Two-pion-exchange three-nucleon potential: Partial wave analysis in momentum space

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We present the complete momentum space three-nucleon potential of the two-pion-exchange type in the partial wave decomposition needed for the Faddeev equations of the three-nucleon bound state. The potential arises from an off-mass-shell model for πN scattering based upon current algebra and a dispersion-theoretical axial vector amplitude dominated by the Δ(1230) isobar. The potential is manifestly Hermitian and defined for all three nucleon momenta. We display some matrix elements of the potential in the five three-body partial waves corresponding to the $^1S_0$ and $^3S_1$-$^3D_1$ states of the two-body subsystem. These matrix elements show a striking contrast to those of an older three-body potential mediated only by the Δ(1230) p-wave resonance.

[NUCLEAR STRUCTURE Three-body potential; few-nucleon system, Faddeev approach, partial wave decomposition in Jacobi variables.]

I. INTRODUCTION

The discrepancy between theoretical and experimental binding energies of light nuclei may be partly due to the neglect of three-body forces. The most studied of these forces is the two-pion-exchange type, illustrated in Fig. 1. If one takes the diagrams in Fig. 1 to be Feynman graphs and attempts a nonrelativistic reduction to obtain a potential, then the nucleon legs are on their mass shells, but the pions are off the mass shell and spacelike. Thus, one needs an off-mass-shell parametrization of the πN scattering amplitude. Current algebra and PCAC constraints can be used to determine this off-mass-shell extrapolation[1-3]; on shell the model agrees very well with the data below,[1,3] at and slightly above threshold.[4] A three-body potential based on this extrapolation was constructed in Ref. 5, which will be called, in the following, I.

In this paper we prepare the three-body potential of I for use in a Faddeev calculation of the triton. In doing so, we found a defect in that potential—a

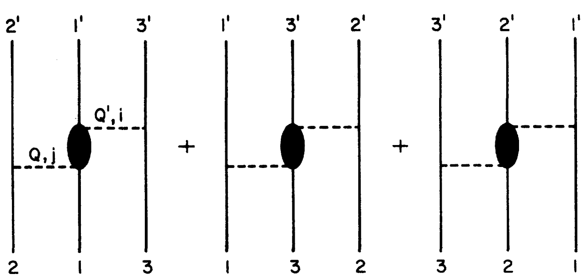

FIG. 1. The Feynman diagrams for the two-pion exchange three-nucleon potential. The shaded oval represents anything except a forward propagating nucleon state.

defect shared by the often used[6-9] Fujita-Miyazawa (FM) potential,[10] the potential derived by Yang,[11] and more recent extensions[12] of the FM potential. In all these potentials the middle nucleon was assumed to be at rest, that is, fixed in one place. This could arguably be a resonable assumption for three-body force effects in nuclear matter or in the effective interaction of shell model theory,[13] but is surely intolerable in the three-body problem where there exist methods for numerically exact solutions of a nonrelativistic Hamiltonian. This assumption, in general, causes a non-Hermitian potential matrix. The non-Hermiticity was not detected in Refs. 6–12 because of the restricted nature of the πN scattering amplitude used there. It did happen to occur in a numerically small portion of the potential of I.

Thus, before proceeding to the triton calculation, it is first necessary to sketch the derivation of the three-body potential of I for the case in which all three nucleons are allowed to move freely. Also, we carry out a consistent expansion in powers of $|\vec{p}|/m$ of the diagrams in Fig. 1. This paper should be read together with I, where the physical ideas were discussed.

To include the three-body force nonperturbatively in a Faddeev calculation of the triton, one needs a partial wave decomposition. This turns out to be highly nontrivial since the Jacobi variables do not occur naturally in the potential of Fig. 1. Another main part of this paper is to carry through that partial wave decomposition.

In Sec. II we rederive the three-body potential matrix in momentum space. The transition to coordinate space is worked out for the dominant parts of the force in Sec. III. In Sec. IV we present the expressions of the three-body force in a partial wave decomposition. Finally, in Sec. V we compare our results based on the general off-shell πN amplitude as given in Refs. 1–5 with the specific model of Δ dominance of Ref. 6, and give a summary and outlook in Sec. VI.

II. THE TWO-PION EXCHANGE THREE-BODY POTENTIAL IN MOMENTUM SPACE

We consider the general expression for the three-three scattering amplitude corresponding to the first diagram of Fig. 1 with particle 1 in the middle. The nonrelaticistic reduction of the Feynman diagram with pseudoscalar coupling yields the potential matrix

$$\langle \vec{p}_1', \vec{p}_2', \vec{p}_3' | W_1 | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle = \text{const}\, \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_1' - \vec{p}_2' - \vec{p}_3') \times \frac{g^2}{4m^2} \vec{\sigma}_2 \cdot \vec{Q} \vec{\sigma}_3 \cdot \vec{Q}' \tau_2^j \tau_3^i \times \frac{H(\vec{Q}^2)}{\vec{Q} + \mu^2} \frac{H(\vec{Q}'^2)}{\vec{Q}'^2 + \mu^2} T_1^{ij}(\text{NR}). \tag{2.1}$$

In this paper we will reserve the three-vectors $\vec{p}$ and $\vec{q}$ for three-body Jacobi variables and, therefore, let $\vec{Q} = \vec{p}_2 - \vec{p}_2'$ and j be the three-momentum and isospin components of the initial pion, and let $\vec{Q}' = \vec{p}_3{}' - \vec{p}_3$ and i describe the final pion.[14] Then, $T_1^{ij}(\text{NR})$ is the nonrelativistic reduction of the amplitude which describes the scattering process $\pi^j(Q) + N(p_1) \to \Pi^i(Q') + N(p_1')$. This amplitude can be written as a sum of s- and u-channel nucleon poles plus background terms. The forward propagating Born term (FPBT) is already included in the Schrödinger equation as the iterate of the one-pion exchange two-nucleon potential (OPEP). Therefore, it must be explicitly calculated and subtracted from the covariant pole term which includes both the forward propagating and backward propagating intermediate nucleons. We fix the overall multiplicative constant in (2.1) by requiring that the FPBT of $T_1^{ij}(\text{NR})$ yield the iteration of OPEP. The form factor $H(\vec{Q}^2) = K^2(Q^2)$, where $K(Q^2)$ is the pionic form factor of the πNN vertex when the nucleons are on the mass shell and the pion is off the mass shell [i.e., $g(Q^2) = gK(Q^2)$ and $g(\mu^2) = g$]. The correction terms neglected in (2.1) and throughout the paper are of order $O((|\vec{p}|/m)^2)$, where $\vec{p}$ is a typical nucleon momentum and m is the nucleon mass. We shall assume $|\vec{p}|$ to be of order μ, the pion mass.

The S matrix for $\pi^j(Q) + N(p_1) \to \pi^i(Q') + N(p_1')$ is

$$(Q' p_1' | S - 1 | Q p_1) = -i(2\pi)^4 \delta^4(p_1' + Q' - p - Q) \times (2p_{10} 2p_{20}' 2Q_0 2Q_0')^{-1/2}(2m) \times T_1^{ij}(\nu, t; Q^2, Q'^2), \tag{2.2}$$

and the transition amplitude has the structure

$$T_1^{ij} = -\bar{u}(p_1' s_1')\left[\delta^{ij}\left(F^{(+)} - \frac{1}{4m} B^{(+)}[\not{Q}', \not{Q}]\right) + i\epsilon^{ijk}\tau^k\left(F^{(-)} - \frac{1}{4m} B^{(-)}[\not{Q}', \not{Q}]\right)\right] \times u(p_1, s_1), \tag{2.3}$$

where $\bar{u}u = 1$ and $(\vec{p}|\vec{p}') = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$. We may decompose the amplitudes in (2.3) into a nucleon pole term plus a background term:

$$F^{(\pm)} = F_p^{(\pm)} + \bar{F}^{(\pm)}, \tag{2.4}$$

$$B^{(\pm)} = B_p^{(\pm)} + \bar{B}^{(\pm)}. \tag{2.5}$$

Once the FPBT's are subtracted from the pole terms, the numerically dominant part of $T^{ij}(\text{NR})$ comes from the background. The nonrelativistic reduction of the background terms is carried through, keeping terms of order $O(\mu/m)$. It turns out that the background combination to (2.1) is basically unchanged from I, even though we calculate (2.3)–(2.5) for any frame and do not restrict the πN amplitude to the rest frame of the incoming nucleon. Thus we have to assume that all three-nucleon momenta $\vec{p}_1, \vec{p}_2, \vec{p}_3$ in (2.1) are of the same order of magnitude, which leads, for instance, to the following estimate for the pion lab energy:

$$\nu = [\vec{p}_2^2 - \vec{p}_2'^2 + \vec{p}_3'^2 - \vec{p}_3{}^2 - (\vec{p}_1 + \vec{p}_1') \cdot (\vec{Q} + \vec{Q}')]/4m = O(\mu^2/m). \tag{2.6}$$

