Configuration space Faddeev calculations. I. Triton ground state properties

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The formulation of Faddeev-type equations in configuration space is discussed. Numerical solutions are obtained using splines and the method of orthogonal collocation. Triton observables and wave-function probabilities are calculated for s-wave NN interaction models of Malfliet and Tjon and the tensor force model of Reid. Comparison with previously published triton results is made; our full five-channel results for the Reid soft-core potential are in excellent agreement with those obtained by Afnan and Birrell using separable expansion methods.

[NUCLEAR STRUCTURE ^{3}H, Faddeev calculations configuration space.]

I. INTRODUCTION

Prior to Faddeev's revolutionary papers[1] on the t-matrix approach to the three-body problem, most of the work on the bound state of the trinucleon system was variational in approach.[2] Mitra,[3] however, had realized that for separable potentials one could solve the Schrödinger equation directly for the triton bound state and n-d scattering length problem. One can, in fact, work directly with the Schrödinger equation for the triton bound-state calculations, employing local potentials as well. Part of the purpose of this paper is to emphasize this point and to remove some of the mystique often associated by the nonexpert with the term "Faddeev calculation" when applied to trinucleon bound-state studies.

We choose to work in configuration space rather than momentum space, because we wish to explore the wave function of the triton bound state in terms of spatial coordinates. These are the coordinates where we are familiar with the deuteron wave function and where our intuition is stronger. We hope to provide that same intuitive feeling for "exact" trinucleon wave functions. Prior to publishing such studies, we report here on our numerical methods and convergence and compare our results with those previously published. We examine triton bound-state properties for the local potential models of Malfliet and Tjon[4] (s-wave potentials) and of Reid[5] [the Reid soft-core (RSC) potential which includes a tensor force in the $^3S_1 - {}^3D_1$ channel]. We compare binding energies, wave function probabilities, and rms radii where available with the momentum space calculations of Malfliet and Tjon and of Kim and coworkers,[6,7] the configuration space calculations of Gignoux and Laverne,[8] and the separable expansion calculations of Afnan and Birrell.[9]

In Sec. II of this paper we discuss the three-body bound-state equations for the specific case of three pairwise interacting bosons in an attempt to make the problem transparent. (The details of the general spin-isospin equations including the tensor force are contained in an appendix.) We emphasize that our equations are equivalent to the usual t-matrix formulation of the Faddeev equations for the bound state. In Sec. III, we outline our numerical methods; in particular, we discuss the use of spline functions in the context of orthogonal collocation. In Secs. IV and V there appears a detailed account of our numerical results for the s-wave potential model and tensor force models, respectively. We summarize our results in Sec. VI.

II. THREE-BODY EQUATIONS IN CONFIGURATION SPACE

The "Faddeev equations" in configuration space were first derived by Noyes[10] for the three-body scattering problem where the Lippmann-Schwinger equation does not yield a unique solution. Subsequently, the equations were applied to numerical studies of the bound-state problem[11,8] as well as the scattering problem.[12] The angular momentum decomposition of the wave function in configuration

space required by these equations is identical to the angular momentum decomposition in momentum space. Since the details of the angular momentum reduction of the equations have been discussed in many papers,[8,11,12,13] we will not repeat the derivation here. Instead, we shall simply review the derivation of the three-body equations for identical bosons interacting via a local two-body potential starting from the Schrödinger equation in configuration space. Equations for particles with spin are given in the Appendix for the particular cases discussed in this paper.

We consider three equal mass, spinless particles with coordinates $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. For the independent center-of-mass coordinates we use the usual Jacobi variables

$$\vec{x}_i = \vec{r}_j - \vec{r}_k \tag{1a}$$

and

$$\vec{y}_i = \tfrac{1}{2}(\vec{r}_j + \vec{r}_k) - \vec{r}_i \,, \tag{1b}$$

where i, j, k imply cyclic permutation. The Hamiltonian for the three-particle system is assumed to be of the form

$$H = T + V(x_1) + V(x_2) + V(x_3) \,, \tag{2}$$

where T is the kinetic energy operator in the center-of-mass coordinates. The total wave function Ψ for the three-particle system can be expressed as the sum of three terms:

$$\Psi = \psi(\vec{x}_1, \vec{y}_1) + \psi(\vec{x}_2, \vec{y}_2) + \psi(\vec{x}_3, \vec{y}_3) \equiv \psi_1 + \psi_2 + \psi_3 \,.$$

(This is, in fact, the key to the Faddeev decomposition of the t-matrix equations.) The functions ψ_1, ψ_2, and ψ_3 all have the same functional form ψ since the particles are identical; the total wave function is the sum of three partial amplitudes which differ only in the permutation of the coordinates describing the three particles. The Schrödinger equation

$$(H - E)\Psi = 0 \tag{3}$$

can then be separated into the three coupled equations

$$[T + V(x_1) - E]\,\psi_1 = -V(x_1)[\psi_2 + \psi_3] \,, \tag{4a}$$

$$[T + V(x_2) - E]\,\psi_2 = -V(x_2)[\psi_1 + \psi_3] \,, \tag{4b}$$

$$[T + V(x_3) - E]\psi_3 = -V(x_3)[\psi_1 + \psi_2] \,. \tag{4c}$$

Adding these three equations gives back the original Schrödinger equation. Since the three particles are identical, the three Faddeev equations are identical in form and it is only necessary to solve one of them. Consequently, one needs to keep only the first equation and to bear in mind that the subscripts on the wave function components indicate which of the coordinate permutations should be used [see Eq.(7) below].

