Relevance of final-state nucleon-nucleon charge exchange in inclusive ($\pi,\pi N$) reactions

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The Sternheim-Silbar linear transport model for calculating the effect of final-state charge exchange of the struck nucleon in ($\pi,\pi N$) reactions has been reexamined. It is demonstrated that for ^{12}C, the effect ranges from only 10% at low pion energies to 4% at the (3,3) resonance and plays a *minor* role in determining the magnitude of the inclusive cross sections. A semiclassical linear transport model is used to calculate the π^-/π^+ ratio for a clean, one-step nucleon knockout. Unlike the Sternheim-Silbar calculation these results are in very good agreement with intranuclear cascade model predictions.

[NUCLEAR REACTIONS $\sigma(\pi,\pi N)$ ratios, (3,3) resonance, final state interaction, nucleon removal, one-step quasi-free and intranuclear cascade calculations.]

I. INTRODUCTION

Recent results[1] for the absolute cross sections of the $^{12}C(\pi^{\pm},\pi N)^{11}C$ reactions between 40 and 600 MeV have been interpreted in terms of a linear transport model constructed by Sternheim and Silbar[2] from a suggestion originating with Hewson.[3] Sternheim and Silbar estimate the probability P that a nucleon struck by an incident pion will undergo charge exchange within the residual nucleus. As a consequence, the ratio of π^- to π^+ induced neutron removal reactions near the (3, 3) isobar resonance energy will be reduced from the value 3.0 predicted by single-step impulse approximation considerations. For a self-conjugate target, the ratio becomes

$$R_n=\frac{\sigma_{\pi^-}}{\sigma_{\pi^+}}=\frac{\sigma_{\pi^-n}(1-P)+\sigma_{\pi^-p\to\pi^-p}P}{\sigma_{\pi^+n}(1-P)+\sigma_{\pi^+p}P}\approx\frac{9-8P}{3+6P}, \qquad (1)$$

where the approximation is valid only near the resonance. Experimentally, $R_n=1.59\pm0.07$ at the resonance. The charge exchange probability P is a function of the nucleon-nucleon charge exchange cross section and the path of the struck nucleon through the nucleus. Success of the Sternheim-Silbar model is predicated on its unmatched ability to account for the behavior of R_n over a wide energy range for several light nuclei[4] subject to a single normalization. Final state charge exchange is allocated a large role in nucleon removal reactions as attested to by the values of P necessary to explain the recent activation measurements. For ^{12}C, P rises from ~16% at 250 MeV to 45% at 40 MeV.

On the other hand, the intranuclear cascade (INC) model, which requires no normalization and which presumably entails all that the Sternheim and Silbar model does and more, has also been used to calculate R_n, yielding a value of $R_n=2.4$ for ^{12}C at the resonance.[5] Dropesky *et al.*,[1] and others have been puzzled by this serious discrepancy. They refer to a conjecture by Silbar *et al.*[6] that the intranuclear cascade model employed probably neglects unspecified quantum mechanical coherence in the nucleon charge exchange interaction.

The following report addresses the discrepancy between the Sternheim-Silbar nucleon charge exchange model and the INC calculations of Harp *et al.*[5] Despite accumulating experimental contraindications, the former is still seriously advocated because there is no outstanding evidence against it.[6] What will emerge here is that the linear transport calculation, when repaired, is in very good agreement with the intranuclear cascade predictions. Furthermore, the final state charge exchange effect turns out, as expected, to be much smaller than claimed.

II. SEMICLASSICAL LINEAR TRANSPORT MODEL

A. Sternheim-Silbar method

The probability that a nucleon, struck within the nucleus by an incident pion, undergoes charge-exchange before emerging is given by Sternheim and Silbar[2,7] as

$$P=\tfrac{1}{2}[1-\exp(-\rho\sigma_{ex}\langle d\rangle)] \qquad (2)$$

for a nucleus of mass A and uniform density $\rho=3/4\pi R^3$. The cross section for nucleon charge exchange is σ_{ex} and $\langle d\rangle$ represents the average distance traveled by the struck nucleon within the nucleus. Assuming the entire reaction process is colinear, the calculation of $\langle d\rangle$ may be visualized by referring to Fig. 1. If $\lambda_\pi=(\sigma_\pi\rho)^{-1}$ is the pion mean free path in the nucleus, then $P_{\pi N}(z)dz=\sigma_\pi\rho\exp[-\sigma_\pi\rho(l+z)]dz$ is the probability at impact pa-

