k}_j+\vec{k}_k) =\frac{\omega(p_i)}{W_i(p_iq_i)\omega_i(q_i)\omega_j(p_i)\omega_k(p_i)}d\vec{p}_i d\vec{q}_i, \quad (9)$$

where

$$\omega_i(k)=(k^2+m_i^2)^{1/2}, \quad (10)$$

$$\omega(p_i)=\omega_j(p_i)+\omega_k(p_i), \quad (11)$$

$$W_i(p_iq_i)=[\omega^2(p_i)+q_i^2]^{1/2}, \quad (12)$$

$$\omega_i+\omega_j+\omega_k=\omega_i(q_i)+W_i(p_iq_i). \quad (13)$$

The momentum $\vec{p}_i$ is the relative momentum between particles $j$ and $k$ measured in the center-of-mass frame of the pair, and $\vec{q}_i$ is the relative momentum between particle $i$ and the pair $j,k$ measured in the three-body center-of-mass frame.

The equivalent expression to Eq. (3) in the relativistic case is

$$\langle\vec{p}_i\vec{q}_i|t_i(W_0)|\vec{p}_i'\vec{q}_i'\rangle=2\omega_i(q_i)\delta(\vec{q}_i-\vec{q}_i')\frac{1}{4\pi}g_i(p_i)\times\tau_i(\omega_0(q_i))g_i(p_i'), \quad (14)$$

where $W_0$ is the invariant mass of the three-body system and $\omega_0(q_i)$ that of the two-body subsystem, which is given by

$$\omega_0(q_i)=[W_0^2+m_i^2-2W_0\omega_i(q_i)]^{1/2}. \quad (15)$$

If we use Eqs. (8)–(14) in Eq. (1), the Faddeev equations for the case of total angular momentum $J=0$ are given by Eq. (4), where the kernels are now

$$K_{ij}(q_iq_j)=\frac{q_j^2}{4\pi}\tau_i(\omega_0(q_i))\times\int_{-1}^{1}d\cos\Theta\frac{\omega(p_i)}{W_i(p_iq_i)\omega_j(p_i)\omega_k(p_i)}\times g_i(p_i)\frac{\omega_i+\omega_j+\omega_k}{W_0^2-(\omega_i+\omega_j+\omega_k)^2+i\epsilon}g_j(p_j), \quad (16)$$

where again $\Theta$ is the angle between $\vec{q}_i$ and $\vec{q}_j$, while[14]

$$p_i^2=(\vec{q}_j+\alpha_{ij}\vec{q}_i)^2=q_j^2+\alpha_{ij}^2q_i^2+2\alpha_{ij}q_iq_j\cos\Theta, \quad (17)$$

$$p_j^2=(\vec{q}_i+\alpha_{ji}\vec{q}_j)^2=q_i^2+\alpha_{ji}^2q_j^2+2\alpha_{ji}q_iq_j\cos\Theta, \quad (18)$$

and the functions $\alpha_{ij}$ and $\alpha_{ji}$ are defined in terms of $q_i$, $q_j$, and $\Theta$, as[14]

$$\alpha_{ij}=\frac{\omega^2(p_i)+m_j^2-m_k^2+2\omega_j(q_j)\omega(p_i)}{2\omega(p_i)[W_i(p_iq_i)+\omega(p_i)]}, \quad (19)$$

$$\alpha_{ji}=\frac{\omega^2(p_j)+m_i^2-m_k^2+2\omega_i(q_i)\omega(p_j)}{2\omega(p_j)[W_j(p_jq_j)+\omega(p_j)]}, \quad (20)$$

$$W_i(p_iq_i)=\omega_j(q_j)+(m_k^2+q_i^2+q_j^2+2q_iq_j\cos\Theta)^{1/2}, \quad (21)$$

$$W_j(p_jq_j)=\omega_i(q_i)+(m_k^2+q_i^2+q_j^2+2q_iq_j\cos\Theta)^{1/2}, \quad (22)$$

$$\omega(p_i)=[W_i^2(p_iq_i)-q_i^2]^{1/2}, \quad (23)$$

$$\omega(p_j)=[W_j^2(p_jq_j)-q_j^2]^{1/2}. \quad (24)$$

The function $\tau_i(\omega_0(q_i))$ in Eq. (16) is obtained by solving the two-body Blankenbecler-Sugar equation,[11] while the function $\tau_i(E-q_i^2/2\nu_i)$ in Eq. (5) on the other hand, is obtained by solving the nonrelativistic Lippmann-Schwinger equation, so that, if one uses the same form for the relativistic and nonrelativistic separable potentials, one has to fit the low-energy nucleon-nucleon data twice. A much simpler way to proceed is to use instead relativistic form factors which are related to the nonrelativistic ones by

$$g_i^{\text{rel}}(p_i)=(m_i^2+p_i^2)^{1/4}g_i^{\text{nonrel}}(p_i). \quad (25)$$

In this case the Blankenbecler-Sugar equation for separable potentials becomes identical to the Lippmann-Schwinger equation, and the functions $\tau_i$ are related to each other as

$$\tau_i^{\text{rel}}(\omega_0)=4\pi M\tau_i^{\text{nonrel}}(E_0), \quad (26)$$

where $M$ is the mass of the nucleon, and the arguments of the two functions are related as

$$\omega_0^2=4M(M+E_0). \quad (27)$$

We should point out that this theory is Lorentz invariant, since we have used a covariant prescription[11] to reduce it from eight to six continuous variables. Also, as we will show next, it has the proper nonrelativistic reduction. We first notice from Eqs. (19)–(24) that in the nonrelativ-

