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Effect of the Pauli principle in elastic scattering

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The effect of imposition of the Pauli principle for two-fragment elastic nuclear scattering is examined. It is shown that the antisymmetrized problem can be cast into the Lippmann-Schwinger form with an effective interaction in which the effect of the Pauli principle is entirely absorbed into the effective interaction potential operator. This result enables the formalism to be developed in analogy with the unsymmetrized formulation. Central to the approach is the choice of the off-shell extension of the transition operator. Comparison is made with a previously proposed treatment based on a different off-shell extension. It is shown that both the antisymmetrized transition operator and the associated optical potential proposed herein are readily expressed as spectator expansions in which the effect of the Pauli principle among the active fermions is incorporated in a physically appealing fashion at each stage of the expansion.

[NUCLEAR REACTIONS Antisymmetrization incorporated in elastic scattering and optical potential theory. Multiple scattering series and spectator expansion.]

I. INTRODUCTION

Recently a treatment of identical fermions in two fragment elastic nuclear scattering has been presented.[1,2] In that development, the imposition of an off-shell unitarity criterion leads to the Alt, Grassberger, and Sandhas (AGS) off-shell extension of the transition operator.[3] A number of benefits are associated with that approach. In this paper we offer an alternate, but related, discussion of the antisymmetrization problem in the multiple scattering formalism. The treatment suggested herein has the advantage that is is completely analogous to the usual unsymmetrized Feshbach,[4] Kerman, McManus, and Thaler (KMT),[5] and spectator[6] results, but with the effects of the Pauli principle explicitly exhibited in a physically compelling fashion. We believe this point of view to be particularly well suited to application of the multiple scattering formalism to intermediate energy problems.

We begin by showing that the fully antisymmetrized problem may be dealt with by means of a Lippmann-Schwinger equation, and hence treated in a manner similar to the usual Feshbach technique for nonidentical particles. To obtain this Lippmann-Schwinger equation we use a particular off-shell extension of the transition operator (T). The Lippmann-Schwinger equation so obtained is necessarily written in terms of an effective interaction $\hat{V}$ which incorporates all of the effects of antisymmetrization, and whose properties we explicate. This result is compared to the treatment of the same problem with the use of the AGS off-shell extension of T.

The effective interaction $\hat{V}$ is then used to provide an expression for an antisymmetrized optical potential operator $\hat{U}$. The same operator is also derived without the intermediary of a Lippmann-Schwinger equation. Similar manipulations lead to the antisymmetrized optical potential operator[1] $\bar{U}$ based on the AGS off-shell extension of T. The two results are compared. The common, crucial ingredient in both optical potentials is that they are defined, as in Ref. 1, through the usual less than fully antisymmetrized projector.

Finally, we show how the antisymmetrized optical potential $P\hat{U}P$ may be readily expressed in terms of a spectator expansion in which the effect of the Pauli principle among the active fermions is fully incorporated in each term of the expansion. The first term of this expansion is the usual first order Watson[7] or KMT (Ref. 8) result, with t considered to be the antisymmetrized two-body transition operator, as expected.

II. A LIPPMANN-SCHWINGER EQUATION FOR THE ANTISYMMETRIZED TWO FRAGMENT PROBLEM

We begin by writing the unsymmetrized scattering eigenstate of the full Hamiltonian H as

$$|\psi\rangle = GG_\alpha^{-1}|\phi^k_{\alpha_0}\rangle , \tag{1}$$

where

$$G_\alpha^{-1} = E - H_\alpha + i\epsilon , \tag{2}$$

$$G^{-1} = E - H + i\epsilon , \tag{3}$$

and $|\phi^k_{\alpha_0}\rangle$ is an eigenstate of H_α with energy E. The state $|\phi^k_{\alpha_0}\rangle$ consists of the internal wave functions of the two bound fragments in their states of lowest excitation and their relative plane-wave motion denoted, respectively, by subscript α_0 and superscript k. The channel α corresponds to a particular arbitrary assignment of the identical nucleons to the two clusters and is referred to as the unsymmetrized elastic channel. The internal wave functions of the fragments are taken to be properly antisymmetrized, that is

$$R_\alpha |\phi^k_{\alpha_0}\rangle = |\phi^k_{\alpha_0}\rangle , \tag{4}$$

where R_α is the normalized ($R_\alpha{}^2 = R_\alpha$) antisymmetrizer internal to (but not between) the two fragments. The difference $H - H_\alpha = V^\alpha$ is the interaction between the fragments. For the special case of nucleon-nucleus scattering, to be discussed in detail later, we have $R_\alpha \equiv R$, where R simply antisymmetrizes the target, and

$$V^\alpha = \sum_{i=1}^{A} v_{0i} = \sum_i v_i , \tag{5}$$

under the circumstance that the projectile, particle (0), interacts through two-body forces with each of the A target nucleons.[9]

The state vector with which we must deal, however, is not that of Eq. (1), but rather the antisymmetrized ket $|\tilde{\psi}\rangle$, given by

$$\begin{aligned} |\tilde{\psi}\rangle &= \mathcal{A}|\psi\rangle = \mathcal{A}\Omega|\phi^k_{\alpha_0}\rangle = \mathcal{A}GG_\alpha{}^{-1}|\phi^k_{\alpha_0}\rangle \\ &= \overline{\mathcal{A}}GG_\alpha{}^{-1}|\phi^k_{\alpha_0}\rangle \equiv \tilde{\Omega}|\phi^k_{\alpha_0}\rangle , \end{aligned} \tag{6}$$

where $\mathcal{A}$,

$$\mathcal{A} = \overline{\mathcal{A}} R_\alpha , \tag{7a}$$

is the full unnormalized antisymmetrizer for the problem,[2] and $\overline{\mathcal{A}}$ is the antisymmetrizer between the two fragments. In the case of nucleon-nucleus scattering, for example,

$$\overline{\mathcal{A}} = 1 - \sum_i E_{0i} , \tag{7b}$$

where E_{0i} is the exchange operator for the projectile (0) and identical fermion (i).