For the purposes of the present discussion, we consider the three bosons to be in relative s states of angular momentum. We define (for convenience and in analogy with the usual prescription for the corresponding two-body bound-state problem) the reduced wave function $\phi(x_1, y_1)$ for the $l = 0$ partial wave by

$$\psi(x_1, y_1) = \phi(x_1, y_1)/(x_1 y_1) \,. \tag{5}$$

Equation (4a) (for particles with mass m) can now be written in the form

$$\frac{1}{x_1 y_1}\left[\frac{\partial^2}{\partial x_1^{\,2}} + \frac{3}{4}\frac{\partial^2}{\partial y_1^{\,2}} - U(x_1) - K^2\right]\phi(x_1, y_1) = U(x_1)\left[\frac{\phi(x_2, y_2)}{x_2 y_2} + \frac{\phi(x_3, y_3)}{x_3 y_3}\right], \tag{6}$$

where $K^2 = -mE/\hbar^2$ and $U(x_1) = mV(x_1)/\hbar^2$. Using Eqs. (1a) and (1b) one can verify that the permuted variables are

$$x_2 = \tfrac{1}{2}(x_1^{\,2} - 4x_1 y_1 \mu + 4y_1^{\,2})^{1/2} \,, \tag{7a}$$

$$y_2 = \tfrac{1}{2}(\tfrac{9}{4}x_1^{\,2} + 3x_1 y_1 \mu + y_1^{\,2})^{1/2} \,, \tag{7b}$$

$$x_3 = \tfrac{1}{2}(x_1^{\,2} + 4x_1 y_1 \mu + 4y_1^{\,2})^{1/2} \,, \tag{7c}$$

$$y_3 = \tfrac{1}{2}(\tfrac{9}{4}x_1^{\,2} - 3x_1 y_1 \mu + y_1^{\,2})^{1/2} \,, \tag{7d}$$

where μ is the cosine of the angle between $\vec{x}_1$ and $\vec{y}_1$. Integrating both sides of Eq. (6) with respect to μ and dividing by 2 (i.e., projecting out the s-wave components of the equation) leads us to the integral-differential equation

$$\left[\frac{\partial^2}{\partial x_1^{\,2}} + \frac{3}{4}\frac{\partial^2}{\partial y_1^{\,2}} - U(x_1) - K^2\right]\phi(x_1, y_1) = \tfrac{1}{2}U(x_1)\int_{-1}^{1} d\mu\left[\frac{x_1 y_1}{x_2 y_2}\,\phi(x_2, y_2) + \frac{x_1 y_1}{x_3 y_3}\,\phi(x_3, y_3)\right] . \tag{8}$$

Replacing μ by $-\mu$ in the second term of the right-hand side of Eq. (8) and using the coordinate relations defined in Eqs. (7a)–(7d) permits one to reduce this to

$$\left[\frac{\partial^2}{\partial x_1^{\,2}} + \frac{3}{4}\frac{\partial^2}{\partial y_1^{\,2}} - U(x_1) - K^2\right]\phi(x_1, y_1) = U(x_1)\int_{-1}^{1} d\mu\,\frac{x_1 y_1}{x_2 y_2}\,\phi(x_2, y_2). \tag{9}$$

Following Noyes[10] we make the convenient polar coordinate change of variables given by

$$x_i = \rho\cos\theta_i \tag{10a}$$

and

$$y_i = \tfrac{1}{2}\sqrt{3}\,\rho\,\sin\theta_i\ , \tag{10b}$$

since the exchange integrals then couple only through the single variable θ (and not ρ). Now Eq. (9) can be written in the form

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta_1^{\,2}}-U(\rho\,\cos\theta_1)-K^2\right]\phi(\rho,\theta_1)$$

$$=\frac{4}{\sqrt{3}}\,U(\rho\,\cos\theta_1)\int_{\theta_-}^{\theta_+}\phi(\rho,\theta_2)\,d\theta_2\ , \tag{11}$$

where the domain of integration θ_2 is shown in Fig. 1 and where θ_2 is defined in terms of θ_1 by the relation $\cos 2\theta_2 = -\frac{1}{2}[\cos 2\theta_1 + \mu\sqrt{3}\,\sin 2\theta_1]$. Equation (11), with the boundary condition that ϕ goes to zero in the asymptotic region as well as along $x_1 = 0$ and $y_1 = 0$ ($\theta_1 = 0$, and $\theta_1 = \pi/2$), is the integral-differential eigenvalue equation for the reduced, bound-state wave function amplitude $\phi(x_1, y_1)$ with eigenvalue K^2. The corresponding equations for particles with spin and higher partial waves have the same form except for the angular factors. In addition, for particles interacting through more than one partial wave, one obtains equations for various angular momentum components, which couple through the exchange terms.