TABLE I. Binding energies obtained by solving the relativistic and nonrelativistic equations.

| Deuteron $D$-state probability $P_D$ | Nonrelativistic calculation $E_B$ (MeV) | Relativistic calculation $E_B$ (MeV) | $\Delta E$ (MeV) |
|---|---|---|---|
| 0% | 10.77 | 11.06 | 0.29 |
| 4% | 9.07 | 9.37 | 0.30 |
| 7% | 7.99 | 8.14 | 0.15 |

istic limit

$$\alpha_{ij} \to \frac{m_j}{m_j+m_k} = \frac{\eta_i}{m_k}, \quad \alpha_{ji} \to \frac{m_i}{m_i+m_k} = \frac{\eta_j}{m_k}, \tag{28}$$

so that Eqs. (17) and (18) become identical to Eqs. (6) and (7). Next, from Eq. (15) we can see that since $W_0 = 3M + E$,

$$\omega_0^2(q_i) \to 4M^2 + 4ME - 3q_i^2, \tag{29}$$

so that comparing with Eq. (27) we get

$$E_0 = E - \frac{3}{4}\frac{q_i^2}{M} = E - \frac{q_i^2}{2\nu_i}, \tag{30}$$

which is just the argument of the nonrelativistic function $\tau_i$ in Eq. (3). Finally, from Eqs. (10)–(13) and (25), one can see that

$$\frac{\omega(p_i)}{W_i(p_i q_i)\omega_j(p_i)\omega_k(p_i)} \to \frac{1}{M^2}, \tag{31}$$

$$\frac{\omega_i + \omega_j + \omega_k}{W_0^2 - (\omega_i + \omega_j + \omega_k)^2 + i\epsilon} \to \frac{1}{2}\frac{1}{E - p_i^2/2\eta_i - q_i^2/2\nu_i + i\epsilon}, \tag{32}$$

$$g_i^{\mathrm{rel}}(p_i) \to M^{1/2} g_i^{\mathrm{nonrel}}(p_i), \tag{33}$$

so that using in addition Eq. (26) we see that the kernel (16) reduces to the kernel (5).

In order to take into account the two $S$-wave nucleon-nucleon channels, we introduce spin and isospin Racah coefficients, so that, for the case of total angular momentum $J = \frac{1}{2}$, the kernels must be multiplied by coefficients $B$, where

$$B_{aa}(^3S_1 \to {}^3S_1) = B_{bb}(^1S_0 \to {}^1S_0) = \tfrac{1}{4}, \tag{34}$$

$$B_{ab}(^3S_1 \to {}^1S_0) = B_{ba}(^1S_0 \to {}^3S_1) = -\tfrac{3}{4}. \tag{35}$$

We solved the integral equations (4) with the kernels (5) and (16), using Yamaguchi form factors which reproduce the low-energy nucleon-nucleon data.[14] We also considered for the $^3S_1$ channel the effect of the tensor force by using Phillips potentials[15] with 4% and 7% deuteron $D$-state probabilities. We present our results in Table I, where we see that the relativistic correction lies between 0.15 and 0.30 MeV, which is in very good agreement with the value of 0.25 MeV found by Jackson and Tjon.[4]

As we mentioned in the Introduction, a larger correction to the binding energy was found by Bawin and Lavine[5] using Vinogradov's formulation[6] of the relativistic three-body problem. In that formulation, one uses instead of the propagator (8) the form

$$G_0(W_0) = \frac{1}{4\pi}\frac{1}{\omega_i\omega_j\omega_k}\frac{1}{W_0 - \omega_i - \omega_j - \omega_k + i\epsilon}, \tag{36}$$

in both the two- and three-body equations. The corresponding relation between the relativistic and nonrelativistic form factors given by Eq. (25) is for this case

$$g_i^{\mathrm{rel}}(p_i q_i) = \left[\frac{2\omega_i(p_i)W_i(p_i q_i)}{W_0 - \omega_i(q_i) + W_i(p_i q_i)}\right]^{1/2} g_i^{\mathrm{nonrel}}(p_i), \tag{37}$$

in terms of the definitions of Eqs. (10)–(13). The reduction to the nonrelativistic limit follows using

TABLE II. Binding energies obtained by solving the relativistic and nonrelativistic equations using Vinogradov's formulation of the relativistic three-body problem.

| Deuteron $D$-state probability $P_D$ | Nonrelativistic calculation $E_B$ (MeV) | Relativistic calculation $E_B$ (MeV) | $\Delta E$ (MeV) |
|---|---|---|---|
| 0% | 10.77 | 11.25 | 0.48 |
| 4% | 9.07 | 9.51 | 0.44 |
| 7% | 7.99 | 8.24 | 0.25 |

the same steps as before.

We show the results of our calculations using Vinogradov's formulation in Table II. We see that the corrections are indeed larger than those of the previous case, although not as large as the 0.7 found by Bawin and Lavine.[5]

To conclude, we have shown that the increase in binding energy due to relativistic effects depends on which relativistic formulation is used as well as on the strength of the tensor force. Our results indicate, however, that the correction is less than 0.5 MeV.

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