PHYSICAL REVIEW C VOLUME 22, NUMBER 3 SEPTEMBER 1980

Particle identity and the optical potential for elastic two-fragment collisions

R. Goldflam and K. L. Kowalski
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
(Received 3 March 1980)

The permutation symmetries resulting from particle identity are incorporated into a complete and consistent set of scattering integral equations which are partition labeled and which also possess a multiple scattering structure. These equations are applied to the investigation of the optical potential for elastic two-fragment collisions including all identity effects. It is found that, among the standard off-shell extensions for the transition operators, only the one proposed by Alt, Grassberger, and Sandhas is entirely satisfactory for the definition of the optical potential. A dynamical integral equation for the symmetrized optical potential is derived. Several alternative forms of this equation are developed. Various low-order approximations to these equations are proposed.

[NUCLEAR REACTIONS Connected-kernel N-particle equations with multiple scattering structure including permutation symmetries. Dynamical equations for the optical potential for elastic collisions of two fragments composed of groups of identical particles.]

I. INTRODUCTION

Optical potentials are widely used for the description of the elastic scattering of two nuclear fragments.[1-3] From the point of view of nuclear reaction theory the extant microscopic formulation of the optical potential possesses two major shortcomings. Until very recently there has been no fully consistent theory for the calculation of this potential.[4,5] Not unrelated to this is the fact that there is no entirely satisfactory incorporation of the permutation symmetries arising from nucleon identity, viz., the Pauli principle.[6,7] In the present article we show how both of these difficulties can be overcome by symmetrizing the theory of the type developed in Ref. 4.[8]

The grafting of permutation symmetries onto the multiparticle scattering equations of Ref. 4 is facilitated by the fact that all of these equations fall into the so-called label-transforming class with respect to permutations of the N particles.[9] This means that the techniques developed in Refs. 9 for deriving symmetrized scattering integral equations can be utilized. A summary of the pertinent aspects of Refs. 9 is presented in Appendix A.

For the sake of simplicity we have confined ourselves, as in Ref. 4, to only pairwise nucleon-nucleon interactions.[10,11] We find it unnecessary, however, to make any commitment as to the nature of the two fragments. All of the cases of nucleon-nucleus, deuteron-nucleus, and nucleus-nucleus scattering are subsumed. We also find it unnecessary to restrict ourselves to a definite species of particle; i.e., we consider the fragments to consist of groups of different species of identical particles. Thus our results apply both to a collection of nucleons which are regarded as identical as well as groups of (identical) neutrons and protons.

This paper is organized as follows. Section II contains mainly a review of the pertinent aspects of multiparticle scattering theory along with an introduction to our notation. We also examine various off-shell extensions of the N-particle transition operators. The appropriate choice of these operators turns out to be a crucial feature of our work. In Sec. III we show how permutation symmetries are imposed upon the multiparticle scattering equations of interest to us. The symmetrized optical potential is defined in Sec. IV using the symmetrized form of the unitarity equation as a guide. Section V is devoted to the derivation of a well-defined dynamical integral equation for this symmetrized optical potential; this equation represents the major result of this paper. In Sec. V a more direct method of derivation is used as compared with the technique outlined in Ref. 8. The details of the latter procedure are also presented here. Several alternative forms of the dynamical optical potential equation are developed. The structure of these equations is explored further in Sec. VI along with the introduction of several low-order approximations. Our work is summarized in Sec. VII. The appendices (A–D) contain the derivations of ancillary results.

II. MANY-BODY SCATTERING EQUATIONS

We review those aspects of the multiparticle scattering theory of Ref. 4 which we require for the remainder of this paper.[10,11] In this section we suppose that all N particles are distinguishable. The modifications which arise when this restriction

is dropped are taken up in Sec. III.

A *partition* a is the grouping of the N particles into n_a distinct clusters. The arrangement of these particles within the clusters and the order of the clusters within the partition are regarded as irrelevant if the particles are distinguishable; modifications of these conventions are sometimes required in the case of identical particles. The unique partitions for which $n_a=1$ and $n_a=N$ are denoted as 1 and 0, respectively. We use Latin letters $a,b,c,\ldots$ to denote arbitrary partitions of the N particles while Greek letters $\alpha,\beta,\gamma,\ldots$, are reserved for two-cluster partitions. Since we are interested in two-fragment elastic collisions the last kind of partitions plays a central role in our work.

The partition indexing of the operators which appear in multiparticle scattering theory is facilitated by a partial ordering of the set of all partitions of N objects. A partition b is said to be *contained in* another partition a, written $b\subset a$, if b can be obtained from a by subdividing one or more of its clusters. We write $b\subseteq a$ when we include the possibility of equality. The alternative possibility, where b is *not contained in or equal to* a is denoted by $b\not\subset a$. These relationships are conveniently represented by the matrices Δ and $\bar{\Delta}$ with the elements[12]

$$\Delta_{a,b}=1, \quad b\subseteq a$$
$$=0, \quad \text{otherwise} \tag{2.1}$$

and

$$\bar{\Delta}_{a,b}=1-\Delta_{a,b}, \tag{2.2}$$

respectively.

The N-particle Hamiltonian H is taken to be the sum of the kinetic energy operator H_0 and the total interaction potential V:

$$H=H_0+V. \tag{2.3}$$

In the present article we confine ourselves to the case where V is a sum of pair interactions $V_{i'}$, namely,

$$V=\sum_{i'} V_{i'}. \tag{2.4}$$

In (2.4) the pair interactions are labeled by the $(N-1)$-cluster partitions i'. The complications which arise when interactions are more general than (2.4) are considered in Ref. 11.

The interaction *internal* to the partition a is defined as

$$V_a\equiv\sum_{i'} \Delta_{a,i'}V_{i'} \tag{2.5}$$

and the partition Hamiltonian is taken to be

$$H_a=H_0+V_a. \tag{2.6}$$

We note that $V_0=0$ so that the designation of the kinetic energy as H_0 is consistent with our notation. H_a generates the dynamics of an N-particle system in which there are no interactions among particles in different clusters of a. If there exist eigenstates $|\phi_a(\nu_a)\rangle$ of H_a which represent the situation where each of the n_a clusters are in bound configurations, then a is called a *stable* partition. These maximally connected eigenstates of H_a correspond to possible asymptotic states of the system and are labeled by channel indices ν_a which refer to the collection of internal quantum numbers of the clusters of a. Here we have not explicitly indicated the dependence of $|\phi_a(\nu_a)\rangle$ upon the center-of-mass (c.m.) momenta of the n_a clusters.

The interaction among clusters of a is characterized by the interaction *external to* the partition a:

$$V^a\equiv H-H_a=\sum_{i'} \bar{\Delta}_{a,i'}V_{i'}. \tag{2.7}$$

Then a transition operator corresponding to the process $b\to a$ is given by

$$T^{(+)}_{a,b}=V^a+V^aGV^b, \tag{2.8}$$

where

$$G=(z-H)^{-1}, \tag{2.9}$$

and z is a parametric energy the dependence upon which we suppress in what follows, except when it is necessary to avoid ambiguity. The scattering amplitude for the process $b\to a$ is the on-shell matrix element $\langle\phi_a(\nu_a)|T^{(+)}_{a,b}|\phi_b(\nu_b)\rangle$. Since only these matrix elements are physically significant there are many other transition operators which one can choose so long as they are on-shell equivalent to $T^{(+)}_{a,b}$ with respect to the appropriate channel states $|\phi_{a,b}(\nu_{a,b})\rangle$. One of these choices, which is distinguished from $T^{(+)}_{a,b}$ by the symmetric role of the partition indices, is

$$T_{a,b}=\bar{\delta}_{a,b}G_b^{-1}+T^{(+)}_{a,b}, \tag{2.10}$$

where

$$G_b=(z-H_b)^{-1}$$

and $\bar{\delta}_{a,b}=1-\delta_{a,b}$. The scattering operator $T_{a,b}$ of Alt, Grassberger, and Sandhas[13] proves to be more satisfactory for our investigation because of its symmetrical relationship to the total Green's function G.[12,13]

In keeping this attribute of partition index symmetry we can express $T_{a,b}$ in terms of the symmetrical part of $T^{(+)}_{a,b}$, viz.,[4,14]

$$T_{a,b}=\bar{\delta}_{a,b}G_b^{-1}+V^a_b+\tau^{a,b}, \tag{2.11}$$

where

$$\tau^{a,b} = V^{a,b} + V^a G V^b \tag{2.12}$$

and

$$V_b^a = \sum_{i'} \bar{\Delta}_{a,i'} V_{i'} \Delta_{b,i'} . \tag{2.13}$$

In the case of only pair interactions it has been shown that the components of $\tau^{\alpha,\beta}$ which refer to two-cluster partitions α, β satisfy the connected-kernel integral equations[4]

$$\tau^{\alpha,\beta} = W_{\rm MS}^{\alpha,\beta} + \sum_{\gamma} W^{\alpha,0}(\gamma) G_0 \tau^{\gamma,\beta} . \tag{2.14}$$

Several aspects of (2.14) require further elaboration. First, the inhomogeneous part of (2.14) has the structure

$$W_{\rm MS}^{\alpha,\beta} = \sum_{a}{}' W^{\alpha,\beta}(a) , \tag{2.15}$$

where $W^{a,b}(c)$ is the c-connected part of $\tau^{a,b}$ and the prime on the summation sign indicates the omission of the partition 1. We recall that an N-particle operator A is said to be *a connected* if the momentum-space matrix elements of A have the form

$$\langle \vec{p}_1', \ldots, \vec{p}_N' | A | \vec{p}_1, \ldots, \vec{p}_N \rangle = A_a(\vec{p}_1', \ldots, \vec{p}_N' | \vec{p}_1, \ldots, \vec{p}_N) \prod_{n=1}^{n_a} \delta(\vec{P}_n' - \vec{P}_n) , \tag{2.16}$$

where

$$\vec{P}_n \equiv \vec{p}_{j_n} + \cdots + \vec{p}_{k_n}$$

is the momentum of the c.m. of the nth cluster, the indices $j_n, \ldots, k_n$ correspond to the particles in the nth cluster, and the functions $A_a(\vec{p}_1', \ldots, \vec{p}_N' | \vec{p}_1, \ldots, \vec{p}_N)$ possess no δ-function singularities. By this criterion the kernels

$$W^{\alpha,0}(\gamma) G_0 = [W^{\alpha,\delta}(\gamma) G_\delta]_{\delta=0} , \tag{2.17}$$

of the coupled equations (2.14) are γ connected. However, a single iteration of Eqs. (2.14) yields kernels which are products of operators of the form

$$W^{\alpha,0}(\gamma) G_0 W^{\gamma,0}(\delta) G_0$$

which either vanish ($\gamma = \delta$) or are connected ($\gamma \neq \delta$) and it is in this sense that we say that Eqs. (2.14) are connected-kernel integral equations.