Using the general forms of the background terms as given in I, we find[15]

$$\bar{F}^{(+)} = \mathcal{A} + \mathcal{B}\vec{Q} \cdot \vec{Q}' + \mathcal{C}(\vec{Q}^2 + \vec{Q}'^2), \tag{2.7}$$

with

$$\mathcal{A} = -\sigma/f_\pi{}^2, \quad \mathcal{B} = 2/\mu^2[\sigma/f_\pi{}^2 - \bar{F}^{(+)}(0, \mu^2; \mu^2, \mu^2)], \quad \mathcal{C} = -\sigma/(\mu^2 f_\pi^2). \tag{2.8}$$

$$\bar{F}^{(-)} = \nu\left[\frac{1}{2f_\pi{}^2} - \frac{g(Q^2)g(Q'^2)}{2m^2} - \frac{g^*(Q^2)g^*(Q'^2)}{9M^2}(M+m)^2\left(\frac{4}{M^2}\frac{E_\pi{}^2 + Q\cdot Q'}{1 - m^2/M^2} - 1\right)\right] = \nu B_1, \tag{2.9}$$

where ν is explicitly written in (2.6), M is the Δ mass, and g^* the $N\Delta\pi$ coupling constant. The expression E_π as given in I is of the order $O(\mu)$. We note that the constant $B_1 \approx -0.1\mu^{-2}$ from this expansion is not the same as the constant coefficient ($f_1 \approx -0.6\mu^{-2}$) of the on-shell expansion in ν^2 and t in Appendix A of I. This discrepancy is an indication that the cancellation of the current algebra and nucleon contact term allows the Δ contribution to control $\bar{F}^{(-)}$ and the further cancellation between the pole and nonpole parts of the isobar is delicate. On the other hand, since ν is $O(\mu^2/m)$, the term $\bar{F}^{(-)}$ is of $O(\mu^2/m^2)$ in the three-body potential and will in the end be dropped.

Thirdly,

$$\bar{B}^{(+)}(\nu, t, Q^2, Q'^2) = \frac{2g^*(Q^2)g^*(Q'^2)}{3m}\,\frac{\nu\beta_\Delta}{\nu_\Delta{}^2 - \nu^2} \qquad (2.10)$$

with $\beta_\Delta = E_\pi{}^2 - 3^{-1}(M+m-E_\pi)^2 - Q\cdot Q'$. One easily estimates $\bar{B}^{(+)} = (1/\mu^2)O[(m\mu/M^2)(\mu/m)]$. Since it is multiplied according to (2.3) by a term of order $O(\mu^2/m)$ we have to neglect it.

Finally, since $\bar{B}^{(-)}$ goes together with a term of order $O(\mu^2/m)$, we end up just with the constant value

$$\begin{aligned}\bar{B}^{(-)} = {} & \frac{4.70}{2f_\pi{}^2} - \frac{g(Q^2)g(Q'^2)}{2m^2} \\ & + \tfrac{2}{9}\, g^*(Q^2)g^*(Q'^2)\,\frac{(M+m)^2}{M^2}\,\frac{1-2E_\pi/(M+m)}{1-m^2/M^2} \\ & + \tfrac{2}{9}\, g^*(Q^2)g^*(Q'^2)\,\frac{m}{M}\left[1 + \frac{M}{2m}(1-m^2/M^2)\right] \\ \equiv {} & C_1\,. \end{aligned} \qquad (2.11)$$

Thereby the nucleon electromagnetic isovector form factor $[F_2^V(0) = \kappa^V = 3.70]$ is approximated by

$$F_1^V(t) + F_2^V(t) = 1 + 3.70 + O(\mu^2/m^2)\,. \qquad (2.12)$$

We note that C_1 agrees with $b_1^{(-)}$ of Appendix A of I; here the current algebra term is large and the isobar pole and nonpole terms add.

Next, we turn to the time-ordered forward propagating nucleon pole terms of Fig. 2 and write down the s-channel [Fig. 2(a)] plus u-channel [Fig. 2(b)] T matrix as in I,

$$\begin{aligned} T^{ij}_{\mathrm{FPBT}} = {} & +g^2\,\frac{m}{E(\vec{p}_a)}\sum \frac{\bar{u}(p')i\gamma_5\tau^i u(p_a,s)\bar{u}(p_a,s)i\gamma_5\tau^j u(p)}{p_0 + Q_0 - E(\vec{p}_a)} \\ & + g^2\,\frac{m}{E(\vec{p}_b)}\sum \frac{\bar{u}(p')i\gamma_5\tau^j u(p_b,s)\bar{u}(p_b,s)i\gamma_5\tau^i u(p)}{p_0 - Q_0' - E(\vec{p}_b)}\,, \end{aligned} \qquad (2.13)$$

where $E(\vec{p}) = (m^2 + p^2)^{1/2}$, $\vec{p}_a = \vec{p} + \vec{Q}$, $\vec{p}_b = \vec{p} - \vec{Q}'$, and the sum is over nucleon spins. This follows the notations of Refs. 2 and 5. We use the Bjorken-Drell

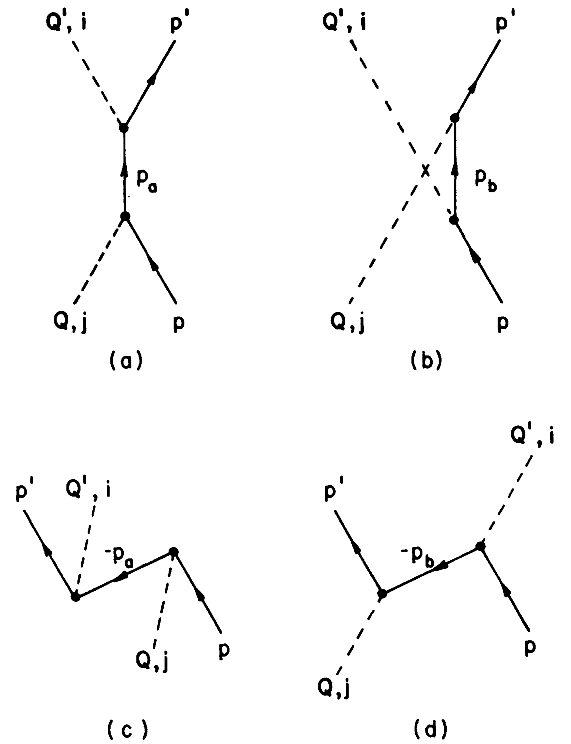

FIG. 2. Time-ordered diagrams corresponding to the s-channel [(a) and (c)] and u-channel [(b) and (d)] nucleon pole terms.

conventions for the γ matrices. To make the intermediate formulas more readable we suppress the Q^2 dependence of g. Equation (2.13) can be written in the form of (2.3) as

$$T^{ij}_{\mathrm{FPBT}} = T^{(+)}_{\mathrm{FPBT}}\,\delta^{ij} + T^{(-)}_{\mathrm{FPBT}}\,i\epsilon^{ijk}\tau^k\,, \qquad (2.14)$$

where

$$T^{(\pm)}_{\mathrm{FPBT}} = -u(p',s')\left(\tilde{A}^{(\pm)}_{\mathrm{FPBT}}\gamma_0 + \nu B^{\pm}_{\mathrm{FPBT}} - \frac{1}{4m} B^{(\pm)}_{\mathrm{FPBT}}[\not{Q}', \not{Q}]\right)u(p,s)\,, \qquad (2.15)$$

and where the amplitudes $\tilde{A}_{\mathrm{FPBT}}$ and B_{FPBT} are given in I.

By expanding the square roots occurring in the expressions for $\tilde{A}_{\mathrm{FPBT}}$ and B_{FPBT} in powers of $(\nu_B \pm \nu)/m = O(\mu^2/m^2)$ and using (2.6), we find to lowest order

$$\tilde{A}^{(+)}_{\mathrm{FPBT}} = \frac{g^2}{2m}\left(2 - \frac{2\vec{p}_1{}^2 + 2\vec{p}_1\cdot\vec{Q} + \vec{Q}^2 - 2\vec{p}_1\cdot\vec{Q}' + \vec{Q}'^2}{2m^2}\right), \qquad (2.16)$$

$$\tilde{A}^{(-)}_{\mathrm{FPBT}} = -\frac{g^2}{4m^3}(\vec{p}_1 + \vec{p}_1')\cdot(\vec{Q} + \vec{Q}')\,, \qquad (2.17)$$

$$B^{(+)}_{\mathrm{FPBT}}=\frac{g^2}{m}\frac{\nu}{\nu_B{}^2-\nu^2}-\frac{g^2}{4m^2}\frac{(\vec{p}_1+\vec{p}_1{}')\cdot(\vec{Q}+\vec{Q}')}{m^2}, \tag{2.18}$$

$$B^{(-)}_{\mathrm{FPBT}}=\frac{g^2}{m}\frac{\nu_B}{\nu_B{}^2-\nu^2}-\frac{g^2}{4m^2}\{2\}. \tag{2.19}$$

