Electromagnetic multipole moments of ground states of stable odd-mass nuclei in the *sd* shell

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Shell-model wave functions for $A = 17$–39 nuclei are used to calculate the one-body densities upon which are based the $M1$, $E2$, $M3$, $E4$, and $M5$ moments of stable ground states with $J^\pi = 3/2^+$ and $5/2^+$. Values of the moments are obtained by combining these densities with single-particle matrix elements calculated with both free space and renormalized expressions for the electromagnetic operators. These results extend previous calculations for the $M1$ and $E2$ moments. The theoretical values are analyzed in terms of deviations from the predictions of the pure-configuration shell model and compared with available experimental data. While several of these data are well described with the present theory, a few other experimental values differ significantly from the corresponding predicted values.

NUCLEAR STRUCTURE ^{17}O, ^{21}Ne, ^{23}Na, ^{25}Mg, ^{27}Al, ^{33}S, ^{35}Cl, ^{37}Cd, ^{39}K; calculations for the $M1$, $E2$, $M3$, $E4$, and $M5$ moments of ground states; complete $0d_{5/2}$-$1s_{1/2}$-$0d_{3/2}$ shell-model wave functions; Chung-Wildenthal Hamiltonians.

I. INTRODUCTION

Extensive data exist for the magnetic dipole and electric quadrupole moments of nuclear states. These data provide qualitative guides to the structure of the associated nuclear state, allow quantitative tests of theoretical wave functions via comparison of measured to predicted moment values, and ultimately, with accurate enough theoretical wave functions, can be analyzed to yield information about the effects of the finite nuclear medium upon the basic nucleonic operators. Measured values of the $M1$ and $E2$ moments of nuclei in the sd shell ($A = 17$–39) have been recently compared with the predictions of a comprehensive and mutually consistent set of wave functions calculated in the sd-shell-model space.[1-3] The results of these comparisons suggest that the significant effects of intra-sd-shell configuration mixing upon the observed moments can be accounted for accurately enough by the theoretical wave functions that meaningful information can be extracted about the renormalizations which are appropriate for this mass region and model space for the $M1$ and $E2$ operators.

In this paper we present predictions from these same model wave functions for $M3$, $E4$, and $M5$ moments. The possibilities for experimental measurements of these higher moments are much more restricted than for the dipole and quadrupole cases. The $M3$ moments of ^{35}Cl and ^{37}Cl have been measured by atomic beam techniques; the accuracy of these measurements are limited by theoretical uncertainties in the electronic wave functions. At present, elastic electron scattering seems to offer the best technique for measuring $M3$, $E4$, and $M5$ moments. We present here shell-model results for the ground states of stable sd-shell nuclei which have such higher moments, namely ground states with $J^\pi = \frac{3}{2}^+$ and $\frac{5}{2}^+$. Even though the higher moments form a less extensive field for study than do the conventional dipole and quadrupole cases, the added dimensions they bring to our perception of nuclear structure, particularly when all facets of the multipole structure can be incorporated into a unified theory, make experimental pursuit of this kind of information highly desirable.

The present paper is organized in an attempt to make clear the different components of theoretical predictions of moment values and their relative importance. We first note the formal relationship by which the shell-model one-body densities and the single-particle matrix elements of the electric and magnetic operators are combined to yield values of the moments. The roles of the single-particle radial wave functions and of operator renormalization in the evaluation of the single-particle matrix elements is then discussed. The results for dipole and quadrupole moments are reviewed to provide a basis for estimating the probable accuracy of the predictions for the higher moments. The effects of intra-sd-shell configuration mixing, which of course are the essence of the present predictions, are illustrated by comparing the moments predicted with either free space or renormalized single-particle matrix elements to the corresponding predictions of the extreme single-particle model.

In Sec. II the calculations of the shell-model transition densities and the single-particle matrix elements and radial integrals are described. Also in this section, the concept of effective single-particle operators is introduced and theoretical and empirical values for the renormalizations are discussed. Comparisons with experiment are discussed in Sec. III.

II. CALCULATION OF GROUND STATE MOMENTS

The information encoded into a shell-model wave function which pertains to the multipole moments of a state J^π, T is expressed in terms of the one-body densities D, the matrix elements of coupled one-nucleon creation and annihilation operators $(a_j^\dagger \times \tilde{a}_{j'})^{\Delta J, \Delta T}$:

$$D^{NJT}_{\Delta J,\Delta T}(j,j')=\frac{\langle\psi^{NJT}|||(a_j^\dagger\times\tilde{a}_{j'})^{\Delta J,\Delta T}|||\psi^{NJT}\rangle}{(2\Delta J+1)^{1/2}(2\Delta T+1)^{1/2}}\,. \quad (1)$$

The values of D for the states under consideration here are presented in Table I, as obtained from the wave functions of Ref. 1. The total isoscalar and isovector matrix elements, which combine to yield the theoretical values of the various observable moments, are constructed from a combination of these one-body densities and the single-particle matrix elements (SPME) of the appropriate operator O of rank $\Delta J, \Delta T$, summed over the N active nucleons, labeled by $k=1$ to N:

$$\left\langle JT\left|\left|\left|\sum_{k=1}^{N} O(\Delta J,\Delta T)_k\right|\right|\right|JT\right\rangle = \sum_{jj'} \mathrm{SPME}[O(\Delta J,\Delta T);j,j']\,D^{NJT}_{\Delta J,\Delta T}(j,j')\,, \quad (2)$$

where

$$\mathrm{SPME}[O(\Delta J,\Delta T);j,j'] = \langle nlj|||O(\Delta J,\Delta T)|||n'l'j'\rangle\,. \quad (3)$$