We remark that the difficulty[14,15] of "truncating" the three-body wave function will not be a problem in this paper; we will limit our consideration of two-body potentials to those which are restricted to interact in no more than the $^1S_0$, $^3S_1 - {}^3D_1$ partial waves since these NN channels provide the main contribution to the binding energy and this restriction permits us to compare with the solution of the same problem by other authors. However, we note that truncation is a nontrivial problem in the study of truly local potentials (e.g., the Coulomb interaction) which couples all partial waves.

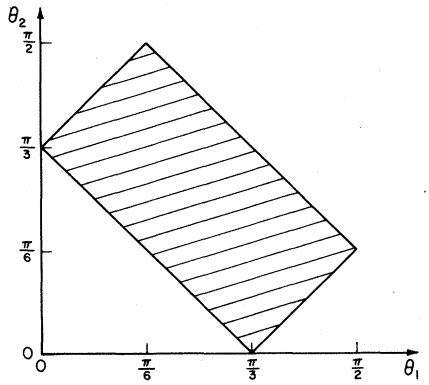

FIG. 1. The domain of integration in Eq. (11) of the text.

III. NUMERICAL METHODS

To solve Eq. (11) of the previous section, we use the technique of orthogonal collocation developed by deBoor and Swartz.[16] The wave function $\phi(\rho, \theta)$ is expressed as a bicubic spline on a rectangular grid in the ρ-θ coordinates

$$\phi(x_1, y_1) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\, s_m(\rho) s_n(\theta)\ . \tag{12}$$

Here we choose for $s_m(\rho)$ and $s_n(\theta)$ piecewise Hermite polynomials.[17] The method of collocation consists of finding the function $\phi(x_1, y_1)$ which satisfies Eq. (11) at M distinct values of ρ_i and N distinct values of θ_j. If one chooses these points to be the two-point Gaussian quadrature points for each partition, the technique is known as orthogonal collocation. The convergence properties of this method are given in Ref. 17. The requirement that the differential equation be satisfied at the collocation points gives a set of $M \times N$ linear equations for the coefficients a_{mn}. Thus, one must solve a matrix eigenvalue problem for the eigenvalue K^2 and the eigenvector a_{mn}.

There are several advantages to using the method of collocation. Splines possess excellent stability properties[17,18] when used to solve differential equations, and the solutions provide a continuous interpolation for all values of ρ and θ. Also, the knots for the splines do not have to be equally spaced; thus one can use more splines in the region where the potential is the strongest. Finally, the resulting matrix equation is a band matrix, which considerably reduces the computer storage requirements. For Eq. (11) one obtains a matrix with $M \times N$ rows and a bandwidth of $6M-1$. Since the matrix is a band matrix, only a portion of the matrix must be stored in the central memory at a given time. This further reduces the memory requirements of the program.

The boundary conditions on the wave function are that $\phi(\rho, \theta)$ is zero at $\theta = 0$ ($y_1 = 0$) and at $\theta = \pi/2$ ($x_1 = 0$), and that the wave function asymptotically goes to zero for large values of ρ. The boundary conditions at $\theta = 0$ and $\theta = \pi/2$ can be used to reduce the number of Hermite splines required for the θ variable, and thereby considerably reduce the size of the final matrix equation. The numerical calculations are facilitated by defining a smoother function $F(\rho, \theta)$ by

$$\phi(\rho, \theta) = F(\rho, \theta)\,\frac{e^{-K\rho}}{\rho^{1/2}}\ .$$

Equation (11) then has the form

$$\frac{\partial^2 F(\rho,\theta)}{\partial\rho^2}+\frac{F(\rho,\theta)}{4\rho^2}+\frac{1}{\rho^2}\frac{\partial^2 F(\rho,\theta)}{\partial\theta^2}-U(\rho\cos\theta)\left[F(\rho,\theta)+\frac{4}{\sqrt{3}}\int_{\theta^-}^{\theta^+}F(\rho,\theta')d\theta'\right]=K\left[\frac{2\partial F(\rho,\theta)}{\partial\rho}\right], \tag{13}$$

with the boundary conditions

$$F(\rho,\theta)=0 \text{ for } \rho=0\,, \tag{14a}$$

$$F(\rho,\theta)=0 \text{ for } \theta=0 \text{ and } \theta=\frac{\pi}{2}\,, \tag{14b}$$

$$\frac{\partial F(\rho,\theta)}{\partial\rho}=0 \text{ for } \rho=\rho_{\max}\,, \tag{14c}$$

where we now write

$$F(\rho,\theta)=\sum_{m=1}^{M}\sum_{n=1}^{N}b_{mn}s_m(\rho)s_n(\theta)\,. \tag{15}$$