With these preliminary definitions disposed of, we now turn to the antisymmetrized Møller wave operator $\tilde{\Omega}$ defined in Eq. (6). We write $\tilde{\Omega}$ as

$$\tilde{\Omega} = G\overline{\mathcal{A}}G_\alpha{}^{-1} = 1 + G_\alpha(G_\alpha{}^{-1}G\overline{\mathcal{A}}G_\alpha{}^{-1} - G_\alpha{}^{-1}) \tag{8a}$$

$$= 1 + G_\alpha[V^\alpha\overline{\mathcal{A}}GG_\alpha{}^{-1} + (\overline{\mathcal{A}} - 1)G_\alpha{}^{-1}] \tag{8b}$$

$$= 1 + G_\alpha\tilde{T} , \tag{8c}$$

where the resolvent identity for G has been used in going from Eq. (8a) to Eq. (8b). The operator $\tilde{T}$ is defined by Eqs. (8) to be

$$\tilde{T} = V^\alpha\overline{\mathcal{A}}GG_\alpha{}^{-1} + (\overline{\mathcal{A}} - 1)G_\alpha{}^{-1} , \tag{9}$$

which we refer to as the (symmetrized) AGS choice[10] of T. This form of transition operator has a different off-shell extension from the standard antisymmetrized form $\hat{T}$ given by

$$\hat{T} = V^\alpha\overline{\mathcal{A}}GG_\alpha{}^{-1} , \tag{10}$$

and it is this difference that was exploited in Refs. 1 and 2. The two choices of T are obviously related by

$$\tilde{T} = \hat{T} + (\overline{\mathcal{A}} - 1)G_\alpha{}^{-1} . \tag{11}$$

The use of the AGS form $\tilde{T}$ in Eqs. (8) arises in a most natural way, in complete analogy with the standard unsymmetrized relation

$$\begin{aligned} \Omega &= GG_\alpha{}^{-1} = 1 + (G - G_\alpha)G_\alpha{}^{-1} \\ &= 1 + G_\alpha V^\alpha GG_\alpha{}^{-1} = 1 + G_\alpha T . \end{aligned} \tag{12}$$

However, since $T = V^\alpha\Omega$, Eq. (12) can be written in the Lippmann-Schwinger form as

$$\Omega = 1 + G_\alpha V^\alpha \Omega , \tag{13}$$

whereas Eq. (8c) cannot be so expressed, since $\tilde{T} \neq V^\alpha\tilde{\Omega}$. The fact that Eq. (8c) does not represent the solution of a Lippmann-Schwinger equation of the usual form is not necessarily a severe limitation. However, the question does arise as to whether the antisymmetrized problem can be managed so that the convenience of the Lippmann-Schwinger equation may be retained. To this end we define a wave operator $\hat{\Omega}$ such that

$$\hat{\Omega} = 1 + G_\alpha\hat{T} , \tag{14}$$

where $\hat{T}$ is given by Eq. (10). In that case one may recognize Eq. (14) as being the solution of

$$\hat{\Omega} = 1 + G_\alpha\hat{T}(1 + G_\alpha\hat{T})^{-1}\hat{\Omega} , \tag{15}$$

which we cast in the form

$$\hat{\Omega} = 1 + G_\alpha\{V^\alpha\overline{\mathcal{A}}[1 + G_\alpha V^\alpha(\overline{\mathcal{A}} - 1)]^{-1}\}\hat{\Omega} \tag{16}$$

by substitution of Eq. (10) into Eq. (15). We then define

$$\hat{V} = V^\alpha\overline{\mathcal{A}}[1 + G_\alpha V^\alpha(\overline{\mathcal{A}} - 1)]^{-1} , \tag{17}$$

so that Eq. (16) takes on the familiar form

$$\hat{\Omega} = 1 + G_\alpha\hat{V}\hat{\Omega} , \tag{18}$$

where all the effects of antisymmetrization have been absorbed into the effective potential $\hat{V}$. The assertion above, that $\tilde{\Omega}$ cannot be cast in this form, can now be easily appreciated. Namely, if we wish to define $\tilde{\Omega}$ by means of the Lippmann-Schwinger equation,

$$\tilde{\Omega} = 1 + G_\alpha\tilde{V}\tilde{\Omega} , \tag{19}$$

then we must define $\tilde{V}$ to be

$$\tilde{V} = \tilde{T}(1 + G_\alpha\tilde{T})^{-1} , \tag{20}$$

in analogy with Eq. (15). The problem is that the operator $(1 + G_\alpha\tilde{T})$ cannot be inverted[11] since

$$(1+G_\alpha\tilde{T})=\overline{\mathfrak{a}}GG_\alpha^{-1}, \tag{21}$$

and $\overline{\mathfrak{a}}$ does not have an inverse.