The one-body operator associated with the interaction of the electromagnetic field with the nucleus in the long wavelength limit ($q\to 0$) can be divided into electric and magnetic multipole operators[4]

TABLE I. One-body transition density values [the $D^{NJT}_{\Delta J,\Delta T}(j,j')$ of Eq. (1)] calculated for $\Delta J=0-J_{\max}$, for the ground states of stable sd-shell nuclei with $J^\pi=\frac{3}{2}^+$ and $\frac{5}{2}^+$ from the wave functions of Refs. 1–3.

	ΔT	j-j' $\frac{5}{2}$-$\frac{5}{2}$	j-j' $\frac{5}{2}$-$\frac{1}{2}$	j-j' $\frac{5}{2}$-$\frac{3}{2}$	j-j' $\frac{1}{2}$-$\frac{1}{2}$	j-j' $\frac{1}{2}$-$\frac{3}{2}$	j-j' $\frac{3}{2}$-$\frac{3}{2}$
^{17}O; $J^\pi=\frac{5}{2}^+$							
$\Delta J=0$–5	0	1.0	0.0	0.0	0.0	0.0	0.0
	1	1.0	0.0	0.0	0.0	0.0	0.0
^{21}Ne; $J^\pi=\frac{3}{2}^+$							
$\Delta J=0$	0	2.8290	0.0	0.0	1.3560	0.0	0.5762
	1	0.7914	0.0	0.0	−0.0731	0.0	0.0824
$\Delta J=1$	0	0.5045	0.0	−0.0766	0.0975	0.0144	0.0869
	1	0.3393	0.0	−0.1067	−0.0719	0.0165	0.0314
$\Delta J=2$	0	−0.4668	−0.5992	−0.1834	0.0	−0.2000	−0.0963
	1	0.0124	−0.0740	−0.0978	0.0	0.0334	0.0015
$\Delta J=3$	0	−0.6183	0.0827	0.0692	0.0	0.0	0.0390
	1	−0.5339	0.0446	0.1071	0.0	0.0	0.0598
^{23}Na; $J^\pi=\frac{3}{2}^+$							
$\Delta J=0$	0	4.2647	0.0	0.0	1.3418	0.0	0.8279
	1	0.6259	0.0	0.0	−0.0019	0.0	0.2349
$\Delta J=1$	0	0.4989	0.0	−0.0277	−0.0945	0.0321	0.0965
	1	−0.2811	0.0	0.0874	0.1002	−0.0790	−0.0213
$\Delta J=2$	0	−0.5307	−0.5789	−0.2759	0.0	−0.1751	−0.0384
	1	−0.1079	0.0930	−0.0697	0.0	−0.0508	0.0070
$\Delta J=3$	0	−0.5251	0.0388	0.0391	0.0	0.0	0.0591
	1	0.5172	−0.0547	−0.0749	0.0	0.0	−0.0826
^{25}Mg; $J^\pi=\frac{5}{2}^+$							
$\Delta J=0$	0	7.0552	0.0	0.0	1.5195	0.0	1.3078
	1	0.9589	0.0	0.0	0.0737	0.0	−0.0017

TABLE I. (*Continued.*)