The boundary conditions (14a) and (14c) can now be used to elimate two of the $s_m(\rho)$. We note that these boundary conditions are not exact.[19] Writing Eq. (13) at the collocation points (ρ_i, θ_j) gives a generalized eigenvalue matrix equation which can then be solved for the eigenvalue K and the eigenvector b_{mn}. The resulting matrix equation has the form

$$A\,b=KB\,b\,, \tag{16}$$

which can be solved by the well known power method.[20]

In the following discussion of numerical results, we have several parameters to vary: the number and distribution of ρ points, the number and distribution of θ points, and the value of ρ beyond which we define the wave function to be asymptotic. The ρ points are distributed according to a scaling algorithm: $(\rho_{n+1}-\rho_n)=(\rho_n-\rho_{n-1})*S_\rho$, where S_ρ is the scale factor. The θ points are distributed in three equally spaced intervals: A, B, C between 0 and $\pi/2$; here A corresponds to the region where the two-body potential $U(\rho\cos\theta)$ is large and therefore contains most of the θ points. A few runs were made in which the θ points were scaled in the same fashion as the ρ points, with a scaling parameter S_θ.

The calculation of probabilities and rms radii is done by means of a direct Simpson's rule integration of the square of the wave function. It is well known that a partial-wave expansion of the total wave function for such purposes converges very slowly. The radial integration limits for the accuracies quoted were 14 fm with steps of 0.1 fm. The θ integral was done with steps of approximately 0.07 rad. These triple integrals were quite fast compared to the time required to solve for the wave function using the analogs of Eq. (13). We refer to Afnan and Birrell[9] for a discussion of the convergence of the probability calculation in terms of partial-wave expansions.

Finally, we remark on a comparison of the use of the spline method discussed here with a more standard finite difference approach. In a single-channel, three-boson problem we found $N_\rho=14$ (28 spline collocation points in the ρ variable) gave an excellent three-body binding energy estimate. In contrast, 40 ρ points in our finite difference calculation left us 2% off in our binding energy estimate.

IV. RESULTS OF s-WAVE MODELS

We begin this study of the accuracy and convergence of our configuration space solution of the triton bound-state problem (see the Appendix for the explicit equations) by examining in detail the s-wave potential models of Malfliet and Tjon.[4] The two-body potentials are sums of Yukawas having a long-range attraction and (in most cases) a short-range repulsion (see Table I):

$$V(r)=V_R e^{-\mu_R r}/r-V_A e^{-\mu_A r}/r\,. \tag{17}$$

These potentials are defined to act only in the $l=0$ two-body partial waves; this implies that the "triton" wave function will have only a symmetric S state when a single average triplet and singlet potential is employed, and S and S' (mixed-symmetry) states when distinct triplet and singlet potentials are used. We have used $\hbar^2/m=41.47$ MeV · fm^2 throughout.

We consider first the average $MT\ V$ potential. Triton binding energies for various parameter values are shown in Table II. We point out the following: Convergence with respect to the number of ρ points is obtained by $N_\rho=12$. Reasonable results are found for the ρ scale factor S_ρ between 1.2 and 1.4; few points are needed outside of 5 fm, but many points are required near the origin. The results are somewhat more sensitive to the number and distribution of θ points in the

TABLE I. Potential parameters for the Malfliet-Tjon models.

Model	V_A (MeV fm)	μ_A (fm^{-1})	V_R (MeV fm)	μ_R (fm^{-1})	B_2 (MeV)
I	513.968	1.550	1438.720	3.110	
II	52.490	0.809	0		
III	626.885	1.550	1438.720	3.110	2.23
IV	65.120	0.663	0		2.23
V	570.316	1.550	1438.4812	3.110	0.35

TABLE II. Triton binding energies for the MT V model as a function of mesh parameters; the binding energy of Ref. 21 is 7.539 MeV for comparison.

N_ρ	N_θ	S_ρ	ρ_{max}	B_3
10	15(9,3,3)	1.3	20.0	7.5296
12	15(9,3,3)	1.3	20.0	7.5425
14	15(9,3,3)	1.3	20.0	7.5415
15	15(9,3,3)	1.3	20.0	7.5412
15	15(9,3,3)	1.3	24.0	7.5414
15	15(9,3,3)	1.2	20.0	7.5411
15	15(9,3,3)	1.4	20.0	7.5420
20	15(9,3,3)	1.3	20.0	7.5407
15	20(12,5,3)	1.3	20.0	7.5405
20	14(7,4,3)	1.3	20.0	7.5400
20	16(9,4,3)	1.3	20.0	7.5397
20	18(11,4,3)	1.3	20.0	7.5401
20	20(13,4,3)	1.3	20.0	7.5403
20	20(12,5,3)	1.3	20.0	7.5399
20	20(12,5,3)	1.3	24.0	7.5400
20	19(11,5,3)	1.3	20.0	7.5398
20	19(12,4,3)	1.3	20.0	7.5402
20	19(12,5,2)	1.3	20.0	7.5385
28	28(17,7,4)	1.3	24.0	7.5402
28	28(17,7,4)	1.3	30.0	7.5403