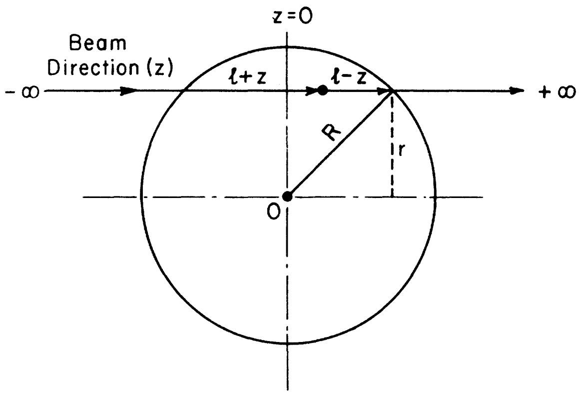

FIG. 1. Cylindrical coordinate system for linear transport at impact parameter r along axis z defined by beam entering from $-\infty$. Uniform nucleus has radius R and $l = (R^2-r^2)^{1/2}$.

rameter r that there is a pion-nucleon interaction between z and $z+dz$ where we have, for convenience, introduced the abbreviation

$$l = l(r) \equiv (R^2 - r^2)^{1/2}.$$

$P_{\pi N}(z)dz$ also represents the probability that the struck nucleon must travel a remaining distance $l-z$ in order to escape. Averaging this distance over all impact parameters corresponds to the expression

$$\langle d\rangle = \frac{2\pi \int r dr \int (l-z) e^{-\sigma_\pi \rho(l+z)} dz}{2\pi \int r dr \int e^{-\sigma_\pi \rho(l+z)} dz}. \tag{3}$$

The limits in Eq. (3) are $0 \le r \le R$ and $-l \le z \le +l$. The denominator in Eq. (3) is necessary in order to exclude transparencies from the averaging procedure. Using the abbreviation

$$x \equiv 2R/\lambda_\pi = 2\sigma_\pi \rho R, \tag{4}$$

the above integrals may be solved exactly to give

$$\langle d\rangle = \frac{4R}{3}\left[1 - \frac{3}{2x} + f(x)\right], \tag{5}$$

in which

$$f(x) = [1 - e^{-x}(1+x)] / [\tfrac{1}{2}x^2 - 1 + e^{-x}(1+x)]. \tag{6}$$

Examination of $\langle d\rangle$ in Eq. (5) shows that in the limit of strong pion-nucleon interactions, e.g., near the (3, 3) resonance, x becomes large and $\langle d\rangle$ approaches $4R/3$. As expected, this is the average chord length of a sphere of radius R, the pions all "reacting" at the surface of the upstream hemisphere. In the weak interaction limit

$$\lim_{x\to 0} f(x) = \frac{3}{2x} - 1$$

or $\langle d\rangle = 0$, again as expected since the pion "never" undergoes its reaction-initiating collision with a nucleon.[8]

However, the difficulty with the Sternheim-Silbar calculation arises elsewhere. The appro- averaging should be done, not for $d(r)$, the distance to be traveled by the struck nucleon at a given impact parameter, but rather for the entire sequential process. This point is most succinctly expressed by the inequality in cylindrical coordinates

$$(1 - e^{-\langle y(r)\rangle}) \neq \langle 1 - e^{y(r)}\rangle.$$

As will be shown, the above fact proves to be especially germane to the ensuing issue. For example, the implication of Eq. (5) that in the strong interaction limit, nucleon removal reactions occur uniformly at the surface of the beamside hemisphere, is incorrect as has been amply demonstrated.[9] Such reactions are localized to the nuclear surface *equatorial* region defined by the beam direction.