We presume the validity of the underlying assumption of N-particle scattering theory[15] regarding scattering integral equations of the connected-kernel type. Namely, we suppose that the connectedness property implies the compactness of the kernel and this in turn implies the possibility of a consistent solution of the integral equations. Equations (2.14), then, constitute a well-defined system of scattering integral equations for the two-cluster partition-indexed operators $\tau^{\alpha,\beta}$. The physical two-cluster to two-cluster transition operators then follow trivially from Eq. (2.11). It is easy to show that the $\alpha \to a$ transition operators with $n_a > 2$ can be obtained from those with $n_a = 2$ by quadrature rules without the solution of any further integral equations.

Equations (2.14) have the desirable feature of possessing inhomogeneous terms with a manifest multiple scattering structure. For example,

$$W^{\alpha,\beta}(i') = \bar{\Delta}_{\alpha,i} t_i \bar{\Delta}_{\beta,i'} , \tag{2.18}$$

where

$$t_{i'} = V_{i'} + V_{i'} G_0 t_{i'} \tag{2.19}$$

is the two-particle transition operator on the N-body space. This structure is a consequence of the direct relationship of (2.10) and (2.11) to the usual definitions of the transition operators in contrast to alternative formulations[16] which involve integral equations with exactly the same kernels as Eqs. (2.14). This relationship is also responsible for the fact that one can use Eqs. (2.14) to derive a consistent theory of the Feshbach[17] optical potential for elastic two-fragment collisions. The object of the present investigation is to extend these results to the case when the fragments are each comprised of one or more species of identical particles.

III. SYMMETRIZED SCATTERING EQUATIONS

The connected-kernel equations (2.14) are partition labeled and possess a manifest multiple scattering structure and are referred to as PLMS equations.[11] Such equations have many desirable features particularly in regard to the formulation of approximations. This is demonstrated in Refs. 4 and 11 for the case of distinguishable particles. We next derive the symmetrized form of (2.14) which is appropriate to the situation where there exist groups of identical particles in our system. This is very easy to do with the aid of the formalism of Ref. 9 which is outlined in Appendix A. This Appendix also contains all of our notational conventions regarding permutations.

In order to apply the results of Ref. 9 we need only establish that the kernels and homogeneous terms of (2.14) are *label transforming*. This property is defined in Appendix A.

The pair potentials $V_{i'}$ transform under the permutation $P \in S$ in the manner

$$U(P)V_{i'}U(P)^{\dagger}=V_{P(i')}\,, \tag{3.1}$$

where $P(i')$ is evidently another pair partition. With the aid of (3.1) and lemma 1 of Appendix A it follows that

$$U(P)V_aU(P)^{\dagger}=V_{P(a)}\,, \tag{3.2a}$$

$$U(P)V^aU(P)^{\dagger}=V^{P(a)}\,, \tag{3.2b}$$

$$U(P)V_b^aU(P)^{\dagger}=V_{P(b)}^{P(a)}\,. \tag{3.2c}$$

Next, let us introduce the auxiliary operator

$$t_f^{a,b}=V_f^{a,b}+V_f^aG_fV_f^b\,, \tag{3.3}$$

where

$$V_f^{a,b}\equiv\sum_{i'}\bar{\Delta}_{a,i'}\bar{\Delta}_{b,i'}V_{i'}\Delta_{f,i'}\,. \tag{3.4}$$

We note that $t_1^{a,b}=\tau^{a,b}$. Both $V_f^{a,b}$ and G_f are easily shown to be label transforming and therefore so is $t_f^{a,b}$. The significance of $t_f^{a,b}$ for our analysis arises from the relation[18]

$$W^{a,b}(c)=\sum_f{}'(\Delta^{-1})_{c,f}t_f^{a,b}\,, \tag{3.5}$$

where $(\Delta^{-1})_{c,f}$ refers to the elements of the inverse of the matrix $\Delta_{a,b}$. One can easily demonstrate using $\delta_{a,b}=\delta_{P(a),P(b)}$, lemma 1, and[4,5,10-12]

$$\sum_c\Delta_{a,c}(\Delta^{-1})_{c,b}=\delta_{a,b}\,, \tag{3.6}$$

that

$$(\Delta^{-1})_{P(c),P(b)}=(\Delta^{-1})_{c,b}\,. \tag{3.7}$$

Then we conclude from (3.5), (3.7), and the label-transforming property of $t_f^{a,b}$ that

$$U(P)W^{a,b}(c)U(P)^{\dagger}=W^{P(a),P(b)}[P(c)]\,, \tag{3.8}$$

namely, $W^{a,b}$ is label transforming as well. The partitions c and $P(c)$ are physically equivalent so that the permutation symmetry transformation preserves the connectivity structure.

From (3.8) and (2.15) we conclude that the inhomogeneous and kernel terms of the PLMS equations (2.14) are label transforming, viz.,

$$U(P)W_{\rm MS}^{\alpha,\beta}U(P)^{\dagger}=W_{\rm MS}^{P(\alpha),P(\beta)}\,, \tag{3.9}$$

$$U(P)W^{\alpha,0}(\gamma)G_0U(P)^{\dagger}=W^{P(\alpha),0}[P(\gamma)]G_0\,. \tag{3.10}$$

Then theorem 1 of Appendix A assures us that the solutions of (2.14) are label transforming as well, although this conclusion is already obvious from definition (2.12). The fact that (2.14) is a label-transforming set of integral equations allows us to obtain from it a set of integral equations for the physically relevant class-labeled (Appendix A) operators $\bar{\tau}^{\hat{\alpha},\hat{\beta}}$. It is important to remark that any approximation to the inhomogeneous or kernel terms of (2.14) which preserves the label-transforming characteristics of (3.9) and (3.10) will generate an approximate but label-transforming solution. This provides a simple criterion for the preservation of permutation symmetries in the course of approximations.

The integral equations for the class-labeled operators $\bar{\tau}^{\hat{\alpha},\hat{\beta}}$ can be written down immediately as a consequence of theorem 2 of Appendix A:

$$\bar{\tau}^{\hat{\alpha},\hat{\beta}}=\tilde{W}_{\rm MS}^{\hat{\alpha},\hat{\beta}}+\sum_{\hat{\gamma}}K^{\hat{\alpha},\hat{\gamma}}\bar{\tau}^{\hat{\gamma},\hat{\beta}}\,, \tag{3.11}$$

$$\bar{\tau}^{\hat{\alpha},\hat{\beta}}=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathcal{R}_\alpha\tau^{\alpha,\bar{\beta}}\,, \tag{3.12}$$

$$\tilde{W}_{\rm MS}^{\hat{\alpha},\hat{\beta}}=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathcal{R}_\alpha W_{\rm MS}^{\alpha,\bar{\beta}}\,, \tag{3.13}$$

$$K^{\hat{\alpha},\hat{\gamma}}=\bar{N}_{\hat{\alpha},\hat{\gamma}}\sum_{\alpha\in\hat{\alpha}}\mathcal{R}_\alpha W^{\alpha,0}(\bar{\gamma})G_0 \tag{3.14}$$

and the remainder of the notation is defined in Appendix A. We recall that partitions $\bar{\beta}$ and $\bar{\gamma}$ refer to some specifically chosen (*canonical*) members of the equivalence classes $\hat{\beta}$ and $\hat{\gamma}$, respectively. Also we note that by theorem 4 of Appendix A the physical scattering amplitudes are obtained from the on-shell matrix elements of the class-labeled AGS operators

$$\bar{T}_{\hat{\alpha},\hat{\beta}}=\hat{G}_{\hat{\alpha},\hat{\beta}}{}^{-1}+\tilde{V}_{\hat{\beta}}^{\hat{\alpha}}+\bar{\tau}^{\hat{\alpha},\hat{\beta}}\,, \tag{3.15}$$

where

$$\tilde{V}_{\hat{\beta}}^{\hat{\alpha}}=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathcal{R}_\alpha V_{\bar{\beta}}^{\alpha} \tag{3.16}$$

and

$$\hat{G}_{\hat{\alpha},\hat{\beta}}{}^{-1}=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathcal{R}_\alpha\bar{\delta}_{\alpha,\bar{\beta}}G_{\bar{\beta}}{}^{-1}\,. \tag{3.17}$$

Equations (3.11)–(3.14) are the symmetrized PLMS equations and they represent the principal result of this section. These equations are used in Sec. V to formulate a consistent theory of the optical potential which incorporates the permutation symmetries of the system which arise from particle identity. These equations can be applied to other aspects of nuclear reactions as well.

We conclude this section by quoting the results of Ref. 9 concerning the values of the normalization constants $N_{\hat{a}}$ and the number of equivalence classes for the case of physical interest, namely, a system of N neutrons and Z protons. (If all of the nucleons are regarded as identical one simply sets $N=0$, e.g., and equates Z with the total number of particles.) Any two-cluster partition consists of a cluster of n neutrons and z protons with $n+z\geq 1$, while the other cluster contains $N-n$ neutrons and $Z-z$ protons. Evidently each pair of

integers (n,z) corresponds to an equivalence class of two-cluster partitions. If $\hat{\alpha} \sim (n,z)$, the number $N_{\hat{\alpha}}$ of distinct partitions in $\hat{\alpha}$ is easily shown to be[9]

$$N_{\hat{\alpha}} = N_{(n,z)} = \binom{N}{n}\binom{Z}{z}(1 - \tfrac{1}{2}\delta_{N/2,n}\delta_{Z/2,z}), \quad (3.18)$$

where the factor involving the δ functions relates to the degenerate situation where both clusters are identical. It is somewhat more difficult to calculate the number of equivalence classes which are contained within the $2^{N+Z-1}-1$ two-cluster partitions. This number is identical with the number $P_=(N,Z)$ of partitions of the pair of integers (N,Z) without regard to order into the two nonempty parts (n,z) and $(N-n,Z-z)$. If one distinguishes the order of the two parts such a partition is called a *composition*.[19] The total number $C(N,Z;2)$ of compositions of (N,Z) into two parts is[20]

$$C(N,Z;2) = (N+1)(Z+1) - 2. \quad (3.19)$$

If N and Z are not both even, the two parts can never be identical and there are two compositions for each partition so

$$P_=(N,Z) = \tfrac{1}{2}C(N,Z;2), \quad (3.20a)$$

while if both N and Z are even there is only one composition corresponding to the partition into two equal clusters. Thus in this case

$$P_=(N,Z) = \tfrac{1}{2}[C(N,Z;2) - 1] + 1. \quad (3.20b)$$

IV. SYMMETRIZED OPTICAL POTENTIAL

We consider the elastic scattering of two nuclear fragments whose asymptotic states $|\phi_{\bar{\beta}}(\nu_{\bar{\beta}})\rangle$ correspond to the channels defined by the equivalence class $\hat{\beta}$ of partitions. The physical scattering amplitudes are identified with the on-shell matrix elements

$$\langle \phi_{\bar{\beta}}^{(f)}(\nu_\beta) | \tilde{T}_{\hat{\beta},\hat{\beta}} | \phi_{\bar{\beta}}^{(i)}(\nu_{\bar{\beta}}) \rangle = \langle \phi_{\bar{\beta}}^{(f)}(\nu_{\bar{\beta}}) | \tilde{T}^{(+)}_{\hat{\beta},\hat{\beta}} | \phi_{\bar{\beta}}^{(i)}(\nu_{\bar{\beta}}) \rangle \quad (4.1)$$

and we have used the superscripts i and f to represent the kinematical states of the two fragments. In the case where all the particles are distinguishable $\hat{\beta} = \bar{\beta}$ and the first two terms in the definition of $\tilde{T}_{\hat{\beta},\hat{\beta}}$, Eq. (3.15), vanish. This has the simplifying consequence that in order to obtain an integral equation for the optical potential one need only find an integral equation for $\tau^{\bar{\beta},\bar{\beta}} = T^{(+)}_{\bar{\beta},\bar{\beta}} = T_{\bar{\beta},\bar{\beta}}$ in which the singularities responsible for the discontinuity across the $\bar{\beta}$-elastic unitarity cut have been explicated.[4]