It remains to be shown that the wave operator $\hat{\Omega}$ possesses useful properties. By comparison of Eq. (18) and Eq. (14), we infer that

$$\hat{T}=\hat{V}\hat{\Omega}. \tag{22}$$

Furthermore, the relation

$$\hat{T}=V^\alpha\overline{\mathfrak{a}}\Omega=\hat{V}\hat{\Omega}=V^\alpha\overline{\mathfrak{a}}[1+G_\alpha V^\alpha(\mathfrak{a}-1)]^{-1}\hat{\Omega}, \tag{23}$$

suggests that

$$\Omega=[1+G_\alpha V^\alpha(\overline{\mathfrak{a}}-1)]^{-1}\hat{\Omega}. \tag{24}$$

In fact, we have from Eq. (14)

$$\hat{\Omega}=1+G_\alpha V^\alpha\overline{\mathfrak{a}}GG_\alpha^{-1}, \tag{25}$$

which with the definition $\Omega=GG_\alpha^{-1}$, gives

$$\hat{\Omega}=(G_\alpha G^{-1}+G_\alpha V^\alpha\overline{\mathfrak{a}})\Omega \tag{26}$$

or

$$\hat{\Omega}=[1+G_\alpha V^\alpha(\overline{\mathfrak{a}}-1)]\Omega, \tag{27}$$

which yields Eq. (24) immediately.

An alternative derivation may help to provide a physical interpretation of Eq. (24). We write

$$\begin{aligned}\Omega&=1+G_\alpha V^\alpha\Omega=1+G_\alpha(V_1+V_2)\Omega\\&=(1-G_\alpha V_2)^{-1}(1+G_\alpha V_1\Omega)\\&=(1-G_\alpha V_2)^{-1}(1+G_\alpha V_1GG_\alpha^{-1}),\end{aligned} \tag{28}$$

and then choose

$$V_1=V^\alpha\overline{\mathfrak{a}} \tag{29}$$

and

$$V_2=-V^\alpha(\overline{\mathfrak{a}}-1). \tag{30}$$

One then finds

$$\Omega=[1+G_\alpha V^\alpha(\overline{\mathfrak{a}}-1)]^{-1}(1+G_\alpha\hat{T}) \tag{31}$$

$$=[1+G_\alpha V^\alpha(\overline{\mathfrak{a}}-1)]^{-1}\hat{\Omega}. \tag{32}$$

The formal relation between the antisymmetrized wave operators $\tilde{\Omega}$ and $\hat{\Omega}$ is also easy to find. We note that Eq. (32) implies that

$$\begin{aligned}\Omega&=[G_\alpha^{-1}+V^\alpha(\overline{\mathfrak{a}}-1)]^{-1}G_\alpha^{-1}\hat{\Omega}\\&=(G^{-1}+V^\alpha\overline{\mathfrak{a}})^{-1}G_\alpha^{-1}\hat{\Omega},\end{aligned} \tag{33}$$

so that

$$\tilde{\Omega}=\overline{\mathfrak{a}}\Omega=\overline{\mathfrak{a}}(G^{-1}+V^\alpha\overline{\mathfrak{a}})^{-1}G_\alpha^{-1}\hat{\Omega}. \tag{34}$$

Again, we remark that the relation between Ω and $\hat{\Omega}$ is such that inversion of the operator in Eq. (32) leads to a Lippmann-Schwinger type of equation for Ω in terms of $\hat{\Omega}$, whereas a similar relation between $\tilde{\Omega}$ and $\hat{\Omega}$ does not exist because the operator to the left of $\hat{\Omega}$ in Eq. (34) may not be inverted. We remark once more that the absence of a Lippmann-Schwinger equation is not a fatal defect, but rather that the existence of such an equation facilitates many further considerations.

III. THE ANTISYMMETRIZED OPTICAL POTENTIAL

A direct consequence of the existence of the Lippmann-Schwinger equation for $\hat{\Omega}$, Eq. (18), is that one can immediately apply well-known techniques to obtain an optical potential for the fully antisymmetrized problem. Following Feshbach,[4] for example, we write Eq. (18) as

$$\hat{\Omega}=1+G_\alpha\hat{V}P\hat{\Omega}+G_\alpha\hat{V}Q\hat{\Omega}, \tag{35}$$

where $P\equiv P_\alpha$ is the usual projector onto the unsymmetrized elastic channel α, with $Q=1-P$. From Eq. (35) one immediately obtains

$$Q\hat{\Omega}P=(1-G_\alpha Q\hat{V})^{-1}G_\alpha Q\hat{V}P\hat{\Omega}P \tag{36}$$

so that

$$P\hat{\Omega}P=P+G_\alpha P[\hat{V}+\hat{V}(G_\alpha^{-1}-Q\hat{V})^{-1}Q\hat{V}]P\hat{\Omega}P \tag{37a}$$