	ΔT	j–j' $\frac{5}{2}$–$\frac{5}{2}$	j–j' $\frac{5}{2}$–$\frac{1}{2}$	j–j' $\frac{5}{2}$–$\frac{3}{2}$	j–j' $\frac{1}{2}$–$\frac{1}{2}$	j–j' $\frac{1}{2}$–$\frac{3}{2}$	j–j' $\frac{3}{2}$–$\frac{3}{2}$
$\Delta J=1$	0	0.9702	0.0	0.0923	0.0287	−0.0091	0.0467
	1	0.6946	0.0	0.0308	0.0217	0.0417	−0.0352
$\Delta J=2$	0	−0.9882	−0.6262	−0.5374	0.0	−0.2778	−0.1637
	1	0.3365	0.0205	−0.0034	0.0	−0.0065	−0.0006
$\Delta J=3$	0	0.4730	−0.0410	0.0324	0.0	0.0	−0.0045
	1	0.2229	−0.0073	−0.0132	0.0	0.0	−0.0116
$\Delta J=4$	0	−0.3336	0.0	−0.0032	0.0	0.0	0.0
	1	−0.0716	0.0	0.0504	0.0	0.0	0.0
$\Delta J=5$	0	0.6799	0.0	0.0	0.0	0.0	0.0
	1	0.6420	0.0	0.0	0.0	0.0	0.0
^{27}Al; $J^\pi=\frac{5}{2}^+$							
$\Delta J=0$	0	8.8251	0.0	0.0	1.8578	0.0	1.3537
	1	0.6738	0.0	0.0	0.2741	0.0	0.2060
$\Delta J=1$	0	0.9466	0.0	0.1400	0.0675	0.0267	0.0793
	1	−0.6770	0.0	−0.1325	−0.1631	−0.0981	0.0448
$\Delta J=2$	0	−0.6687	−0.4073	−0.3770	0.0	−0.2147	−0.1733
	1	0.6098	−0.0738	−0.0104	0.0	0.0796	0.0076
$\Delta J=3$	0	0.6729	−0.1054	0.0505	0.0	0.0	−0.0217
	1	−0.6325	0.1036	−0.0319	0.0	0.0	0.0566
$\Delta J=4$	0	−0.5838	0.0	−0.1027	0.0	0.0	0.0
	1	0.6034	0.0	−0.0736	0.0	0.0	0.0
$\Delta J=5$	0	0.7257	0.0	0.0	0.0	0.0	0.0
	1	−0.7214	0.0	0.0	0.0	0.0	0.0
^{33}S; $J^\pi=\frac{3}{2}^+$							
$\Delta J=0$	0	8.7790	0.0	0.0	4.7039	0.0	5.0782
	1	0.2247	0.0	0.0	0.1169	0.0	0.6420
$\Delta J=1$	0	0.0746	0.0	−0.0309	−0.0127	0.0050	0.8643
	1	0.0044	0.0	−0.1241	−0.0543	0.0580	0.6677
$\Delta J=2$	0	0.1816	0.2089	0.2536	0.0	0.2333	0.3790
	1	−0.0258	0.0072	−0.0707	0.0	−0.0589	0.5313
$\Delta J=3$	0	−0.0221	0.0045	0.0727	0.0	0.0	0.7826
	1	−0.0537	0.0172	0.0433	0.0	0.0	0.7174
^{35}Cl; $J^\pi=\frac{3}{2}^+$							
$\Delta J=0$	0	9.0662	0.0	0.0	4.9683	0.0	4.3833
	1	0.1988	0.0	0.0	0.0152	0.0	0.9649
$\Delta J=1$	0	0.0650	0.0	−0.0556	−0.0027	0.0208	0.8792
	1	−0.0058	0.0	0.0853	0.0228	−0.0118	−0.6781
$\Delta J=2$	0	0.1755	0.1868	0.1757	0.0	0.2190	0.6678
	1	0.0179	−0.0151	−0.0390	0.0	0.0274	0.1419
$\Delta J=3$	0	−0.0250	0.0071	0.0560	0.0	0.0	0.8231
	1	0.0366	−0.0092	−0.0631	0.0	0.0	−0.6991
^{37}Cl; $J^\pi=\frac{3}{2}^+$							
$\Delta J=0$	0	13.7263	0.0	0.0	7.9165	0.0	7.2894
	1	0.0970	0.0	0.0	0.0622	0.0	2.9995
$\Delta J=1$	0	0.0141	0.0	−0.0410	−0.0077	−0.0448	1.3903
	1	−0.0105	0.0	0.0305	0.0058	0.0334	−1.0363
$\Delta J=2$	0	0.0338	0.0348	0.0364	0.0	−0.0232	1.2406
	1	−0.0252	−0.0259	−0.0271	0.0	0.0173	−0.9247
$\Delta J=3$	0	−0.0053	0.0043	0.0349	0.0	0.0	1.3875
	1	0.0039	−0.0032	−0.0260	0.0	0.0	−1.0342
^{39}K; $J^\pi=\frac{3}{2}^+$							
$\Delta J=0$	0	9.7980	0.0	0.0	5.6568	0.0	7.00
	1	0.0	0.0	0.0	0.0	0.0	1.0
$\Delta J=1$	0	0.0	0.0	0.0	0.0	0.0	1.0
	1	0.0	0.0	0.0	0.0	0.0	−1.0
$\Delta J=2$	0	0.0	0.0	0.0	0.0	0.0	−1.0
	1	0.0	0.0	0.0	0.0	0.0	1.0
$\Delta J=3$	0	0.0	0.0	0.0	0.0	0.0	1.0
	1	0.0	0.0	0.0	0.0	0.0	−1.0

$$O(\mathrm{EL}\mu)_k = e_k r^L(k) Y_{L\mu}(\hat{r}(k)) \,, \tag{4}$$

$$O(\mathrm{ML}\mu)_k = \mu_N\left[g_k^s \vec{\mathbf{s}}(k) + \left(\frac{2g_k^l}{L+1}\right)\vec{\mathbf{l}}(k)\right]\vec{\nabla}(k) r^L(k) Y_{L\mu}[\hat{r}(k)]$$

$$= \mu_N[L(L+1)]^{1/2}\left[g_k^s r^{L-1}(k)[Y_{L-1}\times\vec{\mathbf{s}}(k)]_\mu^L + \left(\frac{2g_k^l}{L+1}\right) r^{L-1}(k)[Y_{L-1}\times\vec{\mathbf{l}}(k)]_\mu^L\right], \tag{5}$$

where in free space $e_p = e$, $e_n = 0$, $g_p^s = 5.585$, $g_n^s = -3.826$, $g_p^l = 1$, and $g_n^l = 0$. In terms of these operators the electric and magnetic multipole moments are defined by convention[5] as

$$Q_L = \left(\frac{4\pi}{2L+1}\right)^{1/2}\left\langle JM=J \left| \sum_{K=1}^{A} O(\mathrm{EL}\mu=0)_k \right| JM=J\right\rangle, \tag{6}$$

$$M_L = \left(\frac{4\pi}{2L+1}\right)^{1/2}\left\langle JM=J \left| \sum_{k=1}^{A} O(\mathrm{ML}\mu=0)_k \right| JM=J\right\rangle. \tag{7}$$