When some (or all) of the particles are identical, $\hat{\beta}$ generally includes more than one partition. Since the partitions included in $\hat{\beta}$ are physically equivalent (lemma 2, Appendix A) they must be treated on an equal footing. It is this requirement which complicates the incorporation of permutation symmetries into nuclear reaction theories. This is because if all $\beta \in \hat{\beta}$ are physically equivalent one must deal with all of the operators $T_{\beta,\bar{\beta}}$ democratically, yet from the point of view of the underlying reaction theory each of these operators represents a distinct collision process. In a theory of the optical potential this entails the handling of all of the discontinuities across the unitarity cuts arising from the singularities in the Green's functions G_β, where $\beta \in \hat{\beta}$. Although this does not seem to have been realized in previous discussions of the optical potential, the choice of off-shell extension of the transition operators for $\beta \neq \bar{\beta}$ is a significant aspect of such a singularity analysis.

In order to provide a rationale for a choice of off-shell extension for the transition operators we consider the unitarity condition for the transition operators $T^{(+)}_{a,b}$:

$$T^{(+)}_{a,b}(+) - T^{(+)}_{b,a}(+)^\dagger = V^a - V^b - 2\pi i \sum_{c,\nu_c}{}' T^{(+)}_{a,c}(+)\mathcal{P}_c(\nu_c)\delta(E - H_c)T^{(+)}_{b,c}(+)^\dagger, \quad (4.2)$$

where we have introduced the projector

$$\mathcal{P}_c(\nu_c) = \left\{\prod_{k=1}^{n_c} \int (d\vec{P}_k)\right\} |\phi_c(\nu_c;\vec{P}_1,\ldots,\vec{P}_{n_c})\rangle\langle\phi_c(\nu_c;\vec{P}_1,\ldots,\vec{P}_{n_c})|, \quad (4.3)$$

and the dependence of the asymptotic states on the c.m. momenta $\vec{P}_k$, $1 \leq k \leq n_c$, of the bound clusters has been made explicit. In (4.3), $T^{(+)}_{a,b}(\pm)$, e.g., means $T^{(+)}_{a,b}$ evaluated at $z = E \pm i\epsilon$, $E \geq 0$, $\epsilon \to 0+$ [cf. Eqs. (2.8), (2.9)].

In the discussion of the optical potential we are only interested in matrix elements of (4.2) between the states $\langle\phi_a(\nu_a)|$ and $|\phi_b(\nu_b)\rangle$. However, it should be noted that these matrix elements are not necessarily on-shell. Thus, although we have

$$V^a - V^b = G_a^{-1}(\pm) - G_b^{-1}(\pm) \quad (4.4)$$

and these asymmetric terms do not contribute to (4.2) on-shell, we cannot ignore them off-shell. In the case when all particles are distinguishable, we need to consider (4.2) only when $a = b$ and the term (4.4)

vanishes identically. This is not the case in the physical situation of interest to us and we seek, therefore, a more symmetric expression of multiparticle unitarity.

Specifically, the transition operator $T_{a,b}$ satisfies the unitarity condition

$$T_{a,b}(+)-T_{b,a}(+)^\dagger=2i\epsilon\bar{\delta}_{a,b}-2\pi i\sum_{c,\nu_c}{}'\,[T_{a,c}(+)-\bar{\delta}_{a,c}G_c(+)^{-1}]\,\mathcal{P}_c(\nu_c)\delta(E-H_c)[T_{b,c}(+)^\dagger-\bar{\delta}_{c,b}G_c(-)^{-1}]\,. \tag{4.5}$$

When one considers only the matrix elements of (4.5) of the type specified previously in connection with (4.2), Eq. (4.5) can be reduced to a simpler more symmetric form[21]

$$T_{a,b}(+)-T_{b,a}(+)^\dagger=-2\pi i\sum_{c,\nu_c}{}'\,T_{a,c}(+)\mathcal{P}_c(\nu_c)\delta(E-H_c)T_{b,c}(+)^\dagger\,. \tag{4.6}$$

Now the symmetrized AGS operator is

$$\tilde{T}_{\hat{a},\hat{b}}(+)=\bar{N}_{\hat{a},\hat{b}}\sum_{a\in\hat{a}}\mathcal{R}_aT_{a,\bar{b}}(+)\,, \tag{4.7}$$

where $\bar{b}$ is the canonical partition of the equivalence class $\hat{b}$. From the label-transforming character of $T_{b,a}(+)^\dagger$ and the properties of $\mathcal{R}_a$ it follows that

$$\tilde{T}_{\hat{b},\hat{a}}(+)^\dagger=\bar{N}_{\hat{a},\hat{b}}\sum_{a\in\hat{a}}\mathcal{R}_aT_{\bar{b},a}(+)^\dagger\,. \tag{4.8}$$

Using (4.7) and (4.8) it is straightforward to obtain the permutation symmetrized form of (4.6) if the "kernel," $T_{a,c}(+)\mathcal{P}_c(\nu_c)\delta(E-H_c)$, is label transforming. This is easy to show because $T_{a,c}(+)$ has this property, while the label-transforming character of $\delta(E-H_c)$ is obvious from the identity

$$-2\pi i\delta(E-H_c)=G_c(+)-G_c(-)\,.$$

Also, if $c'\equiv P(c)$ then from (4.3) and (A19) we infer that

$$U(P)\mathcal{P}_c(\nu_c)U(P)^\dagger=\mathcal{P}_{c'}(\nu_c)\,, \tag{4.9}$$

where because c and c' are physically equivalent $\nu_{c'}=\nu_c$. Then using theorem 2 of Appendix A, where the inhomogeneous term in (A16) is identified with $T_{a,b}(+)$, we infer the symmetrized form of the unitarity relation (4.6):

$$\tilde{T}_{\hat{a},\hat{b}}(+)-\tilde{T}_{\hat{b},\hat{a}}(+)^\dagger=-2\pi i\sum_{\bar{c},\nu_{\bar{c}}}{}'\,\tilde{T}_{\hat{a},\hat{c}}(+)\mathcal{P}_{\bar{c}}(\nu_{\bar{c}})\delta(E-H_{\bar{c}})\tilde{T}_{\hat{b},\hat{c}}(+)^\dagger\,. \tag{4.10}$$

We are interested in the elastic scattering of two fragments which are in the ground state configuration $\bar{\nu}_{\bar{\beta}}$ of internal quantum numbers.[22] The form of (4.10) relevant in this case is therefore

$$\tilde{T}_{\hat{\beta},\hat{\beta}}(+)-\tilde{T}_{\hat{\beta},\hat{\beta}}(+)^\dagger=-2\pi i\tilde{T}_{\hat{\beta},\hat{\beta}}(+)\mathcal{P}_{\bar{\beta}}(\bar{\nu}_{\bar{\beta}})\delta(E-H_{\bar{\beta}})\tilde{T}_{\hat{\beta},\hat{\beta}}(+)^\dagger+\text{inelastic terms}\,. \tag{4.11}$$

Equation (4.11) suggests the introduction of the optical potential $\mathcal{U}_{\rm opt}(\hat{\beta})$ via the integral equations

$$\tilde{T}_{\hat{\beta},\hat{\beta}}(+)=\mathcal{U}_{\rm opt}(\hat{\beta})+\mathcal{U}_{\rm opt}(\hat{\beta})g_{\bar{\beta}}(+)\tilde{T}_{\hat{\beta},\hat{\beta}}(+)=\mathcal{U}_{\rm opt}(\hat{\beta})+\tilde{T}_{\hat{\beta},\hat{\beta}}(+)g_{\bar{\beta}}(+)\mathcal{U}_{\rm opt}(\hat{\beta})\,, \tag{4.12}$$

where we define for any two-cluster partition β,

$$g_\beta(\pm)=\mathcal{P}_\beta(\bar{\nu}_\beta)G_\beta(\pm)\,. \tag{4.13}$$

The form (4.12) guarantees that it is possible to describe elastic scattering of the two fragments corresponding to the equivalence class $\hat{\beta}$ in the total c.m. frame by means of the one-body Schrödinger equation. This equation contains a complex, one-body potential in momentum space which is real[23] at energies below the inelastic threshold:

$$\mathcal{U}_{\hat{\beta}}(\vec{k}'|\vec{k})=\langle\phi_{\bar{\beta}}(\vec{k}')|\,\mathcal{U}_{\rm opt}(\hat{\beta})\,|\phi_{\bar{\beta}}(\vec{k})\rangle\,. \tag{4.14}$$

In (4.14) we have suppressed all quantum numbers except for the relative momentum $\vec{k}$ between the two fragments of $\bar{\beta}$. Finally, since all the partitions $\beta\in\hat{\beta}$ are treated on equal footing, Eqs. (4.11) and (4.12) and consequently $\mathcal{U}_{\hat{\beta}}(\vec{k}'|\vec{k})$ are independent of the choice of the canonical partition $\bar{\beta}$.

Equation (4.12) can be regarded as an integral equation for the optical potential, viz.,

$$\mathcal{U}_{\rm opt}(\hat{\beta})=\tilde{T}_{\hat{\beta},\hat{\beta}}(+)[1-g_{\bar{\beta}}(+)\mathcal{U}_{\rm opt}(\hat{\beta})]\,. \tag{4.15}$$

Equation (4.15) is used to obtain dynamical integral equations for $U_{opt}(\hat{\beta})$ in the next section.