On the other hand, it is well known that the covariant nucleon pole terms of (2.4) and (2.5) obtained in the dispersion theory sense are[16]

$$F_P^{(+)}(\nu,t)=\frac{g^2}{m}\frac{\nu_B{}^2}{\nu_B{}^2-\nu^2}, \tag{2.20}$$

$$F_P^{(-)}(\nu,t)=\frac{g^2}{m}\frac{\nu\nu_B}{\nu_B{}^2-\nu^2}, \tag{2.21}$$

$$B_P^{(+)}(\nu,t)=\frac{g^2}{m}\frac{\nu}{\nu_B{}^2-\nu^2}, \tag{2.22}$$

$$B_P^{(-)}(\nu,t)=\frac{g^2}{m}\frac{\nu_B}{\nu_B{}^2-\nu^2}. \tag{2.23}$$

Now we make the subtraction of the forward propagating Born term

$$\Delta F^{(\pm)}\equiv F_p^{(\pm)}-(\tilde{A}^{(\pm)}_{\mathrm{FPBT}}\gamma_0+\nu B^{(\pm)}_{\mathrm{FPBT}}), \tag{2.24}$$

$$\Delta B^{(\pm)}\equiv B_p^{(\pm)}-B^{(\pm)}_{\mathrm{FPBT}}. \tag{2.25}$$

It is a straightforward exercise to find from (2.16)–(2.23) that the expansion in powers of μ/m yields

$$\Delta F^{(+)}=\frac{g(Q^2)g(Q'^2)}{4m}\frac{2\vec{p}_1\cdot\vec{p}_1{}'+\vec{Q}^2+\vec{Q}'^2}{m^2}\gamma_0, \tag{2.26}$$

$$\Delta F^{(-)}=\frac{g(Q^2)g(Q'^2)}{8m}\times\frac{\vec{p}_2{}^2-\vec{p}_2{}'^2+\vec{p}_3{}'^2-\vec{p}_3{}^2+(\vec{p}_1+\vec{p}_1{}')\cdot(\vec{Q}+\vec{Q}')}{m^2}, \tag{2.27}$$

$$\Delta B^{(+)}=-\frac{g(Q^2)g(Q'^2)}{4m^2}\frac{(\vec{p}_1+\vec{p}_1{}')\cdot(\vec{Q}+\vec{Q}')}{m^2}, \tag{2.28}$$

$$\Delta B^{(-)}=\frac{g(Q^2)g(Q'^2)}{2m^2}, \tag{2.29}$$

where we remember that $\Delta B^{(\pm)}$ are multiplied by a term of order $O(\mu^2/m^2)$ according to (2.3).

If we compare (2.26) with (2.29) of I, we note that in (2.26) there is an additional term $2\vec{p}_1\cdot\vec{p}_1{}'$ which could not appear in (2.29) of I because there the amplitude was evaluated in the rest frame $\vec{p}_1=0$ of the incoming nucleon. Although this term is of the same order in μ/m as the $Q^2+Q'^2$ term, the entire $\Delta F^{(+)}$ makes only a small contribution to the nuclear matter results of I [see (2.32) and Table 2 of I] because $g^2/4m^3$ ($\approx 0.15\mu^{-3}$) is much smaller than ($\mathfrak{C}\approx -1\mu^{-3}$) of (2.8). Thus, the neglect of this p-dependent term in I should not matter for nuclear matter calculations.

Expressions (2.27) and (2.28), however, show a more interesting disagreement with the corresponding expressions for $\Delta F^{(-)}$ and $\Delta B^{(+)}$ in Appendix B of I. There the unnecessary rest frame restriction plus the assumption that $Q_0=Q_0'$ could best be set equal to zero led to [instead of (2.6)] $\nu(\text{rest frame})=(Q^2-Q'^2)/4m$, from which followed the $\Delta F^{(+)}$, etc., of I. As we shall see, that unsymmetric combination of $\vec{Q}'^2-\vec{Q}^2$ in T^{ij}(NR) of (2.3) leads to an anti-Hermitian three-body potential. This part of the potential has never been used in a calculation; its nature was noted while preparing the complete potential for a three-body calculation. Finally, the amplitudes $F^{(-)}$ and $B^{(+)}$ will be dropped from the potential in any case. As another example of non-Hermiticity caused by setting $\vec{p}_1=0$, we mention the second term in (2.29) of I. The correct form of that term is anyhow of higher order in μ/m and has always been dropped.

The next step in the nonrelativistic reduction of T_1^{ij} in (2.3) is to note that

$$\bar{u}(p_1',s_1')[\not{Q}',\not{Q}]u(p_1,s_1)=2i\cdot\vec{\sigma}_1(\vec{Q}\times\vec{Q}'), \tag{2.30}$$

$\bar{u}(p_1',s_1')u(p_1,s_1)=1$, and $\gamma^0=1$ to lowest order in μ/m, results which hold whether or not p_1 is set equal to zero. Finally, we insert (2.7), (2.9), (2.11), (2.26)–(2.29), and (2.30) into (2.1), remembering from (2.3) and (2.5) the relative factor (–) between the T matrix and the invariant amplitudes A and B, and get

$$\begin{aligned}\langle\vec{p}_1{}'\vec{p}_2{}'\vec{p}_3{}'|W_1|\vec{p}_1\vec{p}_2\vec{p}_3\rangle&=\text{const}\ \delta^3(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_1{}'-\vec{p}_2{}'-\vec{p}_3{}')\frac{g^2}{4m^2}\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu^2}\frac{H(Q'^2)}{\vec{Q}'^2+\mu^2}\\&\times(\vec{\tau}_2\cdot\vec{\tau}_3\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\{-\mathfrak{a}-\mathfrak{B}\vec{Q}\cdot\vec{Q}'-[\mathfrak{C}+g^2/4m^3-K'(0)\sigma/f_\pi{}^2](\vec{Q}^2+\vec{Q}'^2)-(g^2/2m^3)\vec{p}_1\cdot\vec{p}_1{}'\}\\&+i\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\{(-g^2/8m^3)[\vec{p}_2{}^2-\vec{p}_2{}'^2+\vec{p}_3{}'^2-\vec{p}_3{}^2+(\vec{p}_1+\vec{p}_1{}')\cdot(\vec{Q}+\vec{Q}')]\\&-(B_1/4m)[\vec{p}_2{}^2-\vec{p}_2{}'^2+\vec{p}_3{}'^2-\vec{p}_3{}^2-(\vec{p}_1+\vec{p}_1{}')\ (\vec{Q}+\vec{Q}')]\}\\&+\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\vec{\sigma}_1\cdot(\vec{Q}\times\vec{Q}')(-g^2/4m^3-C_1/2m)).\end{aligned} \tag{2.31}$$

Clearly the three body potential of (2.31) is manifestly Hermitian. If we exchange primed and unprimed nucleon momenta, then $\vec{Q}\to-\vec{Q}$ and $\vec{Q}'\to-\vec{Q}'$. Then the momentum dependent part multiplied by $\vec{\tau}_2\cdot\vec{\tau}_3$ is invariant and $\vec{\tau}_2\cdot\vec{\tau}_3$ is of course Hermitian. The second term is multiplied by the Hermitian $\vec{\tau}_2\times\vec{\tau}_3\cdot\vec{\tau}_1$; because of the factor i, the momentum dependent part must be antisymmetric, which is the case. Finally, the momentum dependent part in the last term is symmetric.

The momentum dependence of the $NN\pi$ and $N\Delta\pi$ coupling constants is assumed to be the same, $g(Q^2) = gK(Q^2)$. Then much of the off-pion-mass-shell extrapolation in (2.31) arises from the products $[g(Q^2)]^2 = g^2H(\vec{Q}^2)$, etc. In the (as we shall see, dominant) $\bar{F}^{(+)}$ amplitude, however, the momentum dependence of the σ term is *not* given by form factor effects on g, see (2.7) and (2.8), so we insert a term proportional to $K'(Q^2=0)$ in the $\mathcal{C}$ part of the potential to undo in lowest order the effect of surrounding T^{ij}(NR) with form factors. We do not make the same correction to $F_1^V + F_2^V$ in $\bar{B}^{(-)}$ because it is multiplied by a term of order $O(\mu^2/m^2)$.

It remains to fix the overall constant (2.31). We first do it formally by calculating the three-body S matrix of Fig. 1(a) and defining W by

$$(\vec{p}_1'\vec{p}_2'\vec{p}_3'|S-1|\vec{p}_1\vec{p}_2\vec{p}_3)_{\rm NR} = -i(2\pi)\delta(p_{10}'+p_{20}'+p_{30}'-p_{10}-p_{20}-p_{30})(\vec{p}_1'\vec{p}_2'\vec{p}_3'|W|\vec{p}_1\vec{p}_2\vec{p}_3)\ . \qquad (2.32)$$

Since the calculation must be the same for any two-body T matrix, we then verify that the forward propagating Born term $T_{\rm FPBT}$ when inserted into (2.1) yields just the iteration of the OPEP. In the notation of I, the Feynman rules for Fig. 1(a) yield

$$(p_1'p_2'p_3'|S-1|p_1p_2p_3) = (2\pi)^4\delta^4(\Sigma p_i - \Sigma p_i')\left[\frac{(2m)^6}{2p_{10}2p_{20}2p_{30}2p_{10}'2p_{20}'2p_{30}'}\right]^{1/2}(-ig)^2\bar{u}(p_2's')i\gamma_5 u(p_2 s)$$
$$\times\frac{i}{Q^2-\mu^2}\tau_2^j(-iT_1^{ij})\tau_3^i\frac{i}{Q'^2-\mu^2}\bar{u}(p_3's')i\gamma_5 u(p_3 s)\ , \qquad (2.33)$$

where T_1^{ij} is related to the two-body S matrix by (2.2). Using

$$\bar{u}(p's')\gamma_5 u(p,s) = \sigma\cdot(\vec{p}-\vec{p}\,')/2m\ , \qquad (2.34)$$

$$Q^2-\mu^2 = -(\vec{Q}^2+\mu^2)\ , \qquad (2.35)$$

to lowest order we get

$$(S-1)_{\rm NR} = -i(2\pi)^4\delta^4(p_1+p_2+p_3-p_1'-p_2'-p_3')\frac{g(Q^2)g(Q'^2)}{4m^2}\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\tau_2^jT_1^{ij}{}_{(\rm NR)}\tau_3^i\frac{1}{Q^2+\mu^2}\frac{1}{Q'^2+\mu^2}\ . \qquad (2.36)$$

Comparing (2.32) and (2.36) we find that the constant in (2.1), with the normalization $(\vec{p}|\vec{p}\,') = (2\pi)^3\delta(\vec{p}-\vec{p}\,')$ of I, would be $(2\pi)^3$.[17] In this paper, we prefer to normalize the momentum space wave functions to the Dirac delta function so that we can more easily take over the technology of momentum space three-body codes. In that case, since $(\vec{p}_1'\vec{p}_2'\vec{p}_3'|\vec{p}_1\vec{p}_2\vec{p}_3) = (2\pi)^9\langle\vec{p}_1'\vec{p}_2'\vec{p}_3'|\vec{p}_1\vec{p}_2\vec{p}_3\rangle$, the constant hereby adopted in (2.1) is $+(2\pi)^{-6}$.