Other notations and definitions of the moments are related to these by the following:

$\mu = M_1$ (magnetic dipole moment) ,
$Q = 2Q_2$ (electric quadrupole moment) ,
Ω [magnetic octupole moment (Schwartz Ref. 6)] $= -M_3$,
Q_4' (Fuller, Ref. 7) $= 8Q_4$ (electric hexadecapole moment) ,
$\Gamma = M_5$ (magnetic triakontadupole moment) . (8)

Equations 6 and 7 can be written in terms of the triply reduced matrix elements as

$$\left\langle JM=JTT_Z \left| \sum_{k=1}^{A} O(L\mu=0)_k \right| JM=JTT_Z\right\rangle$$

$$= \begin{pmatrix} J & L & J \\ J & 0 & -J \end{pmatrix} \left\langle JTT_Z \left|\left| \sum_{k=1}^{A} O(L)_k \left[\frac{1+\tau_3}{2}\right] + \sum_{k=1}^{A} O(L)_k \left[\frac{1+\tau_3}{2}\right] \right|\right| JTT_Z \right\rangle$$

$$= \begin{pmatrix} J & L & J \\ J & 0 & -J \end{pmatrix} \sum_{\Delta T} (-1)^{T-T_Z} \begin{pmatrix} T & \Delta T & T \\ -T_Z & 0 & T_Z \end{pmatrix} \left\langle JT \left|\left|\left| \sum_{k=1}^{A} O(L,\Delta T)_k \right|\right|\right| JT \right\rangle$$

$$= \begin{pmatrix} J & L & J \\ J & 0 & -J \end{pmatrix} \left[\frac{1}{(2T+1)^{1/2}} \left\langle JT \left|\left|\left| \sum_{k=1}^{A} O(L,\Delta T=0)_k \right|\right|\right| JT \right\rangle \right.$$

$$\left. + \frac{T_Z}{[(2T+1)(T)(T+1)]^{1/2}} \left\langle JT \left|\left|\left| \sum_{k=1}^{A} O(L,\Delta T=1)_k \right|\right|\right| JT \right\rangle \right] \tag{9}$$

We have used the convention

$$\tau_3|\text{proton}\rangle = |\text{proton}\rangle \,,$$
$$\tau_3|\text{neutron}\rangle = -|\text{neutron}\rangle \,. \tag{10}$$

The operators $O(L,\Delta T)_k$ are given by

$$O(\mathrm{EL},\Delta T)_k = [1+\Delta T(\tau-1)]\, e_{\Delta T} r^L(k) Y_{L\mu}[\hat{r}(k)] \,, \tag{11}$$

$$O(\mathrm{ML},\Delta T)_k = [1+\Delta T(\tau-1)]\mu_N[L(2L+1)]^{1/2} \times\left[g_{\Delta T}^s r^{L-1}(k)[Y_{L-1}\times\vec{\mathbf{s}}(k)]_\mu^L + \left(\frac{2g_{\Delta T}^l}{L+1}\right) r^{L-1}(k)[Y_{L-1}\times\vec{\mathbf{l}}(k)]_\mu^L\right], \tag{12}$$

where

$$e_{\Delta T} = \frac{e_p + (-1)^{\Delta T} e_n}{2}$$

and

$$g_{\Delta T} = \frac{g_p + (-)^{\Delta T} g_n}{2} \,. \tag{13}$$

The reduced isospin single-particle matrix elements are $\langle t||1||t\rangle = \sqrt{2}$ and $\langle t||\tau||t\rangle = \sqrt{6}$. The reduced single-particle matrix elements of the spherical harmonics are given by Brussaard and Glaudemans (Ref. 4, Eqs. 10.50, 10.70, and 10.72).

In order to make clear the structure of the single-particle matrix elements of the EL and ML operators, they have been evaluated in Table II as products of three factors: the space-spin angular momentum algebra, the nucleonic charges and g factors, and the radial matrix elements. The numerical values for the triply reduced $\Delta T = 0$ and $\Delta T = 1$ matrix elements are evaluated in Table II by using free-space nucleon charges and g factors and harmonic-oscillator radial integrals with $b = 1.83$ fm. Numerical values for the single-par-

TABLE II. Composition of the single-particle matrix elements of the electromagnetic operators, for the single-nucleon orbits $nlj = 0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$. The matrix elements are expressed as products of the form $\langle nlj \,|||\, O(\text{EML}, \Delta T) \,|||\, n'l'j' \rangle = (2\Delta T+1)^{1/2} ABC$, where A are numerical constants, B are the nucleonic charges and moments, and C are the radial integrals $\langle r^L \rangle$ for EL and $\langle r^{L-1} \rangle$ for ML. Values for the complete matrix elements are listed as calculated for the assumptions $b = 1.83$ fm, $e_{\Delta T=0} = 0.5e$, $e_{\Delta T=1} = 0.5e$, $g^l_{\Delta T=0} = 0.5$, $g^l_{\Delta T=1} = 0.5$, $g^s_{\Delta T=0} = 0.880$, and $g^s_{\Delta T=1} = 4.706$. The EL matrix elements are in units of $e\text{fm}^L$ and the ML matrix elements are in units of $\mu_N \text{fm}^{L-1}$.