From (4.12) the unitarity relation satisfied by $\tilde{T}_{\hat{\beta},\hat{\beta}}(+)$ is seen to be[24]

$$\tilde{T}_{\hat{\beta},\hat{\beta}}(+)-\tilde{T}_{\hat{\beta},\hat{\beta}}(+)^{\dagger}=-2\pi i\tilde{T}_{\hat{\beta},\hat{\beta}}(+)\mathcal{P}_{\bar{\beta}}^{-}(\bar{\nu}_{\bar{\beta}})\delta(E-H_{\bar{\beta}})\tilde{T}_{\hat{\beta},\hat{\beta}}(+)^{\dagger}$$
$$+[\tilde{T}_{\hat{\beta},\hat{\beta}}(+)g_{\bar{\beta}}^{-}(+)+1][U_{opt}(\hat{\beta})-U_{opt}(\hat{\beta})^{\dagger}][1+g_{\bar{\beta}}^{-}(-)\tilde{T}_{\hat{\beta},\hat{\beta}}(+)^{\dagger}]\,. \tag{4.16}$$

This indicates that Eqs. (4.12) are consistent with the requirement that $U_{opt}(\hat{\beta})$ has no discontinuities across the $\hat{\beta}$-class elastic unitarity cuts but does not prove it. [Note that the reality[23] of the optical potential below the inelastic threshold obviously implies the vanishing of the inelastic contribution in (4.16) for subthreshold energies. The converse is not necessarily true.] This reality requirement for $U_{\beta}(\vec{k}'|\vec{k})$ appears more difficult to satisfy if we introduce the optical potential via

$$\tilde{T}^{(+)}_{\hat{\beta},\hat{\beta}}(+)=U^{(+)}_{opt}(\hat{\beta})+U^{(+)}_{opt}(\hat{\beta})g_{\bar{\beta}}(+)\tilde{T}^{(+)}_{\hat{\beta},\hat{\beta}}(+)$$
$$=U^{(+)}_{opt}(\hat{\beta})+\tilde{T}^{(+)}_{\hat{\beta},\hat{\beta}}g_{\bar{\beta}}(+)U^{(+)}_{opt}(\hat{\beta})\,. \tag{4.17}$$

The reason for this is that although a relation similar to (4.16) follows from (4.17), the counterpart of (4.12) for $\tilde{T}^{(+)}_{\hat{\beta},\hat{\beta}}(+)$ contains extra asymmetrical terms arising from V^a-V^b. The presence of such terms requires that the second term in the analog of Eq. (4.16) for $T^{(+)}_{\hat{\beta},\hat{\beta}}(+)$ does not vanish even below the inelastic threshold so $U^{(+)}_{opt}(\hat{\beta})$ need not be real in this instance. In this connection we remark that we are unable to prove using the techniques developed in Sec. V that $U^{(+)}_{opt}(\hat{\beta})$ is free of any $\hat{\beta}$-class elastic unitarity cuts. Therefore we discard (4.17) as a useful definition of the optical potential in the case where identical particles are involved.

In Ref. 4 the necessity for developing a consistent dynamical theory for the same off-shell extension of the optical potential as employed in conventional reaction theories[17] was stressed. This was in order that comparisons between the two modes of approach could be facilitated. We have apparently violated this stipulation here. However, it seems evident that a "conventional" optical potential operator for the elastic scattering of two fragments containing identical particles has not been defined previously in the context of standard reaction theory. Thus, we are proposing Eqs. (4.12) as the appropriate starting point for an optical potential formalism under all circumstances. We note that this coincides exactly with the conventional definitions[16] in the case where all the constituent particles are distinguishable.

V. DYNAMICAL EQUATIONS FOR THE OPTICAL POTENTIAL

In this section we address our principal question, namely the development of consistent dynamical equations for the determination of the optical potential $U_{opt}(\hat{\beta})$ defined in Eq. (4.12). We also establish that $U_{opt}(\hat{\beta})$ is free of all $\hat{\beta}$-class elastic unitarity discontinuities and therefore that it possesses the characteristics we conjectured for it in our discussion of Eqs. (4.12) and (4.15) in the preceding section.

Underlying all approaches to the optical potential is the segregation of the elastic unitarity discontinuities from some effective interaction. When there are particle permutation symmetries, singularities in several physical equivalent channels must be handled simultaneously. The present approach is based upon the multichannel generalization of the method of Ref. 4 along with the specification of the AGS off-shell extension for two-cluster transition operators.

We consider the unsymmetrized PLMS equations (2.14) for $\beta=\bar{\beta}$,

$$\tau^{\alpha,\bar{\beta}}=W^{\alpha,\bar{\beta}}_{MS}+\sum_{\gamma}W^{\alpha,0}(\gamma)G_0\tau^{\gamma,\bar{\beta}}\,. \tag{5.1}$$

We are interested in the discontinuity structure of the kernel and the inhomogeneity of Eq. (5.1) with respect to the two-cluster (ground state[22]) elastic unitarity cuts.[25] It is shown in Appendix B that the part[26] of $W^{a,b}(c)$, designated as $\mathfrak{D}[W^{a,b}(c)]_{\gamma}$, which possesses a discontinuity across the γ-elastic cut, is

$$\mathfrak{D}[W^{a,b}(c)]_{\gamma}=V^a_{\gamma}g_{\gamma}V^b_{\gamma}\delta_{c,\gamma}\,. \tag{5.2}$$

We infer from (5.2) and (2.15) that

$$\mathfrak{D}(W^{\alpha,\bar{\beta}}_{MS})_{\gamma}=V^{\alpha}_{\gamma}g_{\gamma}V^{\bar{\beta}}_{\gamma} \tag{5.3}$$

and also

$$\mathfrak{D}[W^{\alpha,0}(\lambda)G_0]_{\gamma}=V^{\alpha}_{\gamma}g_{\gamma}V_{\gamma}G_0\delta_{\gamma,\lambda}\,. \tag{5.4}$$

In connection with (5.4) we note that

$$V^{\alpha}_{\gamma}g_{\gamma}V_{\gamma}G_0=V^{\alpha}_{\gamma}g_{\gamma}-V^{\alpha}_{\gamma}\mathcal{P}_{\gamma}(\bar{\nu}_{\gamma})G_0\,. \tag{5.5}$$

Since the last term in (5.5) is continuous across the γ-elastic cut we can take as an alternative[26] to (5.4)

$$\mathfrak{D}[W^{\alpha,0}(\lambda)G_0]_{\gamma}=V^{\alpha}_{\gamma}g_{\gamma}\delta_{\gamma,\lambda}\,. \tag{5.6}$$

The choice (5.6) is preferable for our purposes, i.e., for dealing with the off-shell extensions of the transition operators used in (4.12).

Now we wish to construct an optical potential $U_{opt}(\hat{\beta})$ free of all $\beta\in\hat{\beta}$ elastic discontinuities. To this end we subtract all of these discontinuities

from the kernels of (5.1) to obtain the $\hat{\beta}$-class nonsingular operators

$$K_\gamma^\alpha(\hat{\beta}) = W^{\alpha,0}(\gamma)G_0 - V_\gamma^\alpha g_\gamma \delta(\gamma \in \hat{\beta}) \tag{5.7}$$

and then we introduce operators $\Lambda_\lambda^\alpha(\hat{\beta})$ as the solutions of

$$\Lambda_\lambda^\alpha(\hat{\beta}) = \delta_{\alpha,\lambda} + \sum_\gamma K_\gamma^\alpha(\hat{\beta})\Lambda_\lambda^\gamma(\hat{\beta}) . \tag{5.8a}$$

Here $\delta(\gamma \in \hat{\beta}) = 1$, if $\gamma \in \hat{\beta}$, and is zero otherwise. Equations (5.7) are well-behaved integral equations since their kernels become connected after one iteration. This is shown in Appendix C. We also observe that from (5.8a) we obtain

$$\Lambda_\lambda^\alpha(\hat{\beta}) = \delta_{\alpha,\lambda} + \sum_\gamma \Lambda_\gamma^\alpha(\hat{\beta})K_\lambda^\gamma(\hat{\beta}) . \tag{5.8b}$$

Equation (5.8b) is of great usefulness in deriving alternative expressions for the kernel and driving terms of the integral equation for $\mathcal{U}_{\rm opt}(\hat{\beta})$.

Using (5.7) and (5.8a) we obtain from (5.1)

$$\tau^{\alpha,\bar{\beta}} = \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta})W_{\rm MS}^{\lambda,\bar{\beta}} + \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) \sum_{\beta\in\hat{\beta}} V_\beta^\lambda g_\beta \tau^{\beta,\bar{\beta}} . \tag{5.9}$$

An important feature of (5.9) is that for $\alpha \in \hat{\beta}$ it couples together only those $\tau^{\gamma,\bar{\beta}}$ with $\gamma \in \hat{\beta}$. This is the set of operators we expect to be relevant to the definition of $\mathcal{U}_{\rm opt}(\hat{\beta})$. If we let $\alpha = \beta \in \hat{\beta}$ and use (2.11) in (5.9) we obtain a set of equations which couple only those $T_{\gamma,\bar{\beta}}$ with $\gamma \in \hat{\beta}$:

$$T_{\beta,\bar{\beta}} = \sum_\gamma \Lambda_\gamma^\beta(\hat{\beta})I_\beta^\gamma(\hat{\beta}) + \sum_\gamma \Lambda_\gamma^\beta(\hat{\beta}) \sum_\lambda \delta(\lambda \in \hat{\beta})V_\lambda^\gamma g_\lambda T_{\lambda,\bar{\beta}} , \tag{5.10}$$

where

$$\sum_\gamma \Lambda_\gamma^\beta(\hat{\beta})I_{\bar{\beta}}^\gamma(\hat{\beta}) = \sum_\gamma \Lambda_\gamma^\beta(\hat{\beta})\left[\left(W_{\rm MS}^{\gamma,\bar{\beta}} - \sum_\lambda V_\lambda^\gamma g_\lambda V_\lambda^{\bar{\beta}}\delta(\lambda \in \hat{\beta})\right) - \sum_\lambda V_\lambda^\gamma \mathcal{P}_\lambda(\bar{\nu}_\lambda)\bar{\delta}_{\lambda,\bar{\beta}}\delta(\lambda \in \hat{\beta})\right] + \bar{\delta}_{\beta,\bar{\beta}}G_{\bar{\beta}}^{-1} + V_{\bar{\beta}}^\beta , \tag{5.11a}$$

and it is shown in Appendix D that

$$I_\beta^\gamma(\hat{\beta}) = \left[\left(\sum_{\alpha\in\hat{\beta}}{}' W^{\gamma,0}(\alpha) + W^{\gamma,0}(\beta)\right)G_0 + \bar{\delta}_{\gamma,\beta}\delta(\gamma \in \hat{\beta})\right] \times \delta(\beta \in \hat{\beta})G_\beta{}^{-1} . \tag{5.11b}$$

Equations (5.11) yield alternative forms for the inhomogeneous term of (5.10) and each suggests different approximations. We return to this point in Sec. VI.