$$=P+G_\alpha P\hat{U}P\hat{\Omega}P \tag{37b}$$

with $\hat{U}$ defined to be

$$\hat{U}=\hat{V}+\hat{V}(G_\alpha^{-1}-Q\hat{V})^{-1}Q\hat{V} \tag{38a}$$

$$\begin{aligned}&=\hat{V}[1+(1-QG_\alpha\hat{V})^{-1}QG_\alpha\hat{V}]\\&=\hat{V}(1-QG_\alpha\hat{V})^{-1}=(1-\hat{V}QG_\alpha)^{-1}\hat{V}\end{aligned} \tag{38b}$$

or

$$\hat{U}=\hat{V}+\hat{V}G_\alpha Q\hat{U}. \tag{39}$$

On the other hand, we have from Eq. (23)

$$\hat{T}=\hat{V}+\hat{V}G_\alpha\hat{T} \tag{40a}$$

so that

$$\hat{T}=\hat{V}+\hat{V}G_\alpha P\hat{T}+\hat{V}G_\alpha Q\hat{T},$$

which when combined with Eq. (38b) yields

$$\hat{T}=\hat{U}(1+G_\alpha P\hat{T}). \tag{40b}$$

For elastic scattering the antisymmetrized transition operator $P\hat{T}P$ is then given by

$$P\hat{T}P=P\hat{U}P+P\hat{U}PG_\alpha P\hat{T}P. \tag{41}$$

The "one-body" operator $P\hat{U}P$ is, of course, an optical potential in which the effects of the Pauli principle have been appropriately taken into account. We shall discuss the practicality of this form of the optical potential shortly.

At this point we wish to obtain the optical potential without the intermediary of a Lippmann-Schwinger equation. Since $\hat{\Omega}$ is given by the expression in Eq. (14) we can immediately write

$$P\hat{\Omega}P=(1+G_\alpha P\hat{T})P, \tag{42}$$

from which the relation

$$P\hat{\Omega}P = P[1 + G_\alpha \hat{T}(1 + G_\alpha P\hat{T})^{-1} P\hat{\Omega}P] \tag{43}$$

follows straightforwardly. We then consider Eq. (43) to be in the form

$$P\hat{\Omega}P = P(1 + G_\alpha \hat{U} P\hat{\Omega}P)\,, \tag{44}$$

from which we identify $\hat{U}$ as

$$\hat{U} = \hat{T}(1 + G_\alpha P\hat{T})^{-1} \tag{45}$$

or

$$\hat{T} = \hat{U} + \hat{U} G_\alpha P\hat{T}\,. \tag{46}$$

Comparison of Eq. (37b) and Eq. (44) or Eq. (40b) and Eq. (46), of course, indicates the identity of the optical potentials defined by the two methods. Insertion of the definition of $\hat{T}$, Eq. (10), into Eq. (46) then yields

$$V^\alpha \overline{\mathfrak{A}} G G_\alpha^{-1} = \hat{U} + \hat{U} G_\alpha P V^\alpha \overline{\mathfrak{A}} G G_\alpha^{-1} \tag{47}$$

or

$$V^\alpha \overline{\mathfrak{A}} = \hat{U} G_\alpha G^{-1} + \hat{U} G_\alpha P V^\alpha \overline{\mathfrak{A}} \tag{48}$$

from which one obtains the explicit result for $\hat{U}$,

$$\hat{U} = V^\alpha \overline{\mathfrak{A}}(G_\alpha G^{-1} + G_\alpha P V^\alpha \overline{\mathfrak{A}})^{-1}\,, \tag{49a}$$

$$= V^\alpha \overline{\mathfrak{A}}[1 + G_\alpha V^\alpha(\overline{\mathfrak{A}} - 1) - G_\alpha Q V^\alpha \overline{\mathfrak{A}}]^{-1} \tag{49b}$$

or alternatively

$$\hat{U} = V^\alpha \overline{\mathfrak{A}}\{1 - G_\alpha[Q V^\alpha - P V^\alpha(\overline{\mathfrak{A}} - 1)]\}^{-1}\,. \tag{50}$$

These forms are easily reconciled with Eqs. (38). The result of Eq. (50) is to be compared with the unsymmetrized Feshbach result analogous to Eq. (38b), viz.,

$$U_F = V^\alpha(1 - G_\alpha Q V^\alpha)^{-1}\,. \tag{51}$$

The physical interpretation of the denominator of Eq. (51) is that it contains scattering to intermediate states in the Q space, whereas the scattering to intermediate states in the P space are included explicitly in the analog of Eq. (46). The additional term in the denominator of Eq. (50) apparently corrects for the fact that scattering via $V^\alpha(\overline{\mathfrak{A}} - 1)$ to the P space is explicitly included in Eq. (46). Equations (49) and (50) may provide useful departure points for approximations at low energies where multiple scattering theories of the KMT type may not be justified. These equations can be written in a variety of suggestive forms and could, for example, be used to obtain corrections to the result of Eq. (51).