			j–j'					
			$\frac{5}{2}$–$\frac{5}{2}$	$\frac{5}{2}$–$\frac{1}{2}$	$\frac{5}{2}$–$\frac{3}{2}$	$\frac{1}{2}$–$\frac{1}{2}$	$\frac{1}{2}$–$\frac{3}{2}$	$\frac{3}{2}$–$\frac{3}{2}$
$M1$		A	1.002		1.070	0.846		0.536
		B	$g^s_{\Delta T} + 4g^l_{\Delta T}$		$-g^s_{\Delta T} + g^l_{\Delta T}$	$g^s_{\Delta T}$		$-g^s_{\Delta T} + 6g^l_{\Delta T}$
		C	1		1	1		1
	m.e.	$(\Delta T=0)$	2.88		−0.41	0.74		1.14
	m.e.	$(\Delta T=1)$	11.63		−7.80	6.90		−1.58
	m.e.	(p)	6.78		−3.47	3.34		0.16
	m.e.	(n)	−2.71		2.89	−2.29		1.45
$E2$		A	−1.044	0.978	−0.522		0.776	−0.798
		B	$e_{\Delta T}$	$e_{\Delta T}$	$e_{\Delta T}$		$e_{\Delta T}$	$e_{\Delta T}$
		C	$3.5b^2$	$-3.162b^2$	$3.5b^2$		$-3.162b^2$	$3.5b^2$
	m.e.	$(\Delta T=0)$	−6.12	−5.17	−3.06		−4.22	−4.67
	m.e.	$(\Delta T=1)$	−10.59	−8.95	−5.30		−7.31	−8.09
	m.e.	(p)	−8.65	−7.31	−4.33		−5.97	−6.60
	m.e.	(n)	0	0	0		0	0
$M3$		A	3.212	3.420	0.874			0.536
		B	$-g^s_{\Delta T} - g^l_{\Delta T}$	$g^s_{\Delta T}$	$2g^s_{\Delta T} - 3^l_{\Delta T}$			$g^s_{\Delta T} - 4^l_{\Delta T}$
		C	$3.5b^2$	$-3.162b^2$	$3.5b^2$			$3.5b^2$
	m.e.	$(\Delta T=0)$	−51.93	−31.85	2.65			−7.03
	m.e.	$(\Delta T=1)$	−339.4	−295.2	140.4			29.4
	m.e.	(p)	−175.3	−143.0	59.2			7.0
	m.e.	(n)	101.8	98.0	−55.4			−17.0
$E4$		A	0.904		1.280			
		B	$e_{\Delta T}$		$e_{\Delta T}$			
		C	$15.75b^4$		$15.75b^4$			
	m.e.	$(\Delta T=0)$	79.9		113.0			
	m.e.	$(\Delta T=1)$	138.9		195.7			
	m.e.	(p)	113.0		159.8			
	m.e.	(n)	0		0			
$M5$		A	8.294					
		B	$g^s_{\Delta T}$					
		C	$15.75b^4$					
	m.e.	$(\Delta T=0)$	1 288					
	m.e.	$(\Delta T=1)$	11 938					
	m.e.	(p)	5 784					
	m.e.	(n)	−3 963					

ticle proton and neutron reduced matrix elements are also given in Table II as obtained from the relation

$$\langle j \,||O||\, j' \rangle_{p/n} = \frac{1}{\sqrt{2}} \langle j \,|||\, O(\Delta T = 0) \,|||\, j' \rangle +/- \frac{1}{\sqrt{6}} \langle j \,|||\, O(\Delta T = 1) \,|||\, j' \rangle \,. \quad (14)$$

In the following sections we will discuss the radial integrals and the introduction of effective charges and g factors.

A. Radial integrals

We have chosen to evaluate the radial integrals independently for each nucleus we consider by choosing harmonic-oscillator wave functions parametrized to reproduce the individual measured values of the rms charge radii. The rms charge radii r_{ch} for essentially all stable sd-shell nuclei are now known to high accuracy, a rather recent development. These values are listed in Table III.

For the harmonic-oscillator potential $V(r) = \frac{1}{2} m\omega^2 r^2$, the point proton rms radius for nuclei in the sd shell is given by[8]

$$r_p^{\,2} = \left[\frac{18 + (Z-8)\frac{7}{2}}{Z} \right] b^2 - \frac{3b^2}{2A} \,, \quad (15a)$$

where

$$b^2 = \hbar / m\omega \,. \quad (15b)$$

TABLE III. Experimentally determined rms charge radii of stable *sd*-shell-nuclei, r_{ch}, and the extracted harmonic-oscillator length parameter b.