The Green's function g_γ is evidently label transforming. Then using the results of Sec. III we find that

$$U(P)K_\gamma^\alpha(\hat{\beta})U(P)^\dagger = W^{P(\alpha)}[P(\gamma)]G_0 - V_{P(\gamma)}^{P(\alpha)}g_{P(\gamma)}\delta(P(\gamma) \in \hat{\beta}) . \tag{5.12}$$

It then follows that

$$U(P)\Lambda_\lambda^\alpha(\hat{\beta})U(P)^\dagger = \Lambda_{P(\lambda)}^{P(\alpha)}(\hat{\beta}) . \tag{5.13}$$

It is important to note that the $\hat{\beta}$-class character of $\Lambda_\lambda^\alpha(\hat{\beta})$ and $K_\lambda^\alpha(\hat{\beta})$ is invariant under permutations. Evidently $I_\beta^\gamma(\hat{\beta})$ is label transforming and so we conclude that the driving terms of (5.9) also possess this property. It then follows from theorem 2 of Appendix A that the symmetrized form of (5.9) is

$$\tilde{T}_{\hat{\beta},\hat{\beta}} = B(\hat{\beta}) + \mathcal{K}(\hat{\beta})\tilde{T}_{\hat{\beta},\hat{\beta}} , \tag{5.14}$$

where

$$\mathcal{K}(\hat{\beta}) = \sum_{\beta\in\hat{\beta}} \mathcal{R}_\beta \sum_\lambda \Lambda_\lambda^\beta(\hat{\beta})V_{\bar{\beta}}^\lambda g_{\bar{\beta}} \tag{5.15}$$

and

$$B(\hat{\beta}) = \sum_{\beta\in\hat{\beta}} \mathcal{R}_\beta \sum_\gamma \Lambda_\gamma^\beta(\hat{\beta})I_{\bar{\beta}}^\gamma(\hat{\beta}) . \tag{5.16}$$

Henceforth we suppose that (5.14)–(5.16), e.g., are referred to the parametric energy $E + i\epsilon$ although we continue to suppress this dependence. Then if we multiply (5.14) on the right by $[1 - g_{\bar{\beta}}\mathcal{U}_{\rm opt}(\hat{\beta})]$ and use (4.15) we obtain a dynamical equation for the optical potential:

$$\mathcal{U}_{\rm opt}(\hat{\beta}) = B(\hat{\beta}) + C(\hat{\beta})\mathcal{U}_{\rm opt}(\hat{\beta}) , \tag{5.17}$$

where

$$C(\hat{\beta}) = \mathcal{K}(\hat{\beta}) - B(\hat{\beta})g_{\bar{\beta}} . \tag{5.18}$$

Special cases of the preceding equations were obtained in Ref. 4 corresponding to $\hat{\beta} = \bar{\beta}$. In particular, the counterpart of (5.17) in this case possesses a driving term and a kernel which are manifestly free of elastic $\hat{\beta}$-class unitarity singularities. It is easy to show that this is the case for (5.17) as well.

The $\hat{\beta}$-class singularity-free character of $B(\hat{\beta})$ is obvious from Eqs. (5.11) and definition (5.16). The structure of $C(\hat{\beta})$ is slightly less evident. We write

$$C(\hat{\beta}) = \sum_{\beta\in\hat{\beta}} \mathcal{R}_\beta C_{\bar{\beta}}^\beta(\hat{\beta})\mathcal{P}_{\bar{\beta}}(\bar{\nu}_{\bar{\beta}}) , \tag{5.19}$$

where

$$C_{\bar{\beta}}^\beta(\hat{\beta}) = \sum_\lambda \Lambda_\lambda^\beta(\hat{\beta})[V_{\bar{\beta}}^\lambda g_{\bar{\beta}} - I_{\bar{\beta}}^\lambda(\hat{\beta})G_{\bar{\beta}}] . \tag{5.20}$$

From (5.8b) we have the identity

$$\sum_{\lambda} \Lambda_{\lambda}^{\beta}(\hat{\beta}) V_{\bar{\beta}}^{\lambda} g_{\bar{\beta}} = \delta_{\beta,\bar{\beta}} - \Lambda_{\bar{\beta}}^{\beta}(\hat{\beta}) + \sum_{\gamma} \Lambda_{\gamma}^{\beta}(\hat{\beta}) W^{\gamma,0}(\bar{\beta}) G_0 , \tag{5.21}$$

which can be used to rewrite (5.20) as

$$C_{\bar{\beta}}^{\beta}(\hat{\beta}) = \Big[\delta_{\beta,\bar{\beta}} - \sum_{\gamma \in \hat{\beta}} \Lambda_{\gamma}^{\beta}(\hat{\beta}) - \sum_{\gamma} \Lambda_{\gamma}^{\beta}(\hat{\beta}) \sum_{\alpha \notin \hat{\beta}}{}' W^{\gamma,0}(\alpha) G_0 \Big] . \tag{5.22}$$

It is now manifest that $C_{\bar{\beta}}^{\beta}(\hat{\beta})$ and thus $C(\hat{\beta})$ are free of elastic $\hat{\beta}$-class singularities.

An alternative derivation of these results employs the multichannel effective interactions considered by Redish,[27] for example.[8] These operators are defined in terms of $T_{\alpha,\beta}$ for $\alpha, \beta \in \hat{\beta}$ by

$$T_{\alpha,\beta} = \mathcal{U}_{\alpha,\beta}(\hat{\beta}) + \sum_{\lambda \in \hat{\beta}} \mathcal{U}_{\alpha,\lambda}(\hat{\beta}) g_{\lambda} T_{\lambda,\beta} , \tag{5.23a}$$

$$T_{\alpha,\beta} = \mathcal{U}_{\alpha,\beta}(\hat{\beta}) + \sum_{\lambda \in \hat{\beta}} T_{\alpha,\lambda} g_{\lambda} \mathcal{U}_{\lambda,\beta}(\hat{\beta}) . \tag{5.23b}$$

Then in a manner similar to the passage from (5.14) to (5.17) we can use (5.23b) to convert (5.10) into a set of dynamical equations for $\mathcal{U}_{\beta,\bar{\beta}}(\hat{\beta})$:

$$\mathcal{U}_{\beta,\bar{\beta}}(\hat{\beta}) = \sum_{\gamma} \Lambda_{\gamma}^{\beta}(\hat{\beta}) I_{\bar{\beta}}^{\gamma}(\hat{\beta}) + \sum_{\lambda \in \hat{\beta}} C_{\lambda}^{\beta}(\hat{\beta}) \mathcal{O}_{\lambda}(\bar{\nu}_{\lambda}) \mathcal{U}_{\lambda,\bar{\beta}}(\hat{\beta}) . \tag{5.24}$$

We remark that in (5.10), (5.20), and (5.22) the label $\bar{\beta}$ can be replaced by an arbitrary partition $\lambda \in \hat{\beta}$.

It is clear that $\mathcal{U}_{\beta,\bar{\beta}}(\hat{\beta})$ is label transforming. Then the application of theorem 2 of Appendix A to (5.24) yields (5.17) with

$$\mathcal{U}_{opt}(\hat{\beta}) = \sum_{\beta \in \hat{\beta}} \mathcal{R}_{\beta} \mathcal{U}_{\beta,\bar{\beta}}(\hat{\beta}) . \tag{5.25}$$

VI. APPROXIMATIONS

In terms of the matrix elements (4.14) of the optical potential (5.17) becomes

$$\mathcal{U}_{\hat{\beta}}(\vec{k}' | \vec{k}) = B_{\hat{\beta}}(\vec{k}' | \vec{k}) + \int (d\vec{q}) C_{\hat{\beta}}(\vec{k}' | \vec{q}) \mathcal{U}_{\hat{\beta}}(\vec{q} | \vec{k}) , \tag{6.1}$$

where

$$B_{\hat{\beta}}(\vec{k}' | \vec{k}) = \langle \phi_{\bar{\beta}}(\vec{k}') | B(\hat{\beta}) | \phi_{\bar{\beta}}(\vec{k}) \rangle , \tag{6.2}$$

$$C_{\hat{\beta}}(\vec{k}' | \vec{k}) = \langle \phi_{\bar{\beta}}(\vec{k}') | C(\hat{\beta}) | \phi_{\bar{\beta}}(\vec{k}) \rangle . \tag{6.3}$$

Equation (6.1) is a relatively innocuous integral equation. After partial wave analysis it reduces to a one-dimensional equation or a finite set of such equations when the fragments possess spins. The principal problem is evidently the determination of the input (6.2) and (6.3).

$B_{\hat{\beta}}(\vec{k}' | \vec{k})$ and $C_{\hat{\beta}}(\vec{k}' | \vec{k})$ are defined in terms of $B_{\bar{\beta}}^{\alpha}(\hat{\beta})$, Eq. (D1), and $C_{\bar{\beta}}^{\beta}(\hat{\beta})$, Eq. (5.19), respectively. We call attention to the four alternative forms (5.11a), (D4), (D6), and (D8) which we have derived for $B_{\beta}^{\alpha}(\hat{\beta})$. Each of these forms generates corresponding alternative expressions for $C_{\bar{\beta}}^{\beta}(\hat{\beta})$ through (5.18). This diversity suggests that these options in expressing (6.2) and (6.3) may be useful in formulating approximations which emphasize different physical aspects of the scattering.

It is important to point out that passage among these forms is possible only by using the *exact* integral equations (5.8) for $\Lambda_{\lambda}^{\alpha}(\hat{\beta})$. These N-particle integral equations, however, will normally be approximated. Once this is done the various expressions for $B_{\hat{\beta}}(\vec{k}' | \vec{k})$ and $C_{\hat{\beta}}(\vec{k}' | \vec{q})$ are no longer equivalent and some may be better approximations than others.

The unfolding of the complete complexity of the full N-particle dynamics embodied in (6.1)–(6.3) is beyond the scope of the present investigation. We confine ourselves to a class of relatively simple approximations which, nevertheless, generalize extant methods for calculating the optical potential and yet are still within the range of practical calculation.

Our major approximation is to ignore all intermediate scatterings contributing to $B_{\bar{\beta}}^{\beta}(\hat{\beta})$ and to $C_{\bar{\beta}}^{\beta}(\hat{\beta})$ which represent more than two-particle clusterings. Thus in (5.8) we take

$$\Lambda_{\lambda}^{\alpha}(\hat{\beta}) \simeq \delta_{\alpha,\lambda} . \tag{6.4}$$

The form for $B_{\bar{\beta}}^{\beta}(\hat{\beta})$ defined by (D1) and (5.11a) is especially convenient for our purposes because of its multiple scattering structure. Then ignoring all but pair-clustered terms we have with (6.4),

$$B_{\bar{\beta}}^{\beta}(\hat{\beta}) \simeq \Big[\sum_{i'} W^{\beta,\bar{\beta}}(i') + V_{\bar{\beta}}^{\beta} \Big] + \bar{\delta}_{\beta,\bar{\beta}} G_{\bar{\beta}}^{-1} . \tag{6.5}$$

Now

$$W^{\beta,\bar{\beta}}(i') = \sum_{i''} \bar{\Delta}_{\beta,i''} t_{i''} \bar{\Delta}_{\bar{\beta},i'} , \tag{6.6}$$

where $t_{i'}$ is the two-particle transition operator (2.19).