Now we check the consistency of the derivation by letting T_1^{ij} in (2.1) be $T_{\rm FPBT}^{ij}$ of (2.13). We find by using $\vec{p}_a = \vec{p}+\vec{Q}$, $\vec{p}_b = \vec{p}-\vec{Q}'$, and (2.34) that

$$\langle\vec{p}_1'\vec{p}_2'\vec{p}_3'|W_1^{\rm FPBT}|\vec{p}_1\vec{p}_2\vec{p}_3\rangle = +{\rm const}\ \delta^3(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_1'-\vec{p}_2'-\vec{p}_3')$$
$$\times\left[\frac{g^2}{4m^2}\frac{\vec{\sigma}_3\cdot\vec{Q}'\vec{\sigma}_1\cdot\vec{Q}'}{\vec{Q}'^2+\mu^2}\vec{\tau}_3\cdot\vec{\tau}_1\frac{H(\vec{Q}'^2)H(\vec{Q}^2)}{p_{10}+Q_0-E(\vec{p}_a)}\frac{g^2}{4m^2}\frac{\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_1\cdot\vec{Q}}{\vec{Q}^2+\mu^2}\vec{\tau}_2\cdot\vec{\tau}_1\right.$$
$$\left.+\frac{g^2}{4m^2}\frac{\vec{\sigma}_1\cdot\vec{Q}\vec{\sigma}_2\cdot\vec{Q}}{\vec{Q}^2+\mu^2}\vec{\tau}_2\cdot\vec{\tau}_1\frac{H(\vec{Q}^2)H(\vec{Q}'^2)}{p_{10}-Q_0'-E(\vec{p}_b)}\frac{g^2}{4m^2}\frac{\vec{\sigma}_1\cdot\vec{Q}'\vec{\sigma}_3\cdot\vec{Q}'}{\vec{Q}'^2+\mu^2}\vec{\tau}_1\cdot\vec{\tau}_3\right]\ . \qquad (2.37)$$

Now the first nucleon propagator in (2.37) is

$$p_{10}+Q_0-E(\vec{p}_a) \approx \vec{p}_1^{\,2}/2m + (\vec{p}_2^{\,2}-\vec{p}_2'^{\,2})/2m - (\vec{p}_1+\vec{p}_2-\vec{p}_2')^2/2m$$
$$= (2m)^{-1}(\vec{p}_1^{\,2}+\vec{p}_2^{\,2}+\vec{p}_3^{\,2}) - (2m)^{-1}(\vec{p}_2+\vec{p}_3+\vec{k}^2)\big|_{\vec{k}=\vec{p}_1+\vec{p}_2-\vec{p}_2'}\ . \qquad (2.38)$$

The second form is just the denominator of the nonrelativistic propagator $G_0 = (E-H_0)^{-1}$ in $V_{31}G_0V_{12}$. Similarly, the nucleon propagator in the second term of (2.37) corresponds to G_0 in $V_{21}G_0V_{13}$. Finally, remembering the OPEP in momentum space is

$$V_{31}^{\rm OPEP} = -\frac{1}{(2\pi)^3}\frac{g^2}{4m^2}\vec{\tau}_3\cdot\vec{\tau}_1\vec{\sigma}_3\cdot\vec{Q}\vec{\sigma}_1\cdot\vec{Q}\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu^2}\ , \qquad (2.39)$$

we get indeed

$$\langle\vec{p}_1'\vec{p}_2'\vec{p}_3'|W_1^{\rm FPBT}|\vec{p}_1\vec{p}_2\vec{p}_3\rangle = {\rm const}\ \delta^3(\vec{p}_1'+\vec{p}_2'+\vec{p}_3'-\vec{p}_1-\vec{p}_2-\vec{p}_3)(2\pi)^6$$
$$\times[\langle\vec{p}_1'\vec{p}_2'\vec{p}_3'|V_{31}^{\rm OPEP}G_0V_{12}^{\rm OPEP}|\vec{p}_1\vec{p}_2\vec{p}_3\rangle + \langle\vec{p}_1\vec{p}_2\vec{p}_3|V_{21}^{\rm OPEP}G_0V_{13}^{\rm OPEP}|\vec{p}_1\vec{p}_2\vec{p}_3\rangle]\ . \qquad (2.40)$$

Again we find the constant in (2.1) to be

$$\text{const} = +(2\pi)^{-6} \ . \tag{2.41}$$

We note that (2.37)–(2.41) are all in the $\vec{p}$-space δ-function normalization adopted. One can repeat the above steps, realizing that, in the normalization of I, V^{OPEP} has no factors of $(2\pi)^{-3}$ and G_0 has a factor of $(2\pi)^3$ in the numerator, to find that the identification of T_{FPBT} with iterated OPEP still holds, only in that normalization the constant is $(2\pi)^9$ greater than (2.41).

As the constant in (2.1) has been shown to be a positive number and the negative strength factors in the potential amplitudes F and B, it is convenient to define new strength parameters associated with the two-body T matrix itself. This will facilitate comparison with earlier work. So we define

$$\begin{aligned} &a=-\mathcal{A}\ ,\quad b=-\mathcal{B}\ ,\quad c=-\mathcal{C}-g^2/4m^3+K'(0)\sigma/f_\pi^{\,2}\ ,\quad d=-g^2/2m^3\ ,\\ &d_1=-g^2/8m^3\ ,\quad d_2=-B_1/4m\ ,\quad d_3=-g^2/4m^3\ ,\quad d_4=-C_1/2m\ ,\end{aligned} \tag{2.42}$$

in terms of which the three-body potential matrix element is given as

$$\begin{aligned}\langle \vec{p}_1{}'\vec{p}_2{}'\vec{p}_3{}'|W_1|\vec{p}_1\vec{p}_2\vec{p}_3\rangle &= \frac{1}{(2\pi)^6}\,\delta^3(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_1{}'-\vec{p}_2{}'-\vec{p}_3{}')\,\frac{g^2}{4m^2}\,\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu^2}\,\frac{H(\vec{Q}'^2)}{\vec{Q}'^2+\mu^2}\\ &\quad\times\{\vec{\tau}_2\cdot\vec{\tau}_3\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'[a+b\vec{Q}\cdot\vec{Q}'+c(\vec{Q}^2+\vec{Q}'^2)+d\vec{p}_1\cdot\vec{p}_1{}']+i\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\\ &\qquad\times[d_1(\vec{p}_2{}^2-\vec{p}_2{}'^2+\vec{p}_3{}'^2-\vec{p}_3{}^2+(\vec{p}_1+\vec{p}_1{}')\cdot(\vec{Q}+\vec{Q}'))+d_2(\vec{p}_2{}^2-\vec{p}_2{}'^2+\vec{p}_3{}'^2-\vec{p}_3{}^2-(\vec{p}_1+\vec{p}_1{}')\cdot(\vec{Q}+\vec{Q}'))]\\ &\qquad+\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\vec{\sigma}_1\cdot(\vec{Q}\times\vec{Q}')(d_3+d_4)\}\ .\end{aligned} \tag{2.43}$$

III. TRANSITION TO COORDINATE SPACE

As an example of the coordinate space representation of the potential and in order to point out corrections to I, we present the form for the abc terms accompanying $\vec{\tau}_2\cdot\vec{\tau}_3$ and for the final term going with (d_3+d_4) in (2.43).

$$\begin{aligned}&\langle \vec{x}_1{}'\vec{x}_2{}'\vec{x}_3{}'|W_1^{abc}+W_1^{d_3d_4}|\vec{x}_1\vec{x}_2\vec{x}_3\rangle\\ &\quad=\delta^3(\vec{x}_1-\vec{x}_1{}')\delta^3(\vec{x}_2-\vec{x}_2{}')\delta^3(\vec{x}_3-\vec{x}_3{}')(g\mu/2m)^2\frac{1}{(4\pi)^2}\\ &\quad\times[\vec{\tau}_2\cdot\vec{\tau}_3(\vec{\sigma}_2\cdot\vec{\nabla}_2\vec{\sigma}_3\cdot\vec{\nabla}_3\{(a-2\mu^2c)Z_1(x_{12})Z_1(x_{13})+c[Z_0(x_{12})Z_1(x_{13})+Z_1(x_{12})Z_0(x_{13})]+b\vec{\nabla}_2\cdot\vec{\nabla}_3Z_1(x_{12})Z_1(x_{13})\})\\ &\quad+\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_2\cdot\vec{\nabla}_2\vec{\sigma}_3\cdot\vec{\nabla}_3\vec{\sigma}_1\cdot\vec{\nabla}_2\times\vec{\nabla}_3(d_3+d_4)Z_1(x_{12})Z_1(x_{13})]\ ,\end{aligned} \tag{3.1}$$

where, as in (3.11) of I, we have defined

$$Z_n(x_{ij})=\frac{4\pi}{\mu}\int\frac{d\vec{Q}}{(2\pi)^3}\,\frac{e^{i(\vec{x}_i-\vec{x}_j)\cdot\vec{Q}}H(\vec{Q}^2)}{(\vec{Q}^2+\mu^2)^n}\ . \tag{3.2}$$

The terms in (2.43) which depend upon nucleon momenta are small and have a more complicated coordinate space representation; they will not be displayed here.