B. Rectified Sternheim-Silbar collinear transport model

Following the above discussion, the linear transport approach is retained, but redeveloped as follows. From the integrand of Eq. (3), one can calculate the residual path length at impact parameter r; that is, the distance to be traveled subsequent to pion nucleon collision.

$$d(r) = \sigma_\pi \rho \int (l-z) e^{-\sigma_\pi \rho(l+z)} dz$$

$$= 2l - \left(\frac{1}{\sigma_\pi \rho}\right) + \frac{e^{-2l\sigma_\pi \rho}}{\sigma_\pi \rho}. \tag{7}$$

Using $d(r)$, we can define the probability at impact parameter r that the struck nucleon undergoes a final state interaction. This probability is

$$P(r) = 1 - \exp\left\{-\sigma\rho\left[2l - \frac{1}{\sigma_\pi \rho} + \frac{\exp(-2l\sigma_\pi \rho)}{\sigma_\pi \rho}\right]\right\}. \tag{8}$$

In Eq. (8) $(\sigma\rho)^{-1} = \lambda_N$ is the nucleon mean free path at some average recoil energy. Integration of $P(r)$ over all impact parameters and azimuthal angle gives the average probability for a final state interaction following an initial pion-nucleon collision.

$$\langle P\rangle = 2\pi \int_0^R P(r) r dr / \pi R^2. \tag{9}$$

Final state interaction of the scattered pion is ignored.[10] However, we are interested in only a fraction of all possible final state interaction channels: those involving charge exchange leading to a particle-stable residual nucleus. At impact parameter r, the charge-exchange probability may be written in terms of the probability of the primary nucleon scattering into a final single particle state of energy U_i ejecting a secondary nucleon which leaves a hole excitation energy E_j. These particle-hole energies are crucial to what follows. For the charge-exchanged residue, the condition of particle stability is that $E_j + U_i \leq S$, where S is the separation energy of the least bound nucleon in the residual nucleus. If $P_j(E_j + U_i \leq S)$ represents the probability of this charge exchange with a nucleon initially in bound state j and ρ_j/ρ represents the fraction of total nucleon density corresponding to nucleons in state j, then $P_{ex}(r)$ is given by

$$P_{ex}(r) = \left[\sum (\rho_j/\rho)P_j(E_j + U_i \leq S)\right]P(r) . \tag{10}$$

Final states, such as both nucleons being in the unbound continuum, are excluded.

In a system such as $^{12}C \to {}^{11}C$, ^{11}B the above sum is replaced by a single term involving the outermost nucleon shell ($1p$), since E_j for the inner shell ($E_{1s} \sim 40$ MeV) already exceeds S (~ 11 MeV). Removing a nucleon from the "available" shell in this system corresponds to $E_j \simeq 0$ and $P_{ex}(r)$ becomes

$$P_{ex}(r) = (\rho_j'/\rho)P_j(U_j \leq S)P(r) . \tag{10'}$$

$P_j(U_j \leq S)$, the appropriate fraction of all possible np scatterings, is specifically the probability that the retained nucleon was scattered into a particle-stable state. It is independent of r as are the densities in the uniform density model employed. Averaging over impact parameters and azimuthal angle gives $\langle P_{ex}\rangle \equiv P_{ex}$. For a *self-conjugate* target, Eq. (1) becomes (in $N \neq Z$ nuclei, more complicated expressions will pertain, see Ref. 7)

$$R_n = \frac{\sigma_{\pi^-}}{\sigma_{\pi^+}} = \frac{\sigma_{\pi^- n}(1 - \langle P\rangle) + \sigma_{\pi^- p \to \pi^- p}\langle P_{ex}\rangle}{\sigma_{\pi^+ n}(1 - \langle P\rangle) + \sigma_{\pi^+ p}\langle P_{ex}\rangle} . \tag{11}$$

Since the crucial nuclear structure requirement omitted in Ref. 7 has been accommodated through the factor ρ_j'/ρ, the scattering may be treated, as done by Sternheim and Silbar, using the free particle-free particle picture

$$P_j(U_j \leq S) = \sigma_{ex}/\sigma ,$$

where we use Sternheim and Silbar's σ_{ex}

$$\sigma_{ex} = 2\pi \int_0^{\theta_{max}} \sin\theta \, d\theta (d\sigma_{ij}/d\Omega) .$$