The reason for the above exercise, the results of which could have been obtained directly from Eqs. (38), is that a similar procedure leads to an analogous result for the optical potential corresponding to $P\tilde{\Omega}P$ or equivalently to the off-shell extension of the transition operator $\tilde{T}$, given by Eq. (9). That is to say, although $\tilde{\Omega}$ does not satisfy a Lippmann-Schwinger equation of the familiar form, $P\tilde{\Omega}P$ presumably does. To see this, we follow the same steps as those above. We write, in analogy to Eq. (42),

$$P\tilde{\Omega}P = (1 + G_\alpha P\tilde{T})P \tag{52}$$

and then

$$\begin{aligned} P\tilde{\Omega}P &= P\{1 + G_\alpha[\tilde{T}(1 + G_\alpha P\tilde{T})^{-1}]P\tilde{\Omega}P\} \\ &\equiv P(1 + G_\alpha \tilde{U} P\tilde{\Omega}P)\,, \end{aligned} \tag{53}$$

from which we identify $\tilde{U}$ as the operator which satisfies the relation

$$\tilde{T} = \tilde{U} + \tilde{U} G_\alpha P\tilde{T} = \tilde{U} + \tilde{T} G_\alpha P\tilde{U}\,. \tag{54}$$

Using Eq. (11) in Eq. (54), one finds

$$(Q + \overline{\mathfrak{A}}P)\tilde{U} = \tilde{T} - \tilde{T} P G_\alpha \tilde{U}\,, \tag{55}$$

which with the aid of the definition of $\tilde{T}$ and the resolvent identity for G gives

$$(Q + G_\alpha^{-1} G \overline{\mathfrak{A}} P)\tilde{U} = \tilde{T}\,. \tag{56}$$

Insertion of the definition of $\tilde{T}$ into Eq. (56) yields

$$\begin{aligned} (G^{-1} G_\alpha Q + \overline{\mathfrak{A}}P)\tilde{U} &= G^{-1} G_\alpha[V^\alpha \overline{\mathfrak{A}} G G_\alpha^{-1} + (\overline{\mathfrak{A}} - 1)G_\alpha^{-1}] \\ &= V^\alpha G_\alpha \overline{\mathfrak{A}} G_\alpha^{-1} + G^{-1} G_\alpha(\overline{\mathfrak{A}} - 1)G_\alpha^{-1} \\ &= V^\alpha + (\overline{\mathfrak{A}} - 1)G_\alpha^{-1}\,, \end{aligned}$$

so

$$(Q + \overline{\mathfrak{A}}P)\tilde{U} = V^\alpha + (\overline{\mathfrak{A}} - 1)G_\alpha^{-1} + V^\alpha G_\alpha Q \tilde{U} \tag{57}$$

or

$$\begin{aligned} \tilde{U} &= \{1 - [V^\alpha Q - (\overline{\mathfrak{A}} - 1)G_\alpha^{-1} P]G_\alpha\}^{-1} \\ &\quad \times [V^\alpha + (\overline{\mathfrak{A}} - 1)G_\alpha^{-1}]\,. \end{aligned} \tag{58}$$

Equation (58) is to be compared with the corresponding equation for $\hat{U}$, Eq. (50). Although the structure of the two equations is identical, the presence of the G_α^{-1} terms tends to obscure the direct physical interpretation of Eq. (58).

The relation connecting $\tilde{U}$ and $\hat{U}$ can easily be obtained with the aid of Eqs. (46) and (54), together with the relationship between $\tilde{T}$ and $\hat{T}$. One finds

$$\begin{aligned} \tilde{T} &= \tilde{U}(1 - G_\alpha P\tilde{U})^{-1} = \hat{T} + (\overline{\mathfrak{A}} - 1)G_\alpha^{-1} \\ &= (1 - \hat{U} P G_\alpha)^{-1}\hat{U} + (\overline{\mathfrak{A}} - 1)G_\alpha^{-1}\,, \end{aligned} \tag{59}$$

from which one obtains the relation

$$\begin{aligned} (1 - \hat{U} P G_\alpha)\tilde{U} &= \hat{U}(1 - G_\alpha P\tilde{U}) + (1 - \hat{U} P G_\alpha) \\ &\quad \times (\overline{\mathfrak{A}} - 1)(G_\alpha^{-1} - P\tilde{U})\,, \end{aligned} \tag{60}$$

or

$$\tilde{U} = \hat{U} + (1 - \hat{U} P G_\alpha)(\overline{\mathfrak{A}} - 1)(G_\alpha^{-1} - P\tilde{U})\,, \tag{61}$$

and of most immediate interest

$$P\tilde{U}P = P\hat{U}P + (G_\alpha^{-1} - P\hat{U}P)PG_\alpha(\overline{\mathfrak{a}} - 1)P \times (G_\alpha^{-1} - P\tilde{U}P)\,. \quad (62)$$

Clearly the two optical potentials $P\tilde{U}P$ and $P\hat{U}P$ differ unless the operator at the extreme right of Eq. (62) vanishes identically. This is not the case. However, we do have

$$P\tilde{U}P|\tilde{\psi}\rangle = P\tilde{U}P\tilde{\Omega}|\phi^k_{\alpha_0}\rangle = P\tilde{T}|\phi^k_{\alpha_0}\rangle \quad (63a)$$

and

$$P\hat{U}P|\hat{\psi}\rangle = P\hat{U}P\hat{\Omega}|\phi^k_{\alpha_0}\rangle = P\hat{T}|\phi^k_{\alpha_0}\rangle\,. \quad (63b)$$