Nucleus	J^π	r_{ch} (fm)	b (fm)	Exp. Ref.	Notes
^{16}O	0^+	2.720(4)	1.769	a	
^{17}O	$\frac{5}{2}^+$	2.712(5)	1.763	a	
^{18}O	0^+	2.794(3)	1.821	a	
^{19}F	$\frac{1}{2}^+$	2.898(10)	1.833	b	
^{20}Ne	0^+	3.020(20)	1.869	c	i
^{21}Ne	$\frac{3}{2}^+$	(2.984)	1.845		j
^{22}Ne	0^+	2.949(21)	1.822	d	k
^{23}Na	$\frac{3}{2}^+$	2.986(9)	1.810	b	
^{24}Mg	0^+	3.035(18)	1.813	e	
^{25}Mg	$\frac{5}{2}^+$	3.003(11)	1.793	f	
^{26}Mg	0^+	3.017(32)	1.802	c	l
^{27}Al	$\frac{5}{2}^+$	3.058(5)	1.804	b	
^{28}Si	0^+	3.125(3)	1.827	b	m
^{29}Si	$\frac{1}{2}^+$	3.122(15)	1.825	g	n
^{30}Si	0^+	3.137(15)	1.835	g	n
^{31}P	$\frac{1}{2}^+$	3.187(3)	1.848	b	
^{32}S	0^+	3.263(2)	1.881	b	
^{33}S	$\frac{3}{2}^+$	(3.264)	1.881		j
^{34}S	0^+	(3.264)	1.881		j
^{35}Cl	$\frac{3}{2}^+$	3.351(16)	1.921	g	o
^{36}Ar	0^+	3.399(5)	1.938	g	o
^{37}Cl	$\frac{3}{2}^+$	3.351(17)	1.921	h	p
^{38}Ar	0^+	3.414(10)	1.948	g	o
^{39}K	$\frac{3}{2}^+$	3.437(2)	1.950	b	
^{40}Ca	0^+	3.474(3)	1.963	e	

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[g]R. Engfer, H. Schneuwly, J. L. Vuilleumier, H. K. Walter, and A. Zehnder, At. Data Nucl. Data Tables 14, 509 (1974).

TABLE III. (*Continued.*)

[h]W. J. Briscoe and H. Crannell, in *Proceedings of the International Conference on Nuclear Physics with Electromagnetic Interactions, Mainz, 1979* (Springer, Berlin, 1979), p. 2.6.

[i]Table I in footnote b.

[j]Interpolated values for r_{ch}.

[k]$r_{ch}(^{22}\mathrm{Ne})-r_{ch}(^{20}\mathrm{Ne})=-0.071$ fm from footnote d.

[l]Table II in footnote b.

[m]For natural Si $r_{ch}=3.129(3)$.

[n]Table IV in footnote g.

[o]Table III in footnote g.

[p]$r_{ch}(^{37}\mathrm{Cl})-r_{ch}(^{35}\mathrm{Cl})=0.000(23)$ from footnote h.

The last term in Eq. (15a) is the correction for center of mass motion.[9] The charge radius is obtained by folding r_p with the rms charge radii for the protons and neutrons and adding relativistic corrections. Ignoring the relativistic spin-orbit correction, which is important only for nuclei with a large neutron or proton excess of spin unsaturated nucleons, the rms charge radii are given by[9]

$$r_{ch}^2=r_p^2+r_{\mathrm{proton}}^2+\frac{N}{Z}r_{\mathrm{neutron}}^2+\frac{3}{4}\left(\frac{\hbar}{mc}\right)^2, \quad (16)$$

where $r_{\mathrm{proton}}^2=(0.86)^2$ fm^2 and $r_{\mathrm{neutron}}^2=-(0.34)^2$ fm^2. The values of b extracted from the experimental charge radii are given in Table III. In terms of b, the radial matrix elements are expressed as

$$\begin{aligned}\langle d|r^2|d\rangle&=3.5b^2,\\ \langle d|r^2|s\rangle&=-\sqrt{10}\,b^2,\\ \langle d|r^4|d\rangle&=15.75b^4.\end{aligned} \quad (17)$$

The procedure we have adopted obviously takes into account all of the known variations in the sizes of *sd*-shell ground states. The conventional prescriptions for either harmonic-oscillator or Saxon-Woods potentials, which assume some smoothly varying mass dependence, e.g., $\hbar\omega=41A^{-1/3}$, fail to account for these variations, sometimes by significant amounts. In our opinion it is appropriate to remove this source of noise in the comparison of theoretical to experimental matrix elements by abandoning such size formulas altogether and using instead the individual measured radii.

We utilize harmonic-oscillator radial dependence rather than Saxon-Woods or Hartree-Fock prescriptions because these latter, supposedly more "realistic," prescriptions do not in fact offer improvements over the harmonic oscillator sufficient to justify the additional complexity and ambiguity their use introduces into the problems at hand. The problem inherent in specifying single-particle wave functions for open-shell nuclei such as concern us here is that the experimental separation energies are inconsistent with the energies obtained from the shell-model single-particle potential; e.g., for ^{28}Si the observed $1s_{1/2}$ separation energy is greater than that for $0d_{5/2}$ and $0d_{3/2}$ greater than $1s_{1/2}$, in inverse order to the shell-model sequence. In the full shell-model calculation this effect comes out of the two-body part of the Hamiltonian, but simple single-particle models are helpless to deal with it. Moreover, by going from the infinite harmonic-oscillator well to finite wells one runs the risk of obtaining more realistic surface behavior in some wave functions at the cost of introducing more serious errors in others.