It is important to note that the term in square brackets in (6.5) reduces to the correct Born approximation. That is, if

$$t_{i'} \simeq V_{i'} , \tag{6.7}$$

then since

$$\sum_{i''} \bar{\Delta}_{\beta,i''} V_{i''} \bar{\Delta}_{\bar{\beta},i'} + V_{\bar{\beta}}^{\beta} = V^{\beta} , \tag{6.8}$$

we have

$$\sum_{i'} W^{\beta,\bar{\beta}}(i') + V^{\beta}_{\bar{\beta}} \simeq V^{\beta}. \tag{6.9}$$

We remark that the other forms for $B^{\alpha}_{\beta}(\hat{\beta})$ that we have considered do not lend themselves to such a ready recovery of the Born limit.

In the approximation (6.5) the Born term of (6.1) becomes

$$B_{\hat{\beta}}(\vec{k}'|\vec{k}) = t_{\hat{\beta}}(\vec{k}'|\vec{k}) + [E_0 - E(\vec{k})]\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{k}), \tag{6.10}$$

where

$$t_{\hat{\beta}}(\vec{k}'|\vec{k}) \equiv \sum_{\beta\in\hat{\beta}} \left\langle \phi_{\bar{\beta}}(\vec{k}') \middle| \mathcal{R}_{\beta}\left[\sum_{i'} \bar{\Delta}_{\beta,i'} t_{i'} \bar{\Delta}_{\bar{\beta},i'} + V^{\beta}_{\bar{\beta}}\right] \middle| \phi_{\bar{\beta}}(\vec{k}) \right\rangle \tag{6.11}$$

and

$$\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{k}) \equiv \sum_{\beta\in\hat{\beta}} \langle \phi_{\bar{\beta}}(\vec{k}') | \mathcal{R}_{\beta} \bar{\delta}_{\beta,\bar{\beta}} | \phi_{\bar{\beta}}(\vec{k}) \rangle. \tag{6.12}$$

E_0 corresponds to the parametric energy appearing in the Green's functions G_{β} of the theory while $E(\vec{k})$ is the energy of the state $|\phi_{\bar{\beta}}(\vec{k})\rangle$ and it includes the binding energies of the fragments and the kinetic energy of relative motion.

Now (A12)

$$\mathcal{R}_{\beta} = R_{\bar{\beta}} \delta(P_{\beta,\bar{\beta}}) U(P_{\beta,\bar{\beta}})^{\dagger}, \tag{6.13}$$

and since we have assumed the nuclear wave functions entering into $|\phi_{\bar{\beta}}(\vec{k})\rangle$ are properly symmetrized we have (A19)

$$R_{\bar{\beta}} |\phi_{\bar{\beta}}(\vec{k})\rangle = |\phi_{\bar{\beta}}(\vec{k})\rangle. \tag{6.14}$$

Also

$$U(P_{\beta,\bar{\beta}}) |\phi_{\bar{\beta}}\rangle = |\phi_{\beta}\rangle. \tag{6.15}$$

Thus, e.g.,

$$\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{k}) = \sum_{\beta\in\hat{\beta}} \delta(P_{\beta,\bar{\beta}}) \langle \phi_{\beta}(\vec{k}') | \phi_{\bar{\beta}}(\vec{k}) \rangle. \tag{6.16}$$

The approximation

$$\mathcal{U}_{\hat{\beta}}(\vec{k}'|\vec{k}) \simeq B_{\hat{\beta}}(\vec{k}'|\vec{k}), \tag{6.17}$$

where the Born term is given by (6.10), includes the modifications to the impulse approximation, (6.11), which arise from particle identity plus the nonorthogonality terms (6.16). We remark that the calculation of the matrix elements of $t_{i'}$ which enter into (6.11) is unequivocal. $t_{i'}$ is uniquely specified solely in terms of two-nucleon information. Also the kinematics relating the total c.m. to the two-nucleon c.m. is uniquely defined—there is no angle transformation problem. In heuristic formulations[28] of the optical potential even without complete treatment of particle identity the recovery of a low-order approximation such as (6.17) in terms of a two-particle transition operator is accomplished only with some attendant ambiguity in the definition of the appropriate kinematics.

If in addition to (6.17) we suppose that (6.7) is valid we obtain an optical potential which is nearly the same as the one generated via the resonating group method (RGM)[29]:

$$\mathcal{U}_{\hat{\beta}}(\vec{k}'|\vec{k}) \simeq \tilde{V}^{\hat{\beta}}(\vec{k}'|\vec{k}) + [E_0 - E(\vec{k})]\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{k}), \tag{6.18}$$

where

$$\tilde{V}^{\hat{\beta}}(\vec{k}'|\vec{k}) = \sum_{\beta\in\hat{\beta}} \langle \phi_{\bar{\beta}}(\vec{k}') | \mathcal{R}_{\beta} V^{\beta} | \phi_{\bar{\beta}}(\vec{k}) \rangle. \tag{6.19}$$

If we use $V^{\beta} = H - H_{\beta}$ for $\beta \neq \bar{\beta}$ in (6.19) we can rewrite (6.18) in the form of the RGM optical potential.[29] One finds that the coefficient of $\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{k})$ then becomes $[E_0 - E(\vec{k}) - E(\vec{k}')]$ as compared to $-E_0$ for the RGM optical potential of Ref. 29. Since the two forms are identical on-shell the recovery of the RGM optical potential involves the additional assumption of the neglect of the off-shell dependence of the coefficient of $\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{k})$.

The simplest generalizations of the approximations (6.17) and (6.18) which retain the same level of calculational difficulty involve using the full integral equation (6.1) with a simplified kernel. If we use (6.4) and neglect all terms except those with two-particle clusterings we obtain from (5.22)

$$C^{\beta}_{\bar{\beta}}(\hat{\beta}) \simeq -\left(\bar{\delta}_{\beta,\bar{\beta}} + \sum_{i'} \bar{\Delta}_{\beta,i'} t_{i'} G_0\right). \tag{6.20}$$

Corresponding to (6.20) we find

$$C_{\hat{\beta}}(\vec{k}'|\vec{q}) \simeq -[\mathcal{R}_{\hat{\beta}}(\vec{k}'|\vec{q}) + \mathcal{K}_{\hat{\beta}}(\vec{k}'|\vec{q})], \tag{6.21}$$

where

$$\mathcal{K}_{\hat{\beta}}(\vec{k}'|\vec{q}) = \sum_{\beta\in\hat{\beta}} \left\langle \phi_{\bar{\beta}}(\vec{k}') \middle| \mathcal{R}_{\beta} \sum_{i'} \bar{\Delta}_{\beta,i'} t_{i'} G_0 \middle| \phi_{\bar{\beta}}(\vec{q}) \right\rangle. \tag{6.22}$$

Equation (6.21) with (6.22) is the counterpart of (6.17). Corresponding to (6.18) we have

$$\mathcal{K}_{\hat{\beta}}(\vec{k}'|\vec{q}) \simeq \sum_{\beta\in\hat{\beta}} \langle \phi_{\bar{\beta}}(\vec{k}') | \mathcal{R}_{\beta} V^{\beta} G_0 | \phi_{\bar{\beta}}(\vec{q}) \rangle. \tag{6.23}$$

When (6.23) is used in (6.21) we obtain from (6.1) a generalization of the resonating group optical potential which does not entail a substantial increase in the level of calculational difficulty for its determination.

The approximations we have considered in this section are expected to be valid at relatively high energies. Our neglect of all but the simplest subsystem clusterings corresponds to a low-density situation. The systematic formulation of physically reasonable low-energy approximations starting from our exact equations remains an open question. We intend to take it up elsewhere.

VII. SUMMARY

We have presented a consistent many-particle theory of the optical potential for nucleus-nucleus scattering which takes into account all the effects of the Pauli principle in a convenient manner. Central to our development are the PLMS equations of Ref. 4 coupled with the use of the AGS[13] transition operator in the definition of the optical potential. The symmetrical structure of the AGS off-shell extension facilitates the simultaneous removal of the elastic singularities in all of the two-cluster channels which are physically equivalent by virtue of particle identity. We obtain dynamical equations for the optical potential which can be approximated to generate potentials which still possess the correct reality properties below the inelastic threshold.

We consider some simple approximations which in lowest order yield the optical potential of the resonating group method.[29] We then find generalization of the resonating group approximation which involves no substantial increase in the level of computational difficulty from that which is ordinarily entailed.[29] Similar approximations which involve folding-type matrix elements of two-particle transition operators rather than the potentials (as in the resonating group formalism) are also proposed. These contain unambiguous statements of the impulse approximation including all identity effects.

The preceding approximations are appropriate to low-density situations and hence relatively high fragment kinetic energies. It is not entirely clear at this time how to construct physically reasonable low-energy approximations starting from our equations. We have derived several alternative forms of these dynamical equations which may be useful in this regard.

On the whole perhaps the most important aspect of the present development is that one can consider the possibility of systematic corrections to various low-order approximations. This is a feature notably absent from previous proposals for the calculation of the optical potential for nucleus-nucleus scattering.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. PHY78-26595. We also thank A. Picklesimer, W. Tobocman, and R. M. Thaler for conversations on the role of nucleon identity in nuclear reactions.

APPENDIX A: PERMUTATION SYMMETRY

We outline here the incorporation of permutation symmetries resulting from particle identity into the types of N-particle scattering equations of interest to us. This is based on the work of Bencze and Redish.[9]

We consider an N-particle system composed of k species of particles with N_i particles of the ith species so that

$$N=\sum_{i=1}^{k} N_i \ .$$

Because all N_i particles of the ith species are identical, the system possesses a symmetry group S which can be identified with the direct product $S_{N_1}\otimes\cdots\otimes S_{N_k}$ of the groups S_{N_i} of permutations on N_i objects. The elements of S are denoted as P and are called permutations, although it should be kept in mind that there is no $P\in S$ which represents the interchange of different species of particles.