Equation (3.1) agrees with (3.9) and (3.10) of I, but disagrees in sign and constants with (3.12) of I. The nuclear matter calculation in I was not made with a three-body potential, but rather an effective two-body potential obtained by summing over nucleon 1 of W_1^{abc}. The effective potential of I has been checked to be consistent with (3.1) and the numerical results and conclusions of I are correct, up to the small factor $d=-g^2/2m^3$, which was neglected in I (see discussion of $\Delta F^{(+)}$ in Sec. II).

IV. PARTIAL WAVE DECOMPOSITION

In a standard Faddeev calculation in a three nucleon system, one needs a partial wave decomposition of the three-body force. We shall provide it in this section. We introduce the standard Jacobi momenta

$$\vec{p}=\tfrac{1}{2}(\vec{p}_2-\vec{p}_3),\quad \vec{q}=\tfrac{2}{3}[\vec{p}_1-\tfrac{1}{2}(\vec{p}_2+\vec{p}_3)]\ , \tag{4.1}$$

which single out particle 1 and are adequate for the first diagram in Fig. 1, which was considered in the two previous sections. We have in terms of $\vec{p}$ and $\vec{q}$ when the total momentum is zero,

$$\vec{p}_1=\vec{q},\quad \vec{p}_2=\vec{p}-\tfrac{1}{2}\vec{q},\quad \vec{p}_3=-\vec{p}-\tfrac{1}{2}\vec{q} \tag{4.2}$$

and the same for the primed quantities; furthermore, the pion momenta are

$$\vec{Q}=\vec{p}-\vec{p}'-\tfrac{1}{2}(\vec{q}-\vec{q}'),\quad \vec{Q}'=\vec{p}-\vec{p}'+\tfrac{1}{2}(\vec{q}-\vec{q}')\ . \tag{4.3}$$

From now on, p, Q, etc., shall refer to the magnitude of the three-vectors, i.e., $Q=|\vec{Q}|$, etc., and not to the four-vectors. It is straightforward to rewrite the expression (2.43) in terms of Jacobi momenta with the results

$$\begin{aligned}\langle \vec{p}'\vec{q}'\,|\,W_1|\vec{p}\vec{q}\rangle = &+\frac{1}{(2\pi)^6}\frac{g^2}{4m^2}\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu^2}\frac{H(\vec{Q}'^2)}{\vec{Q}'^2+\mu^2}\\ &\times(\vec{\tau}_2\cdot\vec{\tau}_3\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\{a+b[\vec{p}^2+\vec{p}'^2-\tfrac{1}{4}(\vec{q}^2+\vec{q}'^2)-2\vec{p}\cdot\vec{p}'+\tfrac{1}{2}\vec{q}\cdot\vec{q}']+c[2(\vec{p}^2+\vec{p}'^2)+\tfrac{1}{2}(\vec{q}^2+\vec{q}'^2)\\ &\qquad -4\vec{p}\cdot\vec{p}'-\vec{q}\cdot\vec{q}']+d\vec{q}\cdot\vec{q}'\}\\ &+i(\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1)\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'[2(d_1+d_2)(\vec{p}'\cdot\vec{q}'-\vec{p}\cdot\vec{q})+2(d_1-d_2)(\vec{q}+\vec{q}')\cdot(\vec{p}-\vec{p}')]\\ &+\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'\vec{\sigma}_1\cdot(\vec{p}-\vec{p}')\times(\vec{q}-\vec{q}')(d_3+d_4))\ .\end{aligned} \tag{4.4}$$

Thus we have to deal with the following types of forces

$$\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu}\frac{H(\vec{Q}'^2)}{\vec{Q}'^2+\mu^2}\vec{\sigma}_2\cdot\vec{Q}\,\sigma_3\cdot\vec{Q}'\begin{cases}1\\ \vec{p}\cdot\vec{p}'\\ \vec{q}\cdot\vec{q}'\\ \vec{p}'\cdot\vec{q}\\ \vec{p}'\cdot\vec{q}'-\vec{p}\cdot\vec{q}\\ \vec{p}\cdot\vec{q}'\\ \vec{\sigma}_1\cdot(\vec{p}-\vec{p}')\times(\vec{q}-\vec{q}')\ .\end{cases} \tag{4.5}$$

The three-body states with respect to which we want to present our force will be

$$|pq\alpha\rangle\equiv|p(ls)j\,q(\lambda\tfrac{1}{2})J\,(jJ)\mathcal{J}\,(\lambda\tfrac{1}{2})T\rangle\ , \tag{4.6}$$

where $lsjt$ are the orbital, total spin, total angular momentum, and total isospin of the 2-3 subsystem, and $\lambda\frac{1}{2}$ $J\frac{1}{2}$ are the corresponding quantum numbers of particle 1; finally $\mathcal{J}$ and T are the total three-body angular momentum and total three-body isospin. For a standard Faddeev calculation, the first five partial wave states dictated by the two-nucleon forces are given in Table I.

The partial wave decomposition is quite tedius but straightforward.[15] We quote only the results for the main terms. Let us define

$$f(\vec{Q}^2)\equiv\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu^2} \tag{4.7}$$

and the three-fold integral

$$H_{\epsilon_1\epsilon_2 l_1 l_2 l_3}=\int_{-1}^{1}dx_1\int_{-1}^{1}dx_2\int_{-1}^{1}dx_3\,P_{l_1}(x)\,P_{l_2}(x_2)\,P_{l_3}(x_3)\,A_1^{\epsilon_1}A_2^{\epsilon_2}f(A_1^2+\tfrac{1}{4}A_2^2-A_1A_2x_1)f(A_1^2+\tfrac{1}{4}A_2^2+A_1A_2x_1) \tag{4.8}$$

with

$$A_1=(p^2+p'^2-2p\,p'x_2)^{1/2},\quad A_2=(q^2+q'^2-2qq'x_3)^{1/2} \tag{4.9}$$

then

$$\begin{aligned}&\langle p'q'\alpha'\,|\,\vec{\sigma}_2\cdot\vec{Q}\,\vec{\sigma}_3\cdot\vec{Q}'\,f(Q^2)f(Q'^2)\,|\,pq\alpha\rangle\\ &\quad=-\sum_{l_1l_2l_3}\ \sum_{\epsilon_1=-l-l'}^{2}\ \sum_{\epsilon_2=-\lambda-\lambda'}^{2}H_{\epsilon_1\epsilon_2 l_1 l_2 l_3}(pp'qq')\sum_{t_3t_4}G^{\alpha'\alpha}_{t_3t_4\epsilon_1\epsilon_2 l_1}\,\phi_{t_3l_2ll'}(pp')\phi_{t_4l_3\lambda\lambda'}(qq')\end{aligned} \tag{4.10}$$

with

$$\phi_{t\bar{l}ll'}(kk')=\sum_{a_1+a_2=t} k^{a_1}k'^{a_2}\left[\frac{(2t+1)!}{(2a_1)!(2a_2)!}\right]^{1/2}\begin{Bmatrix} l' & l & t\\ a_1 & a_2 & \bar{l}\end{Bmatrix}(\bar{l}a_1 l,00)(\bar{l}a_2 l',00)\hat{\bar{l}} \tag{4.11}$$

and

$$G^{\alpha'\alpha}_{t_3t_4\epsilon_1\epsilon_2 l_1}=(4\pi)^2\tfrac{9}{4}(\hat{j}\hat{J}\hat{s})^{1/2}(\hat{j}'\hat{J}'\hat{s}')^{1/2}(-)^{s+1/2+\mathcal{J}}(-)^{l_1}\hat{l}_1(\hat{t}_3\hat{t}_4)^{1/2}\sum_{k_1}\hat{k}_1\begin{Bmatrix}1 & 1 & k_1\\ \frac12 & \frac12 & s\\ \frac12 & \frac12 & s'\end{Bmatrix}\sum_{LS}\hat{L}\hat{S}\begin{Bmatrix} l & s & j\\ \lambda & \frac12 & J\\ L & S & \mathcal{J}\end{Bmatrix}$$