Although the angular limits completely neglect the contribution of the hole energy E_j to the excitation energy, the relationship is applicable for ^{12}C when $E_j = 0$, i.e., for the four $1p$ valence nucleons. As in Ref. 7, σ_{ex} may be approximated by

$$\sigma_{ex} \simeq \beta \frac{4\pi Q}{T_N}\left(a - \frac{bQ}{T_N}\right) , \tag{12}$$

in which β is the Pauli blocking factor; $\beta \sim 0.45$ fits the π, πN data for ^{12}C; $Q = 11$ MeV $\sim S$ is the average nucleon separation energy for ^{11}C; $T_N \sim \frac{1}{3}T_\pi$ is the average recoil kinetic energy of the struck nucleon; and a and b are coefficients of the np charge exchange differential scattering cross section

$$(d\sigma_{ij}/d\Omega) = a - b(1 - \cos\theta) .$$

By specifying the target nucleus, the incident projectile energy, and employing some of the approximations for θ_{max} and $d\sigma/d\Omega$ suggested by Sternheim and Silbar,[7] estimation of the total final state interaction factor $\langle P\rangle$ and the charge-exchange factor P_{ex} becomes straightforward. [For a more extensive discussion of the rationale behind the development of Eq. (10) see Refs. 9 and 11–13.]

For ^{12}C as a target, $\langle P\rangle$ was evaluated by numerical integration of Eq. (9) and was maximum at 0.95 just below resonance, dropping to 0.91 at 50 MeV and 0.68 at 350 MeV. This behavior is not surprising because, for a pion near resonance, λ_π is short and the struck nucleon with $\lambda_N \sim 1.2$ fm travels relatively far within the nucleus, increasing the likelihood of collision. For a pion well below resonance, the struck nucleon's $\lambda_N \sim 0.2$ fm and transmission is still small; but well above resonance, λ_N and λ_π are each ~ 2.3 fm and $\langle P\rangle$ decreases. One would infer from these $\langle P\rangle$'s that the "clean knockout" cross section, which will depend on $1 - \langle P\rangle$, drops to a suspiciously low value. Such behavior, however, is attributable partly to an artifact of the use of a uniform density distribution and partly again to the very large initial projectile-nucleon interaction cross section near resonance. INC calculations have demonstrated this in somewhat comparable systems.[13-15] By employing realistic nucleon density distributions, correct magnitudes are predicted.[9]

For ^{12}C, only the four valence neutrons are available to contribute to charge exchange;

$$P_{ex} = \langle P\rangle \sigma_{ex}\rho_j'/\sigma\rho , \tag{13}$$

where ρ_j' refers to available nucleons.

From Scanlon's np data[16] and Eq. (12) we find

$$\sigma_{ex} \sim 5.8 \times 10^5 \beta T_\pi^{-1.9} \text{ mb} , \tag{14}$$

for carbon, where T_π is in MeV.

The total nucleon-nucleon cross section σ goes roughly as

$$\sigma \sim 9.0 \times 10^3 T_N^{-1.1} \text{ mb} . \tag{15}$$

Using Eqs. (14) and (15) in Eq. (13) we arrive at an expression for the charge-exchange probability in carbon.

$$P_{ex} = 6.4\beta \langle P\rangle T_\pi^{-0.8} . \tag{16}$$

Again following the normalization of Sternheim and

Silbar choosing $\beta \sim 0.45$, Eq. (16) becomes

$$P_{ex} \sim 2.9\langle P\rangle T_\pi^{-0.8} . \tag{17}$$

Before proceeding any further with the above expression, we can lend qualitative support to its approximate validity by looking at the $^{11}B(p,n)^{11}C$ reaction in a comparable energy interval; namely $20 \le T_p \le 150$ MeV. The cross section for the (p,n) reaction can be estimated very roughly by the product of the total reaction cross section for ^{11}B towards protons, times the probability that a charge exchange takes place. Recalling that in Eq. (17) we made use of $T_N \approx \frac{1}{3}T_\pi$, and using $\langle P\rangle \sim 1$, we can write