The difference in Eqs. (63) is $P(\overline{\mathfrak{a}} - 1)G_\alpha^{-1}|\phi^k_{\alpha_0}\rangle$, which expresses the half-shell equality of $\tilde{T}$ and $\hat{T}$ since $|\phi^k_{\alpha_0}\rangle$ is an eigenstate of H_α. It also follows from the definitions of $|\hat{\psi}\rangle$ and $|\tilde{\psi}\rangle$, viz.,

$$|\tilde{\psi}\rangle = (1 + G_\alpha\tilde{T})|\phi^k_{\alpha_0}\rangle \quad (64a)$$

and

$$|\hat{\psi}\rangle = (1 + G_\alpha\hat{T})|\phi^k_{\alpha_0}\rangle\,, \quad (64b)$$

that

$$P|\tilde{\psi}\rangle - P|\hat{\psi}\rangle = PG_\alpha(\overline{\mathfrak{a}} - 1)G_\alpha^{-1}|\phi^k_{\alpha_0}\rangle \quad (65a)$$

or

$$P|\tilde{\psi}\rangle = P|\hat{\psi}\rangle\,, \quad (65b)$$

under those circumstances for which the Lippmann identity term in Eq. (65a) vanishes.[12,13] Equation (65b) must be used with some care[13] due to the delicacy associated with application of the Lippmann identity. A discussion of the Lippmann identity is not our purpose here. For our present purpose it suffices to present Eqs. (63) and to note that

$$\langle\phi^{k'}_{\alpha_0}|P\hat{U}PG_\alpha(\overline{\mathfrak{a}} - 1)G_\alpha^{-1}|\phi^k_{\alpha_0}\rangle = \langle\phi^{k'}_{\alpha_0}|P\tilde{U}PG_\alpha(\overline{\mathfrak{a}} - 1)G_\alpha^{-1}|\phi^k_{\alpha_0}\rangle = 0\,. \quad (66)$$

The Lippmann identity holds for the matrix elements in Eq. (66) because $P\tilde{U}P$ and $P\hat{U}P$ are nonsingular operators, an observation that follows from the fact that $P\tilde{U}P$ and $P\hat{U}P$ satisfy one-body Lippmann-Schwinger equations [obtained by projecting Eqs. (46) and (54) from the left and right with P] in terms of the nonsingular operators $P\tilde{T}P$ and $P\hat{T}P$. This discussion of equivalence of the alternate approaches, represented by $\tilde{U}$ and $\hat{U}$, was primarily to lend credence to the proposition that the choice between the two antisymmetrized optical potentials lies in their calculational practicality or in their feasibility of application to reactions other than elastic scattering. The first of these questions is discussed briefly in the next section.

IV. EXPANSION OF THE ANTISYMMETRIZED OPTICAL POTENTIAL

In this section we establish the relationship of the optical potential $\hat{U}$, given by

$$\hat{T} = \hat{U} + \hat{T}G_\alpha P\hat{U}\,, \quad (67)$$

where

$$\hat{T} = V^\alpha\overline{\mathfrak{a}}GG_\alpha^{-1} = V^\alpha\overline{\mathfrak{a}}\Omega\,, \quad (68)$$

to multiple scattering formalisms of the KMT type which are appropriate to intermediate or higher energies. For the sake of clarity, and since the formalisms themselves are most compelling for this case, we restrict ourselves to nucleon-nucleus scattering.[15] The notation for this has already been indicated in Sec. II.

As a guide to the argument, we shall first deal with the spectator expansion for the unsymmetrized[16] T, where T is given by

$$T = V^\alpha\Omega = \sum_i T_i + \sum_{i<j} S_{ij} + \cdots \quad (69)$$

and

$$\Omega = 1 + G_\alpha T = 1 + G_\alpha\Big(\sum_i T_i + \sum_{i<j} S_{ij} + \cdots\Big)\,, \quad (70)$$

so that we may write

$$\sum_i T_i + \sum_{i<j} S_{ij} + \cdots = \sum_i v_i\Big[1 + G_\alpha\Big(\sum_j T_j + \sum_{j<k} S_{jk} + \cdots\Big)\Big] = \sum_i v_i(1 + G_\alpha T_i) + \sum_{i<j}(v_iG_\alpha T_j + v_jG_\alpha T_i + v_iG_\alpha S_{ij} + v_jG_\alpha S_{ij}) + \cdots\,. \quad (71)$$

Term by term identification[16] then yields

$$T_i = v_i + v_iG_\alpha T_i \quad (72)$$

and

$$S_{ij} = v_iG_\alpha T_j + v_jG_\alpha T_i + (v_i + v_j)G_\alpha S_{ij}\,. \quad (73)$$

From Eq. (72) we identify $T_i \to t_i$, the two-particle transition operator for particles 0 and i, if we approximate G_α by the free two particle propagator. The identification of S_{ij} is most easily accomplished by adding $T_i + T_j$ to both sides of Eq. (73). Upon using Eq. (72) on the right hand side of the resulting equation, one obtains

$$(T_i+T_j+S_{ij})=(v_i+v_j)[1+G_\alpha(T_i+T_j+S_{ij})], \tag{74}$$

which allows the identification

$$S_{ij}=T_{ij}-T_i-T_j \tag{75}$$

with

$$T_{ij}=(v_i+v_j)+(v_i+v_j)G_\alpha T_{ij}. \tag{76}$$

If we approximate G_α by the free three-particle propagator in Eq. (76), $T_{ij}\to t_{ij}$, the three-particle transition operator for scattering of particle 0 from i and j. These identifications provide the expected spectator result[8]

$$T=\sum_i T_i+\sum_{i<j}(T_{ij}-T_i-T_j)+\cdots. \tag{77}$$

Before proceeding to the antisymmetrized case, it is useful to note certain features of the above treatment. In particular, the grouping of the expansion is by no means unique. For example, one could have chosen to expand T in terms of the free $(A+1)$ particle propagator, rather than G_α. In fact, had this been done the result would have been the unique connectivity[14] expansion of T. The result of Eq. (77) can be characterized as the expansion about the propagator G_α. In view of the corresponding treatment of the optical potential to be given shortly, however, the expansion about G_α is suggested by Eq. (67).