$$\times\sum_{L'S'}\hat{L}'\hat{S}'(-)^{S+S'+L}\begin{Bmatrix} l' & s' & j'\\ \lambda' & \frac12 & J'\\ L' & S' & \mathcal{J}\end{Bmatrix}\begin{Bmatrix} l' & l & t_3\\ \lambda' & \lambda & t_4\\ L' & L & k_1\end{Bmatrix}\begin{Bmatrix} S & S' & k_1\\ s' & s & \frac12\end{Bmatrix}\begin{Bmatrix} S' & L' & \mathcal{J}\\ L & S & k_1\end{Bmatrix}$$

$$\times\sum_{r_1+r_2=1}(\tfrac12)^{r_2}(-)^{r_1}\sum_{s_1+s_2=1}(\tfrac12)^{s_2}\delta_{r_1+s_1-t_3,\epsilon_1}\delta_{r_2+s_2-t_4,\epsilon_2}$$

$$\times\sum_{t_1t_2}(\hat{t}_1\hat{t}_2)^{1/2}(r_1s_1t_1,00)(r_2s_2t_2,00)\begin{Bmatrix} r_1 & r_2 & 1\\ s_1 & s_2 & 1\\ t_1 & t_2 & k_1\end{Bmatrix}\begin{Bmatrix} t_2 & t_1 & k_1\\ t_3 & t_4 & l_1\end{Bmatrix}(t_1l_1t_3,00)(t_2l_1t_4,00)\,. \tag{4.12}$$

$$\langle p'q'\alpha'|\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}'f(Q^2)f(Q'^2)\vec{q}\cdot\vec{q}'|pq\alpha\rangle$$

$$=-qq'\sum_{l_1l_2l_3}\sum_{\epsilon_1=-l-l'}^{2}\sum_{\epsilon_2=-\lambda-\lambda'}^{2}H_{\epsilon_1\epsilon_2l_1l_2l_3}(pp'qq')\sum_{t_3t_4}G^{\alpha'\alpha}_{t_3t_4\epsilon_1\epsilon_2l_1}\phi_{t_3l_2ll'}(pp')\bar{\phi}_{t_4l_3\lambda\lambda'}(qq') \tag{4.13}$$

with

$$\bar{\phi}_{t\bar{l}\lambda\lambda'}(qq')=\hat{\bar{l}}\sum_{b_1+b_2=t}q^{b_1}q'^{b_2}F(tb_1b_2)\sum_{k_3k_4}\begin{Bmatrix} b_2 & b_1 & t\\ k_3 & k_4 & 1\end{Bmatrix}(\hat{k}_3\hat{k}_4)^{1/2}\begin{Bmatrix}\lambda' & \lambda & t\\ k_3 & k_4 & \bar{l}\end{Bmatrix}(b_1 1k_3,00)$$

$$\times(b_2 1k_4,00)(\bar{l}k_3\lambda,00)(l_3k_4\lambda',00)\,. \tag{4.14}$$

The expression for $\vec{p}\cdot\vec{p}'$ is analogous to (4.13) and (4.14) with only the changes $qq'\to pp'$, $\phi(pp')\to\bar{\phi}(pp')$ and $\bar{\phi}(qq')\to\phi(qq')$. As one may easily verify, the expressions (4.10) and (4.13) are symmetric under exchange of primed and unprimed quantities. Moreover, they vanish between states 1 and 2–5. This had to be expected since W_1 is symmetric under exchange of particles 2 and 3 can therefore not change the subsystem spin. The same is of course true for the accompanying isospin matrix element ($T=\frac12$)

$$\langle(t'\tfrac12)T|\vec{\tau}_2\cdot\vec{\tau}_3|(t\tfrac12)T\rangle=-\delta_{tt'}6(-)^t\begin{Bmatrix}\frac12 & \frac12 & t\\ \frac12 & \frac12 & 1\end{Bmatrix}\,. \tag{4.15}$$

We skip the case for $\vec{p}'\cdot\vec{q}$, $\vec{p}\cdot\vec{q}'$, and $(\vec{p}'\cdot\vec{q}'-\vec{p}\cdot\vec{q})$ accompanying the coefficients d_1, d_2 since they will be neglected anyhow (see Sec. V). Finally the case $\vec{\sigma}_1\cdot(\vec{p}-\vec{p}')\times(\vec{q}-\vec{q}')$ goes together with an isospin matrix element which switches the two-body system t between 0 and 1:

$$\langle(t'\tfrac12)T|\vec{\tau}_2\times\vec{\tau}_3\cdot\vec{\tau}_1|(t\tfrac12)T\rangle=i\,36(\hat{t}\hat{t}')^{1/2}(-)^{t'+1/2+T}\begin{Bmatrix}\frac12 & t' & T\\ t & \frac12 & 1\end{Bmatrix}\begin{Bmatrix}1 & 1 & 1\\ \frac12 & \frac12 & t\\ \frac12 & \frac12 & t'\end{Bmatrix}. \tag{4.16}$$

We continue with the last term in (4.5) and get

$$\langle p'q'\alpha'|\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\vec{Q}'\vec{\sigma}_1\cdot(\vec{p}-\vec{p}')\times(\vec{q}-\vec{q}')f(Q^2)f(Q'^2)|pq\alpha\rangle$$

$$= -i\sum_{l_1l_2l_3}\sum_{\epsilon_1=-l-l'+1}^{3-|l-l'|}\sum_{\epsilon_2=-\lambda-\lambda'+1}^{3-|\lambda-\lambda'|}H_{\epsilon_1\epsilon_2l_1l_2l_3}\sum_{t_3t_4}\tilde{G}^{\alpha'\alpha}_{t_3t_4\epsilon_1\epsilon_2l_1}\phi_{t_3l_2ll'}(pp')\phi_{t_4l_3\lambda\lambda'}(qq') \quad (4.17)$$

with

$$\tilde{G}^{\alpha'\alpha}_{t_3t_4\epsilon_1\epsilon_2l_1} = (4\pi)^2\,\tfrac{27}{2}(\hat{j}\hat{J}\hat{s})^{1/2}(\hat{j}'\hat{J}'\hat{s}')^{1/2}\hat{l}_1(-)^{l_1}(\hat{t}_3\hat{t}_4)^{1/2}$$

$$\times\sum_{k_1}\hat{k}_1\begin{Bmatrix}1&1&k_1\\ \frac{1}{2}&\frac{1}{2}&s\\ \frac{1}{2}&\frac{1}{2}&s'\end{Bmatrix}\sum_{k_2}\hat{k}_2\sum_{LS}\hat{L}\hat{S}\begin{Bmatrix}l&s&j\\ \lambda&\frac{1}{2}&J\\ L&S&\mathcal{J}\end{Bmatrix}\sum_{L'S'}\hat{L}'\hat{S}'(-)^{L'+S+\mathcal{J}}\begin{Bmatrix}l'&s'&j'\\ \lambda'&\frac{1}{2}&J'\\ L'&S'&\mathcal{J}\end{Bmatrix}$$

$$\times\begin{Bmatrix}S'&L'&\mathcal{J}\\ L&S&k_2\end{Bmatrix}\begin{Bmatrix}k_1&1&k_2\\ s&\frac{1}{2}&S\\ s'&\frac{1}{2}&S'\end{Bmatrix}\begin{Bmatrix}l&l'&t_3\\ \lambda&\lambda'&t_4\\ L&L'&k_2\end{Bmatrix}$$

$$\times\sum_{r_1+r_2=1}(-)^{r_2}(\tfrac{1}{2})^{r_2}\sum_{s_1+s_2=1}(\tfrac{1}{2})^{s_2}\delta_{r_1+s_1+1-t_3,\epsilon_1}\delta_{r_2+s_2+1-t_4,\epsilon_2}$$

$$\times\sum_{t_1t_2}(r_1s_1t_1,00)(r_2s_2t_2,00)\begin{Bmatrix}r_1&r_2&1\\ s_1&s_2&1\\ t_1&t_2&k_1\end{Bmatrix}(\hat{t}_1\hat{t}_2)^{1/2}$$

$$\times\sum_{k_3k_4}\begin{Bmatrix}t_1&t_2&k_1\\ 1&1&1\\ k_3&k_4&k_2\end{Bmatrix}(t_1 1k_3,00)(t_2 1k_4,00)(\hat{k}_3\hat{k}_4)^{1/2}\begin{Bmatrix}k_4&k_3&k_2\\ t_3&t_4&l_1\end{Bmatrix}(k_3l_1t_3,00)(k_4l_1t_4,00)\,. \quad (4.18)$$

One verifies easily the antisymmetry of that expression and again the matrix element is nonzero only for $s\neq s'$. Thus, only state 1 couples with states 2–5.

We would like to remark that all the sums in (4.10), (4.13), and (4.17) are finite. The calculation of the purely geometrical coefficients G and $\tilde{G}$ need be done only once and takes much less time than the calculation of the threefold integral in (4.8).