$$\sigma(p,n) \approx \sigma_R P_{ex} = 1.2\sigma_R T_p^{-0.8} . \tag{18}$$

The total reaction cross section of ^{11}B may be calculated from the "soft spheres model."[17] Both the total reaction cross section and the (p,n) cross section from Eq. (18) are plotted in Fig. 2(a). Also shown are experimental values for the $^{11}B(p,n)^{11}C$ reaction. In Fig. 2(b), this is repeated for $^{89}Y(p,n)^{89}Zr$. Considering the extremely approximate nature of the calculation, the agreement is quite satisfactory. The use of P_{ex} from Ref. 7 gives $\sigma(p,n)$ values five times larger than experiment. More realistic treatment of (p,n) reactions using semiclassical collinear transport is beyond the scope of this work but has been dealt with by Read and Miller.[11]

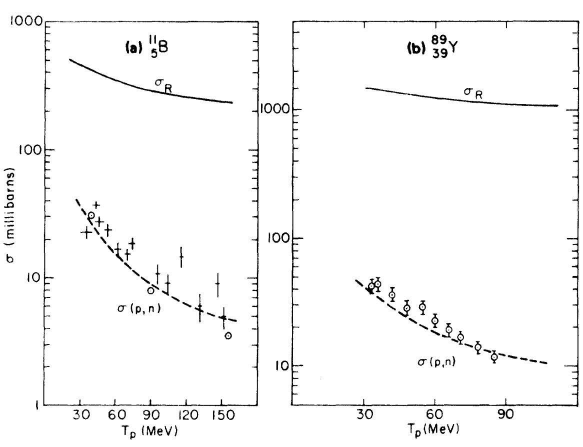

FIG. 2. (a) Experimental (p,n) excitation function for ^{11}B [crosses, G. Albouy *et al.*, J. Phys. Radium 23, 1000 (1962); circles, L. Valentin, Nucl. Phys. 62, 81 (1965)] compared to Eq. (18) (dashed curve) assuming only the uppermost neutron level ($p_{3/2}$) contributes. Solid curve is total reaction cross section. (b) Experimental (p,n) excitation function for ^{89}Y [G. B. Saha *et al.*, Phys. Rev. 144, 962 (1966)] compared to model (dashed curve), assuming only the three uppermost neutron levels ($g_{9/2}$, $p_{1/2}$, $f_{5/2}$) contribute and total reaction cross section (solid curve).

Returning to charge exchange in the pion-induced nucleon-removal reactions expressed by Eq. (17), a plot of P_{ex} vs T_π shown in Fig. 3 reveals that, in the energy range of interest, P_{ex} varies from a maximum of ~11.5% to a minimum of ~2%. In Fig. 4(a) the dashed curve shows that the ratio R_n defined in Eq. (1) does not agree with experiment even if rescaling by parametrization of β is allowed. Letting $P_{re} = P - P_{ex}$ represent the remaining fraction of final state interactions—those not contributing—we can express Eq. (11) in a more illustrative manner

$$R_n = \frac{\sigma_{\pi^- n}(1 - P_{ex} - P_{re}) + \sigma_{\pi^- p \to \pi^- p} P_{ex}}{\sigma_{\pi^+ n}(1 - P_{ex} - P_{re}) + \sigma_{\pi^+ p} P_{ex}} . \tag{11'}$$

If and only if $P_{re} \ll P_{ex}$ do we get the Sternheim and Silbar expression, Eq. (1). The opposite, however, pertains. To recapitulate, the discrepancy is a direct consequence of ignoring two important features: reaction site localization and competing transmission attenuation channels.

C. Semiclassical one-step collinear nucleon knockout

In rectifying the Sternheim-Silbar linear transport model, we were led to a relationship which can be easily adapted to calculating the probability for a clean, one-step nucleon removal by an incident pion. As in Sec. II A, if $P_{\pi N}(z)dz = \sigma_\pi \rho\{\exp[-\sigma_\pi \rho(l+z)]\}dz$ is the probability at impact parameter r that a pion enters the nucleus and strikes a nucleon between z and $z+dz$, then

$$\sigma_\pi \rho\{\exp[-\sigma_\pi \rho(l+z)]\}\{\exp[-(\sigma_\pi' \rho + \sigma' \rho)(l-z)]\}dz$$