In our judgement it is advisable to retain the advantages of harmonic-oscillator dependence, where possible, until rigorous treatments of the single-particle problem—treatments which will presumably follow along the paths explored in the treatment of single-nucleon transfer reactions[10]—become available.

B. Effective charges and g factors

For the calculation of the one-body transition densities it has been assumed that the core is inert and only the motion of the nucleons in the sd shell-model space need be considered. However, it can be shown with perturbation theory that the effects of virtual excitations of nucleons from the $0s$ and $1p$ core orbits into higher orbits and also from the $1s0d$ orbits into higher orbits are important.[11] Microscopic calculations for these effects can be carried out if an effective two-body interaction between the core and valence nucleons is assumed. The results of such microscopic calculations lead in general to the introduction of "effective" single-particle electromagnetic matrix elements, as well as effective two-body electromagnetic matrix elements, which are different from the "free-space" values. The dominant part of these effects might be taken into account by introducing renormalizations of the free-space values of the single-particle matrix elements of the various operators. On a slightly less precise level the renormalization of the single-particle matrix elements might be approximated by introducing j-independent effective charges $\tilde{e}_p$ and $\tilde{e}_n$, and effective g factors $\tilde{g}_p^s$, $\tilde{g}_n^s$, $\tilde{g}_p^l$, and $\tilde{g}_n^l$. The possible uses of effective operators in shell-model calculations has been investigated empirically for $M1$ and $E2$ observables in sd-shell nuclei.[2,3]

The experimental magnetic moments are fairly well described by the sd-shell-model predictions using the free-space g factors. However, the agreement can be improved by using empirically determined (via least-squares fits) effective values for the four g factors, and can be improved even further by using such effective values for the eight individual single-particle matrix elements.[2] The results for the magnetic moments of the nuclei considered in this paper (a small subset of the those considered in Ref. 2) are given in Table IV.

The $E2$ observables [Q moments and $B(E2)$ values] are remarkably well described by using empirical, mass-independent values of the effective charges.[3] Except for the pure neutron or proton configurations, the sd-shell $E2$ matrix elements for the low lying states are very insensitive to the isovector matrix element. Hence only the isoscalar effective charge can be accurately determined empirically, the result being $\tilde{e}_p+\tilde{e}_n=1.7e$. The isovector effective charge in the region $A=20-36$ is consistent with the range $\tilde{e}_p-\tilde{e}_n=1.0e$ to $0.9e$ (Ref. 12). Results from two sets of effective charges will be presented: $\tilde{e}_p=1.35e$ and $\tilde{e}_n=0.35e$, and $\tilde{e}_p=1.3e$ and $\tilde{e}_n=0.4e$. The calculated and experimental Q moments are compared in Table V.

Experimental information on the $M3$, $E4$, and $M5$ observables is scarce, and hence relatively little is known about the empirical effective operators in these cases. With a zero-range interaction between the core and valence particles it can be shown from first-order perturbation theory that the effective $M3$ spin g factors are related to the effective $E2$ charges by the relations[13]

$$\delta_0^s(L=3)=-2\delta_1(L=2)-\tfrac{2}{3}\delta_0(L=2)$$

and

$$\delta_1^s(L=3)=-\tfrac{1}{6}\delta_0(L=2)\,, \tag{18}$$

where the δ are defined by

$$\begin{aligned}\tilde{e}_0(L=2)&=e_0(L=2)[1+\delta_0(L=2)]\,,\\ \tilde{e}_1(L=2)&=e_1(L=2)[1+\delta_1(L=2)]\,,\\ \tilde{g}_0^s(L=3)&=g_0^s(L=3)[1+\delta_0^s(L=3)]\,,\\ \tilde{g}_1^s(L=3)&=g_1^s(L=3)[1+\delta_1^s(L=3)]\,.\end{aligned} \tag{19}$$

The orbital g factors are not renormalized by a zero-range interaction. Recently the $M3$ gamma decays strengths in ^{24}Al, ^{24}Na, and ^{34}Cl have been compared with the sd-shell-model calculations using free space and effective g factors.[13] The empirical quenching of the spin g factors for these cases are $\delta_1^s=-0.13\pm0.05$ and $\delta_0^s=-0.30\pm0.05$. Using $E2$ effective charges of $\tilde{e}_p=1.35e$ and $\tilde{e}_n=0.35e$, the relations (18) give $\delta_1^s=-0.12$ and $\delta_0^s=-0.46$, whereas $E2$ effective charges of $\tilde{e}_p=1.3e$ and $\tilde{e}_n=0.4e$ give $\delta_1^s=-0.12$ and $\delta_0^s=-0.27$, in better agreement with the empirical values. The $M3$ moments will be calculated with both sets of

TABLE IV. Magnetic dipole moments μ.