V. QUANTITATIVE REMARKS AND COMPARISON WITH OTHER WORK

The potentials given in (2.43) and (4.4) of Sec. II and IV, respectively, contain strength parameters. They are expressed in terms of the physical constants in the πN amplitude which we take as follows: $\sigma/f_\pi{}^2=1.13\,\mu^{-1}$, $\bar{F}^{(+)}(0,\ \mu^2;\mu^2,\ \mu^2)=-0.16\ \mu^{-1}$, $g^{*2}(\mu^2)=3.32\ \mu^{-2}$, $g^2(\mu^2)=179.7$, $M=8.825\ \mu$, $m=6.726\ \mu$, and $\mu=139.6$ MeV. Then we find the strength parameters in the first part of the potential which is associated with $\vec{\tau}_2\cdot\vec{\tau}_3$ (and $F^{(+)}$ as

$$a=+1.13\ \mu^{-1},\quad b=-2.58\ \mu^{-3},$$
$$c=+1.00\ \mu^{-3},\quad d=-0.295\ \mu^{-3}\,. \quad (5.1)$$

The second part (from $F^{(-)}$) of the potential will be neglected, since

$$d_1=-0.074\ \mu^{-3},\quad d_2=+0.003\ \mu^{-3} \quad (5.2)$$

which is of the order $O(\mu^2/m^2)$. We have already

TABLE I. Quantum numbers of the first five partial wave states of the three-body system. See text for definitions.

No.	l	s	j	λ	J	t
1	0	0	0	0	$\frac{1}{2}$	1
2	0	1	1	0	$\frac{1}{2}$	0
3	2	1	1	0	$\frac{1}{2}$	0
4	0	1	1	2	$\frac{3}{2}$	0
5	2	1	1	2	$\frac{3}{2}$	0

shown that any contribution from $B^{(+)}$ is of order $O(\mu^2/m^2)$ and has already been neglected in (2.31) of Sec. II. Finally, the strength of the third part [from $B^{(-)}$] is

$$d_3 = -0.148\ \mu^{-3},\quad d_4 = -0.605\ \mu^{-3}\ . \tag{5.3}$$

At this point, it is instructive to compare the potential discussed here with other three-body potentials. The oldest is the Fujita-Miyazawa force, which is based on a theory of Ref. 18. According to Bhaduri *et al.*[6] (BNR) one ends up with almost the same form for the potential, except that the second part, multiplied by $d_1 + d_2$, is altogether absent. The choice of $C_p = +0.61$ MeV leads to

$$a^{\mathrm{BNR}} \approx 0,\quad b^{\mathrm{BNR}} = -1.39\ \mu^{-3},\quad c^{\mathrm{BNR}} = 0,$$
$$d_1^{\mathrm{BNR}} = 0,\quad d_2^{\mathrm{BNR}} = 0,\quad d_3^{\mathrm{BNR}} = 0,\quad d_4^{\mathrm{BNR}} = -0.347\ \mu^{-3}\ . \tag{5.4}$$

Loiseau[12] has recently remarked that the s-wave component (due to a and c) considered here is in good qualitative agreement with an earlier[7] s-wave component arising from a "direct" s-wave πN interaction (already appearing in Fujita-Miyazawa's work) and from a t-channel pole (the ϵ or σ meson) first considered by Harrington.[19] Although the remark referred to the effective two-nucleon potentials shown in Loiseau *et al.*[7] and in I, we can compare s waves already at the level of the πN amplitude in (2.1) and (2.3) of Sec. II. The ϵ pole appears only in the $\overline{F}^{(+)}$ amplitude and has the form $[(\vec{Q}+\vec{Q}')^2 + m_\epsilon{}^2]^{-1}$. Loiseau *et al.* attempted to find the strength of this contribution in terms of the effective ranges of s-wave πN scattering. One can expand their ϵ pole to $O(\mu/m)$ and find that their potential has the following strength parameters (in our notation)

$$a_s = 0.043\ \mu^{-1},\quad a_\epsilon = -0.046\ \mu^{-1},$$
$$b_\epsilon = -0.0037\ \mu^{-3},\quad c_\epsilon = -0.0018\ \mu^{-3} \tag{5.5}$$

which are *not* in good qualitative agreement with (5.1). This disagreement is weaker at the effective $2N$ potential level when we compare Fig. 12 of I and Figs. 10 and 11 of Loiseau *et al.*[7] at, for example, $r = 1.5$ fm and see that $V_a + V_c = +0.8$ MeV and $V_s + V_\epsilon \approx +0.13$ in the central part and $V_a + V_c \approx -1.96$ MeV and $V_s + V_\epsilon \approx -0.27$ in the tensor part. The type of extrapolation from s-wave scattering lengths and effective ranges ignores the current algebra ("soft-pion") constraints and cannot be considered realistic nowadays. On the other hand, the πN amplitude which yields strength parameters (5.1)–(5.3) does successfully predict πN scattering lengths and effective ranges.[4]

Yang[11] has derived a three-body potential from a model of the πN amplitude obtained from a chiral invariant Lagrangian of interacting π's, ρ's, N's, and Δ's. He, in contrast to Refs. 6–10, finds a nonzero result for the subtraction of the forward propagating Born term, but his quoted result differs from (2.26)–(2.29) of Sec. II. His Δ contribution is of the same form as that of (5.4) and he claims strength parameters similar to theirs. In order to arrive at this result, he must evaluate his s-channel Δ pole in the frame in which the middle nucleon is at rest. This is easily seen by comparing (2.14) and (2.20) of Ref. 11.

Yang also considers a ρ-meson t-channel pole in the πN amplitude which ultimately acts, in the triton, to oppose the effect of the Δ with about one sixth the strength. It is difficult to compare the ρ part of Yang's potential with (5.1)–(5.3) as his final form is different. We remark, however, that in the vector dominance model of electromagnetic form factors of the nucleon, the terms $F_1^V(t)/2f_\pi{}^2$ of $\overline{F}^{(-)}$ and $(F_1^V(t) + F_2^V(t)/2f_\pi{}^2$ of $\overline{B}^{(-)}$ in (2.12) of Sec. II) correspond to a ρ pole in the t channel. It would seem from (2.13) of Ref. 11 that Yang's ρ contribution corresponds only to the charge coupling $F_1^V(t)/2f_\pi{}^2$ of $\overline{F}^{(-)}$. We have seen in Sec. II that this current algebra term is largely canceled by the Δ isobar and the nucleon pole already at the level of $\overline{F}^{(-)}$ and was therefore dropped in this paper. On the other hand, the Pauli moment coupling of the ρ, which is not in Yang's Lagrangian, makes a sizable contribution to $\overline{B}^{(-)}$ [see (2.11) of Sec. II]. That is, the term $[F_1^V(t) + F_2^V(t)]/2f_\pi{}^2$ contributes $-0.4\ \mu^{-3}$ to the strength parameter $d_4 = -0.6\ \mu^{-3}$ of (5.3). Thus, Yang's potential is about half as strong in b and d_4, has no a or c terms, and seemingly has a stronger d_2 than the strength parameters of (5.1)–(5.3).

To calculate the potential matrix elements of Sec. III, we have to evaluate the threefold integral (4.8) numerically. For large values of p and p', the integrand varies strongly with x_2 because of the small pion mass. Thus, it is necessary to subtract the poles in x_2 which lie close to $x_2 = 1$. It is convenient to choose a monopole form factor so that

$$H(\vec{Q}^2) = \left(\frac{\Lambda^2 - \mu^2}{\Lambda^2 + \vec{Q}^2}\right)^2\ . \tag{5.6}$$

The value of Λ as determined by the observed Goldberger-Treiman discrepancy of 6% is $\Lambda \approx 6\ \mu$. Such a small value, although indicated by the data and theoretical determinations (see Ref. 23 for a discussion), would necessitate a second subtraction. For illustrative purposes only, we have made calculations with $\Lambda = 7.1\ \mu$ (or $\Lambda^2 = 25$ fm^{-2}). In that case, a second subtraction turned out to be not necessary. A typical number of Gaussian quadrature points is 10 for each dimension.

Since the potential matrix elements depend on the four variables $p'q'pq$ and on the quantum numbers $\alpha'\alpha$ of the various three-body states, we present a dependence on one variable which, however, can give a first feeling for the relative importance of the various terms. For fixed q, q', and p' we present the dependence on p and label the matrix elements between those states of Table I as $\langle\alpha'|\alpha\rangle$. In Figs. 3(a)–3(c) we show some sample matrix elements from the first (*abcd*) part of the potential. We remember that for this part of the potential there is no coupling between state 1 and the states 2–5. It is interesting to note that the diagonal matrix elements $\langle 1|1\rangle$ and $\langle 2|2\rangle$ are identical. That is easily verified by an inspection of the expressions (4.10)–(4.15). We see from Fig. 3(a) that the b and c terms are of opposite sign and strongly cancel each other. The d term is smaller than the a term and is not plotted. Figure 3(b) shows matrix elements in the same channels for a large value of the spectator momenta $q = q' = 3$ fm^{-1}. In this case the b and c contributions are constructive at low p, destructive at higher p, and the potential is quite different in shape from the b term alone (which corresponds in shape to the FM potential). Figure 3(c) shows the very strong coupling [note change of vertical scale on Fig. 3(c)] between states 2 and 3 which correspond to $^3S_1$ and $^3D_1$ partial waves in the two-body subsystem. Again a strong cancellation occurs resulting in a potential quite different from that of FM. All the remaining matrix elements of the first part of the potential are much smaller. The third $(d_3 + d_4)$ part of the potential is sketched in Fig. 4. That part connects only the state 1 with the states 2–5. It appears to be less important than the first part.