is the probability that a pion-nucleon collision

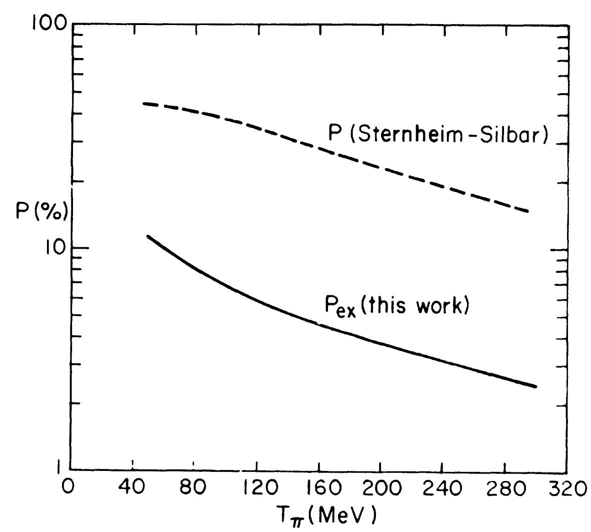

FIG. 3. Dashed curve: Sternheim-Silbar final-state charge exchange probability P for ^{12}C. Solid curve: reevaluated probability P_{ex}.

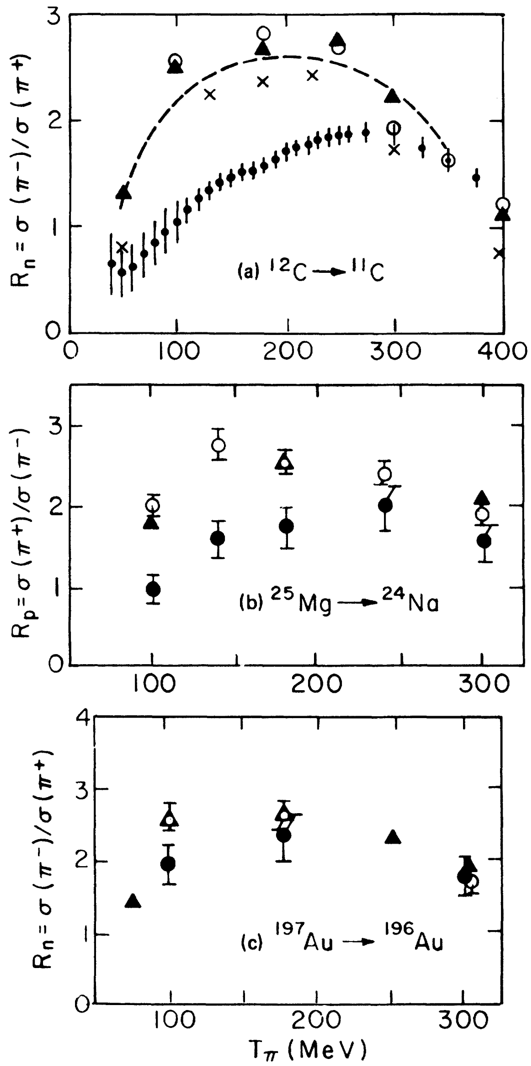

FIG. 4. (a) $R_n = \sigma(\pi^-)/\sigma(\pi^+)$ for $^{12}C \rightarrow {}^{11}C$. Experiment with error bars, and INC calculations as crosses (Ref. 1); triangles, Benioff-type calculation (this work); circles, uniform density (this work); dashed curve, corrected Sternheim-Silbar model (this work). (b) $R_p = \sigma(\pi^+)/\sigma(\pi^-)$ for $^{25}Mg \rightarrow {}^{24}Na$. Experiment, closed circles; and INC results, open circles (Ref. 19); triangles, Benioff-type calculation (this work). (c) $R_n = \sigma(\pi^-)/\sigma(\pi^+)$ for $^{197}Au \rightarrow {}^{196}Au$ as in (b).