grid. As few as 14 θ points could be utilized to obtain a reasonable binding energy *if* at least 7 of these points were placed in the region A where the potential is large. (For the more repulsive RSC potential discussed in the next section, it was found that 7 was insufficient for the accuracy desired and that at least 9 were required.) In general, our results ($B_3 = 7.540 \pm 0.001$ MeV) compare favorably with the separable expansion result ($B_3 = 7.539$ MeV) of Ref. 21. We note that our binding energy differs slightly from the original 7.3 MeV result by Malfliet and Tjon.[4] It is necessary to multiply $V(r)$ in Ref. 4 by a factor to obtain the two-body binding energy of 0.35 MeV quoted in Table I of the reference[21]; note that the quoted potential strengths for Eq. (13) of Ref. 21 were erroneously divided by π.

We next consider the singlet and triplet potential models MT II–IV (no repulsion) and MT I–III (soft repulsion). For these models we quote representative results in Table III. The MT II–IV model is of interest only for completeness: The one-term Yukawa does not do a good job of reproducing either the $^1S_0$ or $^3S_1$ phase shifts (in contrast to one-term separable potential models) although parametrized to reproduce the correct scattering length and deuteron binding energy, respectively. We note that the binding energy obtained is slightly lower than the value of 12.1 MeV quoted by Malfliet and Tjon.[4] The rms charge radii (R) of the triton and ^{3}He (for point nucleons and without a Coulomb force) are small compared to experiment due to the overbinding. The value of $P_{S'}$ is somewhat smaller than might be expected.

The MT I–III results are much more interesting. The binding energy is slightly larger than the 8.3 MeV quoted by Malfliet and Tjon.[4] We found it necessary to multiply the potential quoted in Table I of Ref, 4 by 0.9866 to ensure a deuteron binding energy of 2.23 MeV. (We note that the binding energy appears to be more sensitive to the N_θ distribution than some of the wave function properties.) The model binding of $8.536 \pm .003$ MeV is very close to the experimental value but the rms radii are slightly large compared to the experimental values. Thus, such an s-wave model can come close to reproducing experimental wave function properties in addition to the binding energy, but it is not completely adequate. As one might anticipate, the value of $P_{S'}$ (= 1.95%) in this model is slightly larger than the corresponding quantity in a model which includes a tensor force. We close this section with the remark that the MT I–III appears to be the best local, s-wave potential model presently available for trinucleon studies.

V. RESULTS FOR THE RSC MODEL

The RSC potential model as defined by Reid[5] has many partial waves. We shall limit our consideration, as mentioned above, to the $^1S_0$ and $^3S_1 - {}^3D_1$ components since they provide the bulk of the triton binding and we can compare directly with previously published results. In the results quoted below, the "best estimates" do not necessarily correspond to any one set of parameters, but may correspond to a synthesis of several sets. In addition, the "errors" are clearly an estimate. Both are obviously subjective and may be slightly incorrect. The reader is free to play the same game with our numbers.

We shall consider first the three-channel approximation to the full five-channel calculation. In this approximation the tensor component of the wave function is retained only for the interacting pair; that is, the spectator nucleon is assumed to have zero angular momentum with respect to the c.m. of the interacting pair. The results of sample calculations in this approximation are shown in Table IV. We estimate our binding energy to be $B_3 = 6.380 \pm 0.005$ MeV. This result is essentially independent of the number of ρ points down to $N_\rho = 14$; it is sensitive to the distribution of θ points. A typical run with scaled θ points ($S_\theta = 1.35$) gave 6.379 MeV, $P_{S'} = 1.90\%$ and values of P_D and R identical to those in (18) and (19) below. We note that P_D appears to be so well determined

TABLE III. Triton results for the triplet-singlet models of Malfliet and Tjon for typical mesh parameters.

	MT II–IV			*MT I–III*			
N_ρ	16	20	16	16	20	20	24
S_ρ	1.3	1.3	1.3	1.3	1.3	1.3	1.3
N_θ	16(9,4,3)	20(13,5,2)	16(9,4,3)	16(9,4,3)	20(13,5,2)	20(12,5,3)	24(S_θ=1.15)
ρ_{max}	20.0	20.0	24.0	20.0	24.0	24.0	24.0
B_3 (MeV)	11.880	11.880	8.537	8.537	8.535	8.536	8.536
$R(^3\mathrm{H})$(fm)	1.29	1.29	1.62	1.62	1.62	1.62	1.62
$R(^3\mathrm{He})$(fm)	1.37	1.37	1.77	1.77	1.77	1.77	1.77
P_S(%)	99.32	99.32	98.05	98.05	98.05	98.05	98.05
$P_{S'}$(%)	0.68	0.68	1.95	1.95	1.95	1.95	1.95

that model parameter variations were unable to budge that quantity from 8.01%, although $P_{S'}$ was model sensitive. Our best estimates of the wave function probabilities (in percent) are

$$P_S = 90.10 \pm 0.02 ,$$
$$P_{S'} = 1.89 \pm 0.02 , \quad (18)$$
$$P_D = 8.01 \pm 0.01 .$$