The preliminary treatment of the unsymmetrized T shows us how to treat the symmetrized case. We write the antisymmetrized transition operator $\hat{T}$ as

$$\hat{T}=V^\alpha\overline{\mathfrak{A}}\Omega=V^\alpha\overline{\mathfrak{A}}\left[1+G_\alpha\left(\sum_i T_i+\sum_{i<j}S_{ij}+\cdots\right)\right], \tag{78}$$

where the unsymmetrized Ω is taken directly from Eq. (70) and T_i and S_{ij} are given by Eqs. (72) and Eq. (75), respectively. We then express $V^\alpha\overline{\mathfrak{A}}$ as

$$V^\alpha\overline{\mathfrak{A}}=\sum_i v_i\left(1-\sum_j E_{0j}\right)\equiv\sum_i w_i+\sum_{i<j}X_{ij} \tag{79}$$

with

$$w_i=v_i(1-E_{0i}) \tag{80}$$

and

$$X_{ij}=-(v_iE_{0j}+v_jE_{0i}). \tag{81}$$

We then write, in analogy with Eq. (69),

$$\hat{T}=\sum_i\hat{T}_i+\sum_{i<j}\hat{S}_{ij}+\cdots, \tag{82}$$

and insert Eqs. (79)–(82) into Eq. (78) to obtain

$$\sum_i\hat{T}_i+\sum_{i<j}\hat{S}_{ij}+\cdots=\left(\sum_i w_i+\sum_{i<j}X_{ij}\right)\times\left(1+G_\alpha\sum_k T_k+G_\alpha\sum_{k<l}S_{kl}+\cdots\right), \tag{83}$$

from which we obtain, through an obvious identification,

$$\hat{T}_i=w_i(1+G_\alpha T_i)=v_i(1-E_{0i})(1+G_\alpha T_i). \tag{84}$$

The quantity $\hat{T}_i$ becomes, if G_α is replaced by the free two-particle propagator, the antisymmetrized two-body transition operator. The general case is given by Eq. (68) for an arbitrary number of target particles. From Eq. (83) one may then identify $\hat{S}_{ij}$ as

$$\begin{aligned}\hat{S}_{ij}&=w_iG_\alpha T_j+w_jG_\alpha T_i+(w_i+w_j)G_\alpha S_{ij}+X_{ij}[1+G_\alpha(T_i+T_j+S_{ij})]\\&=(w_i+w_j+X_{ij})[1+G_\alpha(T_i+T_j+S_{ij})]-w_i(1+G_\alpha T_i)-w_j(1+G_\alpha T_j)\\&=(w_i+w_j+X_{ij})[1+G_\alpha(T_i+T_j+S_{ij})]-(\hat{T}_i+\hat{T}_j).\end{aligned} \tag{85}$$

From this one may recognize that

$$\hat{T}_i+\hat{T}_j+\hat{S}_{ij}=\hat{T}_{ij}, \tag{86}$$

where $\hat{T}_{ij}$ satisfies

$$\hat{T}_{ij}=(v_i+v_j)(1-E_{0i}-E_{0j})(1+G_\alpha T_{ij}), \tag{87}$$

which becomes, upon approximation of the propagator in analogy with Eq. (76), the antisymmetrized three-body transition operator.[17] Combining results, we have the not entirely unexpected result,

$$\hat{T}=V^\alpha\overline{\mathfrak{A}}\Omega=\sum_i(V^\alpha\overline{\mathfrak{A}}\Omega)_i+\sum_{i<j}[(V^\alpha\overline{\mathfrak{A}}\Omega)_{ij}-(V\overline{\mathfrak{A}}\Omega)_i-(V\overline{\mathfrak{A}}\Omega)_j]+\cdots \tag{88}$$

or

$$\hat{T}=\sum_i\hat{T}_i+\sum_{i<j}(\hat{T}_{ij}-\hat{T}_i-\hat{T}_j)+\cdots. \tag{89}$$