occurs between z and $z+dz$, followed by escape of both forward scattered particles with mean free paths $\lambda'_\pi = (\sigma'_\pi\rho)^{-1}$ and $\lambda'_N = (\sigma'\rho)^{-1}$. The cross section for nucleon removal from any state in this collinear model is

$$\sigma(\pi, \pi' N) = 2\pi \int_0^R r\,dr \int_{-l}^{+l} \sigma_\pi \rho e^{-\sigma_\pi \rho (l+z)} e^{-(\sigma'_\pi \rho + \sigma' \rho)(l-z)} dz\,. \tag{19}$$

A similar expression results for a collinear trajectory with backward scattering of the pion. Upon integration Eq. (19) gives

$$\sigma(\pi, \pi' N) = [\sigma_R(\pi' + N') - \sigma_R(\pi)] \frac{\sigma_\pi}{\sigma_\pi - (\sigma'_\pi + \sigma')[(A-1)/A]}\,, \tag{20}$$

in which $\sigma_R(\pi)$ and $\sigma_R(\pi' + N')$ are the total reaction cross sections for a uniform density sphere of mass number A towards a "single particle" with mean free paths λ equal to $(\sigma_\pi\rho)^{-1}$ and $(\sigma'_\pi\rho + \sigma'\rho)^{-1}$, respectively. $\sigma_R(\nu)$ is identical to the expression derived by Fernbach, Serber, and Taylor[18]

$$\sigma_R(\nu) = \pi R^2 \left[1 - \frac{1 - e^{-2R/\lambda_\nu}(1 + 2R/\lambda_\nu)}{2(R/\lambda_\nu)^2}\right]. \tag{21}$$

Equation (20), if multiplied by the fraction of pion-nucleon collisions with available nucleons (for example, the four $1p$ neutrons in carbon) provides an expression for the appropriate nucleon removal leading only to particle-stable final states and an exiting πN pair. Within this collinear model, the cross section for neutron removal can be approximated as a function of energy from free particle cross sections and target radius. We have done this as a function of pion energy for π^+ and π^- and the results for $R_n = \sigma_{\pi^-}/\sigma_{\pi^+}$ are shown as circles for ^{12}C in Fig. 4(a). Because of the sensitivity of nucleon removal reactions to the diffuseness of the nuclear "surface" alluded to earlier, calculations involving a uniform density distribution should not be taken too seriously.[13]

A much more realistic assessment of nucleon removal reactions was made by Benioff[9] using collinear transport and a harmonic oscillator shell model. Maintaining the linearity approach, Benioff's equations are readily adaptable to pion induced reactions, parametrized by incident pion-nucleon, escaping pion-nucleon and nucleon-nucleon collision cross sections and density distributions for nucleons. In essence, Benioff's treatment goes one step further than that embodied in Eq. (19) by using a nucleon density distribution function that varies realistically with r and z, with different forms associated with different shell states. Excitation functions for $\pi^\pm$ reactions $^{12}C \rightarrow {}^{11}C$, $^{25}Mg \rightarrow {}^{24}Na$, and $^{197}Au \rightarrow {}^{196}Au$, and ratios R_n and R_p calculated according to Benioff's prescription are illustrated in Figs. 4(a)–4(c) and compared to recent experimental data[1,19] and intranuclear cascade calculations.[1,19] Agreement with the more sophisticated Monte Carlo method is very good.

A review of the literature on nucleon removal reactions shows that the inability of the intranuclear cascade calculations and the impulse approximation to reproduce observed cross sections to better than 50% accuracy has not been confined to pion-induced systems. Cross sections for proton-induced proton-removal reactions[4] and neutron-removal reactions[13,20] as well as anti-proton-induced neutron-removal reactions[21] have been fairly consistently misjudged, although not by large factors.

III. CONCLUSION

The straightforward model of Sternheim and Silbar for calculating the effect of a final state nucleon charge exchange in pion-induced nucleon removal reactions has been reexamined. It has been shown that, within the probably oversimplified but now corrected treatment, the final state interaction in question accounts for only a small percentage of observed cross sections. A similarly simplified linear transport model of clean, one-step quasi-free nucleon removal reactions gives results that are in good agreement with Monte Carlo intranuclear cascade calculations. The Sternheim-Silbar explanation for the major cause of the discrepancy between experiment and impulse approximation predictions is apparently invalid, leaving the matter open for further speculation.

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