The corresponding estimates of the ^{3}H and ^{3}He rms radii (without a Coulomb force and for point nucleons) for this underbound model are

$$R(^3\mathrm{H}) = 1.77 \pm 0.01 \text{ fm} ,$$
$$R(^3\mathrm{He}) = 1.99 \pm 0.01 \text{ fm} . \quad (19)$$

Our binding energy estimate agrees reasonably well with the previous results of 6.4 MeV by four different groups.[22,23] However, our $P_{S'}$ and P_D differ to some extent with the quoted values of (1.8%, 7.6%) and (2.0%, 7.8%) of Refs. 6 and 14, respectively. Our radii values differ only slightly from the values 1.76 and 1.99 fm of Ref. 14, but this implies a difference in our isoscalar radii as well as isovector radii. Based upon the results discussed below for the full five-channel calculation, we believe our results to be more accurate. Unfortunately, we do not have a separable expansion result to compare with here.

Sample results for the full five-channel calculation are shown in Table V. Unfortunately, the size limitation on our matrices did not permit us to explore as completely as before the dependence of the results upon the number of ρ and θ points. However, the results quoted are for parameter sets which proved adequate in the three-channel approximation just discussed. We list our best estimates of the binding energy, wave function probabilities, and rms radii in Table VI along with the corresponding quantities of Brandenburg, Kim, and Tubis,[7] of Laverne and Gignoux,[8] and of Afnan and Birrell[9] for comparison. We note that the induced P-state probabilities are largely meaningless as we have no p-wave potentials; however, we include them in order to give a total accounting of our probability. We note that our overall agreement appears to be best with the separable expansion work of Ref. 9. We essentially confirm the results of Laverne and Gignoux,[8] although we disagree slightly with their $P_{S'}$ value and our radii differ outside of the estimated error limits. In fact, the apparent agreement for $R(^3\mathrm{He})$ is fortuitous in that both the isoscalar and isovector radii differ. We refer to Afnan and Birrell[9] for a discussion of the P_D disagreement (and corresponding P_S difference) with Brandenburg *et al.*;

TABLE IV. Triton results for the RSC ($^1S_0$, $^3S_1 - ^3D_1$) potential model in the truncated three-channel approximation for typical mesh parameters. Probabilities are given in percent and radii in fm.

N_ρ	20	20	20	20	20	15	15
S_ρ	1.2	1.3	1.4	1.4	1.4	1.3	1.3
N_θ	20(12,5,3)	20(12,5,3)	20(12,5,3)	20(12,5,3)	20(13,4,3)	15(10,3,2)	15(11,3,1)
ρ_{max}	24.0	24.0	24.0	30.0	24.0	24.0	24.0
B_3	6.383	6.383	6.382	6.382	6.383	6.381	6.376
$R(^3\mathrm{H})$	1.77	1.77	1.77	1.77	1.77	1.77	1.77
$R(^3\mathrm{He})$	1.99	1.99	1.99	1.98	1.99	1.99	1.99
P_S	90.10	90.10	90.11	90.11	90.10	90.09	90.11
$P_{S'}$	1.89	1.89	1.88	1.88	1.89	1.90	1.89
P_D	8.01	8.01	8.01	8.01	8.01	8.01	8.01

TABLE V. Triton results for the RSC potential ($^1S_0$, $^3S_1$-$^3D_1$) in the full five-channel calculation for typical mesh parameters. Probabilities are given in percent and radii in fm.

N_ρ	14	14	14	14
S_ρ	1.3	1.3	1.3	1.3
N_θ	14(8,4,2)	14(9,3,2)	14(10,3,1)	14(10,3,1)
ρ_{max}	20.0	20.0	20.0	18.0
B_3	7.02	7.02	7.01	7.01
$R(^3H)$	1.71	1.70	1.70	1.70
$R(^3He)$	1.93	1.90	1.90	1.89
P_S	89.06	89.15	89.16	89.16
$P_{S'}$	1.75	1.66	1.66	1.66
P_{2P}	0.04	0.04	0.04	0.04
P_{4P}	0.04	0.04	0.04	0.04
P_D	9.11	9.11	9.10	9.10

they have omitted certain channels from the probability calculation. Our experience has shown us that P_D is one of the most firmly fixed predictions of the model, reasonably insensitive to any mesh parameter variations. Finally, a single scaled-θ run ($S_\theta = 1.35$) gave the values listed in Table VI.