A realization of S consists of the mappings of the set $\tilde{\mathfrak{A}}_0$ of all the partitions of N-particles onto itself:

$$P(a)=a', \ P\in S, \ a, \text{ with } a, a'\in\tilde{\mathfrak{A}}_0 \ . \tag{A1}$$

It is clear that:

Lemma 1.

$$n_{P(a)}=n_a \ , \tag{A2a}$$

$$\Delta_{a,b}=\Delta_{P(a),P(b)} \ , \tag{A2b}$$

$$\overline{\Delta}_{a,b}=\overline{\Delta}_{P(a),P(b)} \ , \tag{A2c}$$

for any $a, b\in\tilde{\mathfrak{A}}_0, P\in S$. Also:

Lemma 2. If $P(a)=a'$, then the partitions a and a' correspond to physically equivalent groupings of particles of the system. $\tilde{\mathfrak{A}}_0$ decomposes into disjoint equivalence classes $\hat{a}$ which consist of physically equivalent partitions:

$$\hat{a}=\{P(a)\,|\,\text{all } P\in S\}. \tag{A3}$$

(The equivalence classes $\hat{a},\hat{b},\ldots,$ are labeled in terms of some characteristic elements $a,b,\ldots,\in S$.)

The physical significance of the decomposition of $\tilde{\mathfrak{A}}_0$ into equivalence classes is that true physical observables are class functions. Thus, for example, those transition operators which correspond to the scattering of groups of identical particles are labeled by classes $\hat{a}$ rather than partitions.

The group symmetry S is realized on the N-particle Hilbert space by a group of unitary operators $U(P), P\in S$. The action of these symmetry transformations upon the types of operators which appear in multiparticle scattering theory is of prime concern to us. In particular we consider those sets of operators which are defined by means of partition labels. Such a set, e.g., $\mathcal{O}^{a,b,c,\cdots}$ is said to be *label transforming*[9] if for any $P\in S$

$$U(P)\mathcal{O}^{a,b,c,\cdots,}U(P)^{-1}=\mathcal{O}^{P(a),P(b),P(c),\cdots,}. \quad \text{(A4)}$$

Consider a generic partition-labeled scattering integral equation

$$T^{a,b}=I^{a,b}+\sum_c K^{a,c}T^{c,b}. \quad \text{(A5)}$$

Then it is easy to show the following[9]:

Theorem 1. If $I^{a,b}$ and $K^{a,c}$ are label transforming and if the solution of (A5) is unique, then these solutions $T^{a,b}$ are label transforming as well.

Next for any $a\in\tilde{\mathfrak{A}}_0$ we consider the following nonempty subgroups of S:

$$S_a\equiv\{P\,|\,P\in S,\ni P(a)=a\}, \quad \text{(A6)}$$

that is, S_a consists of all permutations which leave the partition a unchanged. Then[9]:

Lemma 3. The N_a left (or right) cosets of S_a for any $a\in\tilde{\mathfrak{A}}_0$ are all distinct and they exhaust S so that

$$N_{\hat{a}}=|S|/|S_a|, \quad \text{(A7)}$$

where, e.g., $|S|$ is the order of S.

Next we define the symmetrizers with respect to the subgroups S_a:

$$R_a\equiv\frac{1}{|S_a|}\sum_{P\in S_a}\overline{U}(P), \quad \text{(A8)}$$

where $\overline{U}(P)=\delta(P)\,U(P)$ and $\delta(P)$ is the parity of the permutation with respect to fermion exchange, i.e., $\delta(P)=+1$ (-1) if P contains an even (odd) number of fermion transpositions. Since $S_0=S_1=S$ the symmetrizer with respect to S is $R\equiv R_0=R_1$. It is obvious that for any label-transforming operator $\mathcal{O}^a$, $[\mathcal{O}^a,R_a]=0$. One can show[9] that for any $a\in\hat{a}$

$$R=\left(\frac{1}{N_{\hat{a}}}\right)\sum_{a'\in\hat{a}}R_a\overline{U}(P_{a',a})^{\dagger}=\left(\frac{1}{N_{\hat{a}}}\right)\sum_{a'\in\hat{a}}\overline{U}(P_{a',a})R_a, \quad \text{(A9)}$$

where $P_{a',a}\in S$ and $P_{a',a}(a)=a'$.

Now let us collect the major results of Ref. 9 in regard to the set of coupled integral equations (A5) where it is presumed that the conditions of theorem 1 obtain. To each equivalence class $\hat{a}$ we assign a definite (canonical) partition $\bar{a}\in\hat{a}$. Then we define the class-labeled transition operators

$$\tilde{T}^{\hat{a},\hat{b}}=\overline{N}_{\hat{a},\hat{b}}\sum_{a\in\hat{a}}\mathcal{R}_aT^{a,\bar{b}}, \quad \text{(A10)}$$

where

$$\overline{N}_{\hat{a},\hat{b}}=(N_{\hat{b}}/N_{\hat{a}})^{1/2} \quad \text{(A11)}$$

and

$$\mathcal{R}_a\equiv R_{\bar{a}}\overline{U}(P_{a,\bar{a}})^{\dagger}, \quad \text{(A12)}$$

with a similar definition for $\tilde{I}^{\hat{a},\hat{b}}$ in terms of $I^{a,\bar{b}}$. The class-labeled kernels are defined somewhat differently. Let

$$X^{\hat{a},c}\equiv\overline{N}_{\hat{a},\hat{b}}\sum_{a\in\hat{a}}\mathcal{R}_aK^{a,c}, \quad \text{(A13)}$$

where we note the parametric dependence upon $\hat{b}$ through $\overline{N}_{\hat{a},\hat{b}}$. It then follows from the preceding development that[9]

$$X^{\hat{a},c}\overline{U}(P_{c',c})^{\dagger}=X^{\hat{a},c'}, \quad \text{(A14)}$$

$$X^{\hat{a},c}R_c=X^{\hat{a},c}, \quad \text{(A15)}$$

and consequently:

Theorem 2.

$$\tilde{T}^{\hat{a},\hat{b}}=\tilde{I}^{\hat{a},\hat{b}}+\sum_{\hat{c}}\kappa^{\hat{a},\hat{c}}\tilde{T}^{\hat{c},\hat{b}}, \quad \text{(A16)}$$

where

$$\kappa^{\hat{a},\hat{c}}=\overline{N}_{\hat{b},\hat{c}}X^{\hat{a},\bar{c}}=\overline{N}_{\hat{a},\hat{c}}\sum_{a\in\hat{a}}\mathcal{R}_aK^{a,\bar{c}}. \quad \text{(A17)}$$

Also[9]:

Theorem 3. If, in an obvious matrix notation, K^m is a connected operator for some $m\geq 1$, then so is κ^m.

Theorem 4. If the operators $T^{a,b}$ are transition operators corresponding to the reaction $b\to a$ in the case of distinguishable particles and if S is a symmetry group of the N particle system, then the physical scattering amplitudes are the on-shell matrix elements

$$\langle\phi_{\bar{a}}(\nu_{\bar{a}})\,|\,\tilde{T}^{\hat{a},\hat{b}}\,|\,\phi_{\bar{b}}(\nu_{\bar{b}})\rangle. \quad \text{(A18)}$$

Here $\nu_{\bar{b}}$, e.g., represents the internal quantum numbers of the bound clusters of $\bar{b}$ and we have suppressed the dependence on the momenta for the c.m. motion of the clusters of $\bar{a}$ and $\bar{b}$. The asymptotic states are presumed to be properly symmetrized with regard to their internal structure, e.g.,

$$|\,\phi_{\bar{a}}(\nu_{\bar{a}})\rangle=R_{\bar{a}}\,|\,\phi_{\bar{a}}(\nu_{\bar{a}})\rangle. \quad \text{(A19)}$$

APPENDIX B: STRUCTURE OF $W^{a,b}(c)$

Let $[\mathcal{O}]_a$ denote the a-connected part of an operator $\mathcal{O}$. Then by definition $(c\neq 1)$

$$W^{a,b}(c)=[V^{a,b}+V^aGV^b]_c=[V^{a,b}_c+V^a_cG_cV^b_c]_c, \quad \text{(B1)}$$

where the second equality follows using the methods of Ref. 11 (cf. also Ref. 5). $t^{a,b}_f$ possesses a cluster decomposition

$$t^{a,b}_f=\sum_d[t^{a,b}_f]_d, \quad \text{(B2)}$$

where

$$[t_f^{a,b}]_d = [V_f^{a,b} + V_f^a G_f V_f^b]_d = \Delta_{f,d}[V_d^{a,b} + V_d^a G_d V_d^b]_d . \tag{B3}$$

The combination of (B1) and (B3) yields for $f \neq 1$

$$t_f^{a,b} = \sum_d{}' \Delta_{f,d} W^{a,b}(d) . \tag{B4}$$

We recover (3.5) from (B4) and the fact that Δ^{-1} also exists when the partition 1 is not allowed as one of the matrix indices.

We note that the only discontinuous part of $W^{a,b}(c)$ is $[V_c^a G_c V_c^b]_c$ and G_c has no discontinuities across the γ-elastic unitarity cut unless $c=\gamma$. The part of G_γ which is discontinuous across that unitarity cut is the γ-connected pole term g_γ. We therefore infer (5.2).

APPENDIX C: CONNECTED-KERNEL PROPERTY

If (5.8a) is iterated once the kernel of the resulting equation is

$$[K^2(\hat{\beta})]_\lambda^\alpha = \sum_\gamma K_\gamma^\alpha(\hat{\beta}) K_\lambda^\gamma(\hat{\beta}) . \tag{C1}$$

Let us examine the various combinations of operators appearing in the sum on the right side of (C1). We keep in mind that $W^{\gamma,0}(\lambda)$ is either λ connected or zero (if $\lambda=\gamma$) while g_λ is λ connected. Also $W^{\gamma,0}(\lambda)$ and V_β^γ contain interactions external to γ and therefore their products with γ-connected operators are connected (or zero). We see then that the products $W^{\alpha,0}(\gamma)G_0 W^{\gamma,0}(\lambda)G_0$, $V_\gamma^\alpha g_\gamma V_\lambda^\gamma g_\lambda$, $W^{\alpha,0}(\gamma)G_0 V_\lambda^\gamma g_\lambda$, and $V_\gamma^\alpha g_\gamma W^{\gamma,0}(\lambda)G_0$ are either connected or they vanish depending on the indices. The sum (C1) is then a nontrivial connected operator.