Clearly there is a striking difference to the FM force of Refs. 6–10 which according to (5.4) has no a, c, or d term and the b and d_4 terms are smaller by roughly a factor of 2. This difference has been repeatedly emphasized at the level of the underlying πN amplitude in the past,[2,5] but has been somewhat obscured at the potential level because of the common practice of displaying and calculating with *effective two-body* potentials. Figures 3 and 4 give, for the first time, a hint at the complexity of a *three-body* potential and show in a compelling fashion the differences between the PCAC-current algebra three-body force and the FM force.

VI. SUMMARY AND OUTLOOK

We have examined the momentum space three-nucleon potential of the two-pion-exchange type for the purpose of including it in a Faddeev calculation of the triton. The potential is based on an off-pion-mass-shell πN scattering amplitude obtained by a PCAC extrapolation subject to the constraints of current algebra with background corrections dominated by the Δ isobar. We, in contrast to an earlier study, did not evaluate the amplitude in the rest frame of the nucleon, but for general values of $\vec{p}$. We did make a consistent expansion in powers of $|\vec{p}|/m$ to arrive at a manifestly Hermitian potential. The Hermiticity of the potential thus derived separates it from other three-nucleon potentials already included in perturbative[11,20] and variational[9] calculations of the triton. The latter potentials were derived by assuming one nucleon to be always fixed in place. This latter assumption, in general, causes a non-Hermitian potential matrix.

We have made a partial wave decomposition of the three-nucleon potential and calculated momentum space matrix elements between the five lowest partial waves which dominate the triton binding energy. It turns out that the often used Fujita-Miyazawa force[6-10] can be cast into the form of a special case of the potential considered by us. Therefore we can see that the matrix elements of the two potentials are in some important instances rather different. We, at this stage, will not speculate about the effects of these differences in the two potentials upon the calculated properties of the triton.

Let us end with a brief remark about using the three-nucleon potential in a Faddeev calculation. One can introduce either a new Faddeev component corresponding to the fully symmetrical three-body force of Fig. 1 or one can keep to three components and pay the penalty of living with a more complicated amplitude in the Faddeev kernel. Let us look at the second possibility.

The Schrödinger equation in integral form is

$$\Psi = G_0 \sum_{i=1}^{3} (V_i + W_i)\Psi \ , \tag{6.1}$$

where V_i are the pair interactions where i labels the particle which is not in the pair (i.e., $V_1 \equiv V_{23}$) and W_i is that piece of the fully symmetrical three-body force with particle i in the middle. Introducing the two cyclic permutation operators

$$P_1 = P_{12}P_{23}, \quad P_2 = P_{13}P_{23} \tag{6.2}$$

together with $P \equiv P_1 + P_2$, and using the antisymmetry of Ψ we can write (6.1) as

$$\Psi = G_0(1+P)(V_1 + W_1)\Psi \ . \tag{6.3}$$

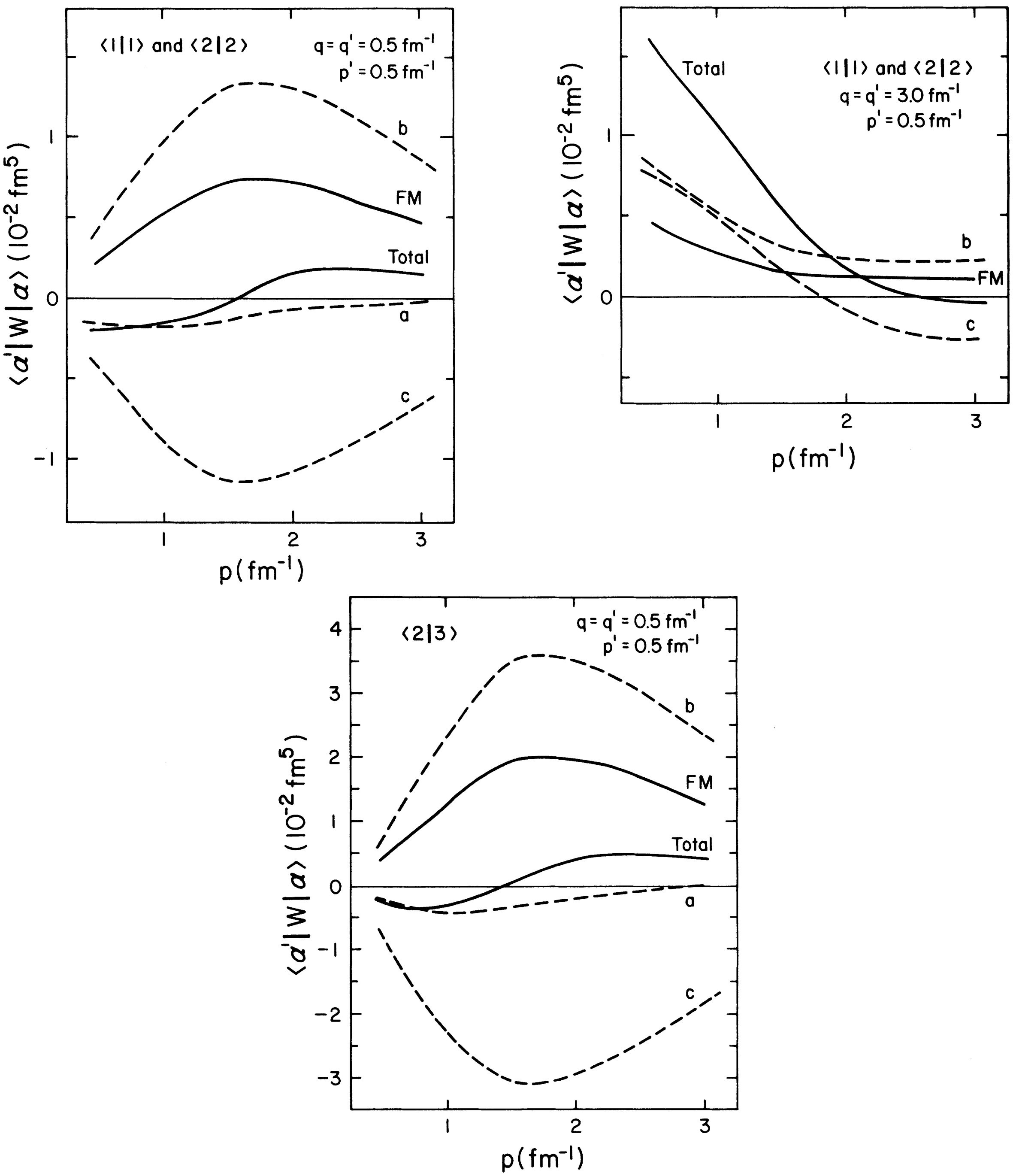

FIG. 3. Expectation values $\langle p'q'd'|W_1^{abcd}|pqd\rangle$ where the Jacobi momenta p, q, and q' are held fixed. The three-body partial wave states of Table I are labeled by α and α'. The contributions from strength parameters abc and d of Ref. 1 are summed to yield the curve labeled Total. The Fujita-Miyazawa force (labeled FM) with the strength parameters of (5.4) is also shown.

Because G_0 is symmetric, one can commute $1+P$ with G_0 and define the Faddeev amplitude ψ_1 by

$$\Psi \equiv (1+P)\psi_1 \qquad (6.4)$$

or

$$\psi_1 = G_0(V_1 + W_1)\Psi \,. \qquad (6.5)$$

We get then the Faddeev equation

$$\psi_1 = G_0 TP\psi_1 \qquad (6.6)$$

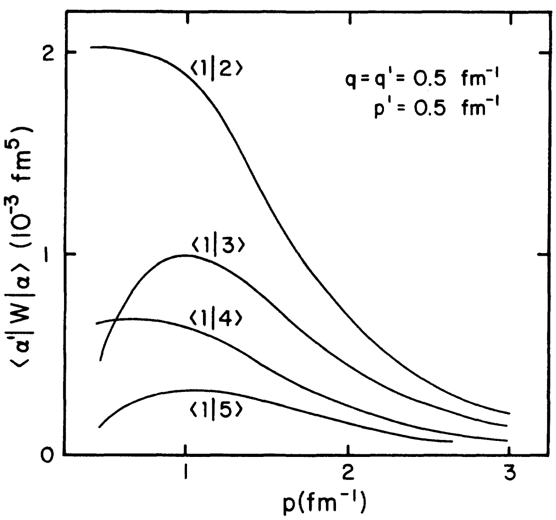

FIG. 4. Nonzero expectation values of $W_1^{d_3+d_4}$ in the $\langle\alpha'|\alpha\rangle$ states labeled as in Fig. 3. The corresponding FM curves are not plotted; they have about one-half the strength of those shown.

with

$$T = V_1 + W_1 + (V_1 + W_1)G_0T \ . \tag{6.7}$$

If W_1 can be considered as a perturbation, the alternate form of (6.7) appears to be convenient[21]

$$T = t + (1 + tG_0)W_1(1 + G_0T) \ , \tag{6.8}$$

where t is the two-body t matrix corresponding to V_1. The calculation of the operators P and V_1 between the states of Table I is by now standard (see, for example, Ref. 22) and will not be given here. A solution of the set (6.6) and (6.7) is in progress and will be reported elsewhere.

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