VI. SUMMARY

We have shown that our computational procedure is sufficiently precise as to permit us to carry out our intended studies of the ground state trinucleon wave functions. Our results for the Malfliet-Tjon s-wave potential models agree with their original published numbers to better than 5%. We find agreement with the separable expansion estimate of the binding for an average singlet-triplet potential with replusion to about 1 keV out of 7.540 MeV. For the RSC potential ($^1S_0$, $^3S_1 - {}^3D_1$) in the truncated three-channel approximation, we find agreement with the binding energy estimates of previous local potential calculations to within 20 keV of our 6.380 MeV. However, we obtain slightly different wave function probabilities. For the full five-channel RSC calculation, we find a binding energy of 7.02 ± 0.02 MeV, which is in reasonable agreement with the three published estimates. Our overall results appear to agree best with the separable expansion work,[9] although we are in significant disagreement with only the P_D estimate of Ref. 7. Our $P_{S'}$ results are the least well converged and the most uncertain.

We note that our rms radii calculations bear the expected relation with the model binding energy: Overbinding produces radii that are too small and underbinding produces radii which are too large. We emphasize that obtaining the correct triton binding with two-body s-wave potentials that approximate reasonably the two-body scattering data does not ensure that the correct rms radius will result.

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APPENDIX: SPIN-ISOSPIN EQUATIONS FOR THE TRITON

We consider the case of local nucleon-nucleon interactions acting only in the $^1S_0$ and $^3S_1 - {}^3D_1$ states. For the partial-wave expansion of the wave function we use the J-j coupling scheme and write the wave function in the form

$$\psi(\vec{x}_1, \vec{y}_1) = \sum_{\alpha=1}^{5} \psi_\alpha(x_1, y_1) \times |[l_\alpha s_\alpha) j_\alpha, (L_\alpha S_\alpha) J_\alpha] JM; (t_\alpha T_\alpha) T\rangle , \tag{A1}$$

where

l_α = orbital angular momentum of particles 2 and 3,
s_α = spin angular momentum of particles 2 and 3,
j_α = total angular momentum of particles 2 and 3,
L_α = orbital angular momentum of particle 1 relative to the center-of-mass of particles 2 and 3,
S_α = spin of particle 1 ($S = \frac{1}{2}$),
J_α = total angular momentum of particle 1,

TABLE VI. Comparison of five-channel RSC triton results with those previously published. Probabilities are given in percent and radii in fm.

	Present calculations	Afnan and Birrell	Laverne and Gignoux	Brandenburg, Kim, and Tubis
B_3	7.02 ± 0.02	7.02	7.0	6.98
$R(^3He)$	1.90 ± 0.01	1.90	1.90	
$R(^3H)$	1.70 ± 0.01		1.65	
P_S	89.13 ± 0.04	89.1	89.2	90.2
$P_{S'}$	1.68 ± 0.04	1.7	1.8	1.7
P_P	0.08			
P_D	9.11 ± 0.01	9.1	9.0	8.1

TABLE VII. The 5 states in J-j coupling which compose the triton wave function when the N-N interaction is limited to $^1S_0$ and $^3S_1$-$^3D_1$.

α	$(l_\alpha, s_\alpha)j_\alpha$	$(L_\alpha, S_\alpha)J_\alpha$	$(t_\alpha, T_\alpha)T$
1	$(0,0)0$	$(0,\frac{1}{2})\frac{1}{2}$	$(1,\frac{1}{2})\frac{1}{2}$
2	$(0,1)1$	$(0,\frac{1}{2})\frac{1}{2}$	$(0,\frac{1}{2})\frac{1}{2}$
3	$(2,1)1$	$(0,\frac{1}{2})\frac{1}{2}$	$(0,\frac{1}{2})\frac{1}{2}$
4	$(0,1)1$	$(2,\frac{1}{2})\frac{3}{2}$	$(0,\frac{1}{2})\frac{1}{2}$
5	$(2,1)1$	$(2,\frac{1}{2})\frac{3}{2}$	$(0,\frac{1}{2})\frac{1}{2}$

J = total angular momentum of the triton,
t_α = total isospin of particles 2 and 3,
T_α = isospin of particle 1, and
T = total isospin of the triton.

For such an interaction effective *just* in the $^1S_0$ and $^3S_1 - ^3D_1$ states, only the five states listed in Table VII are needed. These states are the same as those discussed by Harper, Kim, and Tubis.[13] The "three channel" approximation referred to in Sec. V of the test corresponds to retaining only the first three components listed in the table. For a model in which there exists just $^1S_0$ and $^3S_1$ potentials, only the first two states listed in Table VII are retained.