The expansion given in Eq. (88) is, of course, in complete analogy with the unsymmetrized expansion of Eq. (77). This fact, coupled with the fact that the definition of the symmetrized optical potential operator $\hat{U}$ by Eq. (67) exactly parallels the definition of the unsymmetrized operator U, viz.,

$$U = T - TG_\alpha PU \,, \tag{90}$$

allows one to write the expansion for $\hat{U}$ by inspection of the expansion of U. In order to take advantage of this, we now derive the expansion[18] for U in a manner which makes the preceding assertion apparent. We begin by writing U (or analogously $\hat{U}$) as

$$U = \sum_i U_i + \sum_{i<j} W_{ij} + \cdots \,, \tag{91}$$

and insert this equation into Eq. (90) to obtain

$$\begin{aligned}\sum_i U_i + \sum_{i<j} W_{ij} + \cdots &= \sum_i T_i(1 - G_\alpha PU_i) - \sum_{i<j}[T_i G_\alpha PU_j + T_j G_\alpha PU_i + (T_i + T_j)G_\alpha PW_{ij} + S_{ij}G_\alpha P(U_i + U_j + W_{ij}) - S_{ij}] + \cdots \\ &= \sum_i T_i(1 - G_\alpha PU_i) + \sum_{i<j}\{(T_i + T_j + S_{ij})[1 - G_\alpha(U_i + U_j + W_{ij}) - U_i - U_j]\} + \cdots \,.\end{aligned} \tag{92}$$

From Eq. (92) one may identify

$$U_i = T_i - T_i G_\alpha PU_i \tag{93}$$

and

$$(U_i + U_j + W_{ij}) = (T_i + T_j + S_{ij}) \times [1 - G_\alpha P(U_i + U_j + W_{ij})] \,, \tag{94}$$

which with the definition

$$U_i + U_j + W_{ij} = U_{ij} \,, \tag{95}$$

leads to

$$U_{ij} = (T_i + T_j + S_{ij})(1 - G_\alpha PU_{ij}) \tag{96a}$$

$$= T_{ij}(1 - G_\alpha PU_{ij}) \,. \tag{96b}$$

The corresponding "one-body" optical potential is then given by

$$PUP = \sum_i PU_iP + \sum_{i<j} P(U_{ij} - U_i - U_j)P + \cdots \,. \tag{97}$$

The analogous symmetrized results are obtained by replacement of the operators in Eqs. (93)–(97) by the corresponding "hat" operators. We note that Eqs. (93) and (96b) allow us to write

$$PU_iP = PT_iP - PT_iPG_\alpha PU_iP \tag{98a}$$

and

$$PU_{ij}P = PT_{ij}P - PT_{ij}PG_\alpha PU_{ij}P \,. \tag{98b}$$

The result given by Eqs. (97) and (98) is most striking. It implies that

$$PUP = A(1 + PT_1PG_\alpha)^{-1}PT_1P + \tfrac{1}{2}A(A-1) \times [(1 + PT_{12}PG_\alpha)^{-1}PT_{12}P - 2(1 + PT_1PG_\alpha)^{-1}PT_1P] + \cdots \,. \tag{99}$$

That is to say, the optical potential, to second order in the spectator expansion, requires only PT_1P and $PT_{12}P$ as input. If the approximation of the propagators discussed earlier is used, these reduce to an overlap integral of solutions of two- and three-body equations, respectively. The expressions $(1 + PT_1PG_\alpha)^{-1}PT_1P$ and $(1 + PT_{12}PG_{\alpha 0})^{-1}PT_{12}P$ are merely the solutions of the one-body integral equations given by Eqs. (98).

For the antisymmetrized problem the analogous result is simply

$$P\hat{U}P = A(1 + P\hat{T}_1PG_\alpha)^{-1}P\hat{T}_1P + \tfrac{1}{2}A(A-1) \times [(1 + P\hat{T}_{12}G_\alpha)^{-1}P\hat{T}_{12}P - 2(1 + P\hat{T}_1PG_\alpha)^{-1}P\hat{T}_1P] + \cdots \,, \tag{100}$$

where the requisite input is $P\hat{T}_1P$ and $P\hat{T}_{12}P$. With the usual approximation of the propagator this simply prescribes the use of the appropriately symmetrized two- and three-body quantities.

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[9]The restriction to two-body forces is by no means necessary and is adopted for the sake of simplicity only. In fact, since the Pauli principle is effectively treated as a multibody force in Sec. IV, the extension to many-body forces will become evident. See also Ref. 6.

[10]Actually the definition of the symmetrized AGS operator used previously (Refs. 1 and 2) is given by $\tilde{T}R_\alpha$. We use the definition of Eq. (8) for the sake of simplicity. The modifications necessary to incorporate R_α somewhat obscure the central features of our treatment. Thus it must be understood that the nuclear states we use are already antisymmetric with respect to the interchange of any pair of identical fermions within one or the other cluster.

[11]The nature of the difficulty with $\tilde{V}$ can also be seen, perhaps more clearly, if we were to set $P=1$, $Q=0$ in Eq. (57), which would then yield $\tilde{V}$, if it existed.

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[15]Generalization, however, appears to be amenable to the methods of Sec. 2 of Ref. 14, for example.

[16]The general technique to be used below has its origin in Ref. 6.

[17]The presence of the antisymmetrizer between particles i and j is implicit in Eq. (87), as it is also in Eq. (76). In fact, since the operators are defined to operate on states $|\phi_\alpha\rangle$ for which $R_\alpha|\phi_\alpha\rangle=|\phi_\alpha\rangle$, the antisymmetrizer between the active target particles can always be abstracted to the operators. The fact that the antisymmetrizer commutes with both G_α and the potential sums then establishes the interpretation of Eq. (87).

[18]The spectator expansion for the unsymmetrized optical potential given here is that of P. C. Tandy and R. M. Thaler, Phys. Rev. C 22, 2321 (1980).