APPENDIX D: ALTERNATIVE BORN TERMS

We establish the validity of (5.11b). We also develop some alternative forms for the operators ($\beta \in \hat{\beta}$)

$$B_\beta^\alpha(\hat{\beta}) = \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) I_\beta^\lambda(\hat{\beta}) \tag{D1}$$

which appear as Born terms of the integral equations (5.10). We require the identity

$$W_{\rm MS}^{\lambda,\beta} = \sum_a{}' W^{\lambda,0}(a) G_0 G_\beta^{-1} + \sum_\gamma W^{\lambda,0}(\gamma) G_0 V_\beta^\gamma - V_\beta^\lambda , \tag{D2}$$

which is established in Ref. 4. Also from (5.8b) it follows that

$$V_\beta^\alpha = -\sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) \left[\sum_\gamma W^{\lambda,0}(\gamma) G_0 V_\beta^\gamma - V_\beta^\lambda - \sum_\gamma V_\gamma^\lambda g_\gamma \delta(\gamma \in \hat{\beta}) V_\beta^\gamma \right] . \tag{D3}$$

When (D2) and (D3) are combined with the expression (5.11a) for $B_\beta^\alpha(\hat{\beta})$ we find

$$B_\beta^\alpha(\hat{\beta}) = \bar{\delta}_{\alpha,\beta} G_\beta^{-1} + \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) \left[\sum_a{}' W^{\lambda,0}(a) G_0 - \sum_\gamma V_\gamma^\lambda g_\gamma \bar{\delta}_{\gamma,\beta} \delta(\gamma \in \hat{\beta}) \right] G_\beta^{-1} . \tag{D4}$$

Equation (D4) can be rewritten as

$$B_\beta^\alpha(\hat{\beta}) = \left(\bar{\delta}_{\alpha,\beta} + \sum_{\gamma,\lambda} \Lambda_\lambda^\alpha(\hat{\beta}) [W^{\lambda,0}(\gamma) G_0 - V_\gamma^\lambda g_\gamma] \bar{\delta}_{\gamma,\beta} \delta(\gamma \in \hat{\beta}) \right) G_\beta^{-1} + \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) \left[\sum_{a \in \hat{\beta}}{}' W^{\lambda,0}(a) G_0 + W^{\lambda,0}(\beta) G_0 \right] G_\beta^{-1} . \tag{D5}$$

Again using (5.8b) we can reduce the first term on the right side of (D5) to

$$\left[\sum_{\gamma \in \hat{\beta}} \Lambda_\gamma^\alpha(\hat{\beta}) \bar{\delta}_{\gamma,\beta} + \bar{\delta}_{\alpha,\beta} \delta(\alpha \notin \hat{\beta}) \right] G_\beta^{-1} .$$

Then

$$B_\beta^\alpha(\hat{\beta}) = \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) \left[\left(\sum_{a \in \hat{\beta}}{}' W^{\lambda,0}(a) + W^{\lambda,0}(\beta) \right) G_0 + \delta(\lambda \in \beta) \bar{\delta}_{\lambda,\beta} \right] G_\beta^{-1} + \bar{\delta}_{\alpha,\beta} \delta(\alpha \notin \hat{\beta}) G_\beta^{-1} , \tag{D6}$$

where

$$\delta(\alpha \notin \hat{\beta}) = 1 - \delta(\alpha \in \hat{\beta}) .$$

The form (5.11b) of $I_\beta^\lambda(\hat{\beta})$ is obvious from (D6) when $\alpha, \beta \in \hat{\beta}$.

Another expression for $I_\beta^\lambda(\hat{\beta})$ which is valid for all λ can be obtained by using (5.8b) again to rewrite (D5) as

$$B_\beta^\alpha(\hat{\beta}) = \sum_\lambda \Lambda_\lambda^\alpha(\hat{\beta}) \left(W_{\rm MS}^{\lambda,\beta} + \bar{\delta}_{\lambda,\beta} G_\beta^{-1} + V_\beta^\lambda - \sum_\gamma W^{\lambda,0}(\gamma) G_0 \bar{\delta}_{\gamma,\beta} [G_\beta^{-1} + V_\beta^\gamma] \right) . \tag{D7}$$

If we insert the expression (D2) into (D7) we find the particularly interesting expression

$$B^{\alpha}_{\beta}(\hat{\beta}) = \sum_{\lambda} \Lambda^{\alpha}_{\lambda}(\hat{\beta}) \left[\left(\sum_{a}{}' W^{\lambda,0}(a) \bar{\delta}_{n_a,2} + W^{\lambda,0}(\beta) \right) G_0 + \bar{\delta}_{\lambda,\beta} \right] G_{\beta}^{-1} , \tag{D8}$$

which also defines $I^{\lambda}_{\beta}(\hat{\beta})$ for all λ. The apparent disparity between (D6) and (D8) is easily resolved with the help of the identity

$$\sum_{\lambda} \Lambda^{\alpha}_{\lambda}(\hat{\beta}) \left[\bar{\delta}_{\lambda,\beta} \delta(\lambda \in \hat{\beta}) - \sum_{\gamma \in \hat{\beta}} W^{\lambda,0}(\gamma) G_0 \right] = \bar{\delta}_{\alpha,\beta} \delta(\alpha \in \hat{\beta}) , \tag{D9}$$

which follows from (5.8b).

---

[1]Reference 2 reviews the development of the optical potential until circa 1965. Progress since that time on the theory of the optical potential is typified in the work reported in Ref. 3. Several of the reviews contained in Ref. 3 include extensive references to the extant literature.

[2]A. L. Fetter and K. M. Watson, in *Advances in Theoretical Physics*, edited by K. A. Brueckner (Academic, New York, 1965), p. 113.

[3]*Microscopic Optical Potentials*, edited by H. V. v. Geramb, Lecture Notes in Physics, 89 (Springer, Berlin, 1979).

[4]K. L. Kowalski, Ann. Phys. (N.Y.) 120, 328 (1979).

[5]W. N. Polyzou, University of Maryland thesis, 1979 (unpublished).

[6]There are numerous investigations of the optical potential in nuclear physics using the technique of second quantization both for nuclear matter and finite nuclei. (See Refs. 2, 3, and the literature cited therein.) This approach is especially well suited to the description of nucleon scattering in infinite nuclear matter and we have nothing to add in this instance. The same degree of effectiveness of these methods does not survive the transition to finite nuclei unless they are coupled with the ideas of modern multiparticle scattering theory. (See Ref. 7.)

[7]V. V. Komarov and A. M. Popova, Phys. Lett. 28B, 476 (1969); Fiz. Elem. Chastits At. Yadra 5, 1075 (1974) [Sov. J. Part. Nucl. 5, 1075 (1974)]; Yad. Fiz. 10, 1089 (1969) [Sov. J. Nucl. Phys. 10, 621 (1970)]; J. L. Ballot and F. Becker, Phys. Rev. 164, 1285 (1967); R. D. Amado, *ibid.* 132, 485 (1963); A. C. Fonseca and P. F. Shanley, Phys. Rev. C 14, 1343 (1976); A. G. Sitenko, in *Few-Body Systems and Nuclear Forces I*, edited by H. Zingl, M. Haftel, and H. Zankel, Lecture Notes in Physics, 82 (Springer, Berlin, 1978), p. 387 and references therein.

[8]R. Goldflam and K. L. Kowalski, Phys. Rev. Lett. 44, 1044 (1980) contains a preliminary account of some aspects of the present work.

[9]G. Bencze and E. F. Redish, Nucl. Phys. A238, 240 (1975); J. Math. Phys. 19, 1909 (1978). See also G. Bencze, in *Few-Body Nuclear Physics* (IAEA, Vienna, 1978).

[10]The inclusion of many-particle forces involves no difficulties in principle but is technically more complex. Equations including general interactions which provide a starting point for the incorporation of identity effects are derived in Ref. 11. Reference 11 also contains a detailed description of contemporary techniques in many-particle scattering theory; some of this material can be taken as background for the present article.

[11]R. Goldflam and K. L. Kowalski, Phys. Rev. C 21, 483 (1980).

[12]K. L. Kowalski, Phys. Rev. C 16, 7 (1977).

[13]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. 2B, 167 (1967); P. Grassberger and W. Sandhas, *ibid.* 181 (1967).

[14]V. Vanzani, Nuovo Cimento 2A, 525 (1971); Lett. Nuovo Cimento 16, 1 (1976).

[15]For a review of this field see K. L. Kowalski, in *Few-Body Systems and Nuclear Forces*, edited by H. Zingl, M. Haftel, and H. Zankel, Lecture Notes in Physics, 87 (Springer, Berlin, 1978), Vol. II, p. 393.

[16]I. H. Sloan, Phys. Rev. C 16, 1945 (1972); G. Bencze, Nucl. Phys. A210, 568 (1973); E. F. Redish, *ibid.* A225, 16 (1974); A235, 82 (1974); Phys. Rev. C 10, 67 (1974).

[17]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); 19, 287 (1962).

[18]The validity of (3.5) is immediately clear from the work in Refs. 4 and 11 although this equation does not appear in quite this form in either reference. Therefore, we have presented a proof of (3.5) in Appendix B for the sake of completeness.

[19]G. E. Andrews, in *Encyclopedia of Mathematics and its Applications* (Addison-Wesley, Reading, Mass., 1976), Vol. 2, p. 57–59.

[20]Ref. 19, p. 59, Eq. (4.3.3).

[21]For an indication of what happens when this is not the case see K. L. Kowalski, Phys. Rev. 188, 2235 (1969).

[22]There are a number of obvious extensions of our development if this assumption is relaxed. For example, one could just as easily distinguish some excited configuration of the two fragments and consider the optical potential for the elastic scattering where the fragments are in the same excited state both initially and finally. This is of interest, for example, in constructing potentials for use in distorted-wave impulse approximation calculations of inelastic scattering. Also one can open up the idea of the optical potential by distinguishing some subset of the possible values of the internal quantum numbers $\nu_{\bar{\beta}}$. In this case Eqs. (4.18) below become effectively a set of coupled channel equations since $g_{\bar{\beta}}(+)$ is replaced by $[\Sigma_{\nu_{\bar{\beta}}} \mathcal{G}_{\bar{\beta}}(\nu_{\bar{\beta}})]$ $G_{\bar{\beta}}(+)$ where the sum is over some or all possible $\nu_{\bar{\beta}}$. The modifications in our development which are required to deal with these cases are relatively trivial

and therefore we do not consider them in detail here.

[23]We mean by the reality of $v_{\hat{\beta}}(\vec{k}' \mid \vec{k})$ the continuity of $v_{opt}(\hat{\beta})$ across all of the elastic unitarity cuts associated with the physically equivalent channels $\beta \in \hat{\beta}$.

[24]D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43, 248 (1966).

[25]For the sake of brevity, henceforth, we refer to these cuts as γ-*elastic cuts* and the discontinuities across them as γ-*elastic discontinuities*. Evidently there are additional cuts generated by the inelastic contributions to the Green's functions (cf. Ref. 22), but their structure is not relevant to our discussion.

[26]This definition is obviously ambiguous. For any relevant operator $\mathcal{O}$ we demand that $\mathfrak{D}[\mathcal{O}]_\gamma$ be of connectivity γ and that $\mathcal{O} - \mathfrak{D}[\mathcal{O}]_\gamma$ possesses no discontinuity across the γ-elastic cut. Thus, $\mathfrak{D}[\mathcal{O}]_\gamma$ is nonunique to within the addition of any γ-connected term which is continuous across the γ-elastic cut.

[27]E. F. Redish, Phys. Rev. C 10, 67 (1974).

[28]A. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) 8, 551 (1959).

[29]M. LeMere and Y. C. Tang, Phys. Rev. C 19, 391 (1979); Y. C. Tang, M. LeMere, and D. R. Thompson, Phys. Rep. 47, 167 (1978).