For the states listed in Table VII, the resultant set of coupled Faddeev equations corresponding to Eq. (6) of Sec. I are

$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{3}{4}\frac{\partial^2}{\partial y_1^2}-\frac{l_\alpha(l_\alpha+1)}{x_1^2}-\frac{3}{4}\frac{L_\alpha(L_\alpha+1)}{x_1^2}-K^2\right]\phi_\alpha(x_1,y_1)-\sum_{\alpha'=1}^{5}v_{\alpha\alpha'}(x_1)\phi_{\alpha'}(x_1,y_1)$$

$$=\sum_{\alpha'=1}^{5}v_{\alpha\alpha'}(x_1)\sum_{\beta=1}^{5}\int_{-1}^{1}d\mu\,\frac{x_1y_1}{x_2y_2}\left(\frac{x_1}{x_2}\right)^{l_\beta}\left(\frac{y_1}{y_2}\right)^{L_\beta}K_{\alpha',\beta}(x_1,y_1,\mu)\phi_\beta(x_2,y_2), \quad \text{(A2)}$$

TABLE VIII. Kernels for triton Faddeev equations.

$$K_{1,1}=\tfrac{1}{4}$$

$$K_{1,2}=-\tfrac{3}{4}$$

$$K_{1,3}=K_{1,4}=0$$

$$K_{1,5}=\tfrac{3}{32}\left[-(\tfrac{3}{2}\xi^2-5+\tfrac{3}{2}\xi^{-2})+2\sqrt{3}\,(\xi^{-1}-\xi)P_1(\mu)-4P_2(\mu)\right]$$

$$K_{2,1}=-\tfrac{3}{4}$$

$$K_{2,2}=\tfrac{1}{4}$$

$$K_{2,3}=K_{2,4}=0$$

$$K_{2,5}=-\tfrac{1}{3}K_{1,5}$$

$$K_{3,1}=K_{3,2}=0$$

$$K_{3,3}=-\tfrac{1}{8}\left[1-2\sqrt{3}\,\xi P_1(\mu)+3\xi^2P_2(\mu)\right]$$

$$K_{3,4}=-\tfrac{1}{8}\left[3\xi^{-2}+2\sqrt{3}\,\xi^{-1}P_1(\mu)+P_2(\mu)\right]$$

$$K_{3,5}=\frac{1}{32\sqrt{2}}\left[(3\xi^{-2}-7)-2\sqrt{3}(2\xi^{-1}+\tfrac{7}{5}\xi)P_1(\mu)+(5+3\xi^2)P_2(\mu)+\frac{6\sqrt{3}}{5}\xi P_3(\mu)\right]$$

$$K_{4,1}=K_{4,2}=0$$

$$K_{4,3}=-\tfrac{1}{8}\left[3\xi^2-2\sqrt{3}\xi P_1(\mu)+P_2(\mu)\right]$$

$$K_{4,4}=-\tfrac{1}{8}\left[1+2\sqrt{3}\xi^{-1}P_1(\mu)+3\xi^{-2}P_2(\mu)\right]$$

$$K_{4,5}=\frac{1}{32\sqrt{2}}\left[(3\xi^2-7)+2\sqrt{3}(2\xi-\tfrac{7}{5}\xi^{-1})P_1(\mu)+(5+3)\xi^{-2})P_2(\mu)-\frac{6\sqrt{3}}{5}\xi^{-1}P_3(\mu)\right]$$

$$K_{5,1}=-\tfrac{3}{4}P_2(\mu)$$

$$K_{5,2}=\tfrac{1}{4}P_2(\mu)$$

$$K_{5,3}=\frac{-1}{8\sqrt{2}}\left[\frac{7\sqrt{3}}{5}\xi P_1(\mu)-(1+3\xi^2)P_2(\mu)+\frac{3\sqrt{3}}{5}\xi P_3(\mu)\right]$$

$$K_{5,4}=\frac{1}{8\sqrt{2}}\left[\frac{7\sqrt{3}}{5}\xi^{-1}P_1(\mu)+(1+3\xi^{-2})P_2(\mu)+\frac{3\sqrt{3}}{5}\xi^{-1}P_3(\mu)\right]$$

$$K_{5,5}=\tfrac{1}{8}\left[-\frac{17}{20}+\frac{3\sqrt{3}}{40}(\xi^{-1}-\xi)P_1(\mu)+\frac{31}{28}P_2(\mu)+\frac{3\sqrt{3}}{40}(\xi-\xi^{-1})P_3(\mu)-\frac{9}{35}P_4(\mu)\right]$$

$$\xi=\frac{2}{\sqrt{3}}\frac{y_1}{x_1}=\tan\theta_1$$

where the reduced wave function ϕ_α is defined by

$$\psi_\alpha(x_1, y_1) = \frac{\phi_\alpha(x_1, y_1)}{x_1 y_1} \quad . \tag{A3}$$

The explicit expressions for $K_{\alpha',\beta}$ are given in Table VIII. Again the "three-channel" approximation corresponds to dropping components 4 and 5 in Table VII and setting $K_{3\alpha}$, $K_{4\alpha}$, $K_{\alpha 4}$, $K_{\alpha 3}$ to zero in Table VIII. For the s-wave potential model (having only $^1S_0$ and $^3S_1$ potentials), only components 1 and 2 of the wave function are retained along with K_{11}, K_{12}, K_{22}, K_{21}.

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