	ψ(e.s.p.)[a]	Theory μ(e.s.p.)[b] (μ_N)	$\mu(sd)$[c] (μ_N)	$\mu(sd)$[d] (μ_N)	Experiment $\mu(q=0)$[e] (μ_N)
^{17}O $\frac{5}{2}^+$	$\nu d_{5/2}$	−1.91	−1.91	−1.88	−1.893
^{21}Ne $\frac{3}{2}^+$	$\nu d_{3/2}$	1.15	−0.77	−0.66	−0.662
^{23}Na $\frac{3}{2}^+$	$(\pi d_{3/2})^{-1}$	0.126	2.10	2.05	2.218
^{25}Mg $\frac{5}{2}^+$	$\nu d_{5/2}$	−1.91	−0.85	−0.84	−0.855
^{27}Al $\frac{5}{2}^+$	$(\pi d_{5/2})^{-1}$	4.79	3.39	3.50	3.642
^{33}S $\frac{3}{2}^+$	$\nu d_{3/2}$	1.15	0.50	0.58	0.644
^{35}Cl $\frac{3}{2}^+$	$\pi d_{3/2}$	0.126	0.74	0.88	0.822
^{37}Cl $\frac{3}{2}^+$	$\pi d_{3/2}$	0.126	0.32	0.57	0.684
^{39}K $\frac{3}{2}^+$	$(\pi d_{3/2})^{-1}$	0.126	0.13	0.40	0.391

[a]The wave functions of a single proton (π) or neutron (ν) suggested for the states by the extreme single-particle (e.s.p.) shell model.

[b]Calculated magnetic moments based on the e.s.p. wave function and the "free-nucleon" g factors.

[c]Calculated magnetic moments based on the complete sd-shell space wave functions of Refs. 1–3 and the "free-nucleon" g factors.

[d]Calculated magnetic moments based on the complete sd-shell space wave functions of Refs. 1–3 and the empirical sd-shell $M1$ single-particle matrix elements of Ref. 2.

[e]References 25–27; these measurements were made in the long-wavelength limit [at zero momentum (q) transfer].

TABLE V. Electric quadrupole moments Q.

	b (fm)	ψ(e.s.p.)	Theory[a] Q(e.s.p.)[b] ($e$fm^2)	$Q(sd)$[b] ($e$fm^2)	$Q(sd)$[c] ($e$fm^2)	$Q(sd)$[d] ($e$fm^2)	Experiment $Q(q=0)$[e] ($e$fm^2)	$Q(e,e)$[f] ($e$fm^2)
^{17}O $\frac{5}{2}^+$	1.763	$\nu d_{5/2}$	0	0	−2.18	−2.49	−2.58	
^{21}Ne $\frac{3}{2}^+$	1.845	$\nu d_{3/2}$	0	5.77	10.17	10.22	10.3 ± 0.8	
^{23}Na $\frac{3}{2}^+$	1.810	$(\pi d_{3/2})^{-1}$	4.59	5.92	10.24	10.27	10.8 ± 0.8	
^{25}Mg $\frac{5}{2}^+$	1.793	$\nu d_{5/2}$	0	11.09	18.05	17.93	22	$\pm(24.4^{+0.8}_{-4.0})$[g]
^{27}Al $\frac{5}{2}^+$	1.804	$(\pi d_{5/2})^{-1}$	6.51	8.96	13.89	13.70	14.0 ± 0.02	± (15.2 ± 1.6)[h]
^{33}S $\frac{3}{2}^+$	1.881	$\nu d_{3/2}$	0	−3.76	−6.94	−7.02	−6.4 ± 1.0	
^{35}Cl $\frac{3}{2}^+$	1.921	$\pi d_{3/2}$	−5.17	−4.67	−8.17	−8.20	−8.2	± (7.8)[i]
^{37}Cl $\frac{3}{2}^+$	1.921	$\pi d_{3/2}$	−5.17	−4.99	−6.74	−6.49	−6.5	± (6.2)[i]
^{39}K $\frac{3}{2}^+$	1.950	$(\pi d_{3/2})^{-1}$	5.32	5.32	7.19	6.92	5.4 ± 0.2	

[a]The nomenclature of the column headings is consistent with that of Table IV.

[b]$e_p=1$, $e_n=0$.

[c]$\tilde{e}_p=1.35$, $\tilde{e}_n=0.35$.

[d]$\tilde{e}_p=1.30$, $\tilde{e}_n=0.40$.

[e]References 25–27; these measurements were made in the long-wavelength limit [at zero momentum (q) transfer].

[f]From elastic electron scattering (nonzero momentum transfer) measurements.

[g]Reference 16.

[h]Reference 21.

[i]Reference 22.

TABLE VI. Magnetic octupole moments Ω.

	b (fm)	ψ(e.s.p.) [a]	Theory Ω(e.s.p.) [b] ($\mu_N$fm²)	Ω(sd) [b] ($\mu_N$fm²)	Ω(sd) [c] ($\mu_N$fm²)	Ω(sd) [d] ($\mu_N$fm²)	Experiment Ω(q=0) ($\mu_N$fm²)	Ω(e,e) ($\mu_N$fm²)
^{17}O $\frac{5}{2}^+$	1.763	$\nu d_{5/2}$	−17.84	−17.84	−17.19	−16.37		~±10 [e]
^{21}Ne $\frac{3}{2}^+$	1.845	$\nu d_{3/2}$	1.95	6.58	6.29	6.00		
^{23}Na $\frac{3}{2}^+$	1.810	$(\pi d_{3/2})^{-1}$	−0.78	−9.47	−8.04	−8.30		
^{25}Mg $\frac{5}{2}^+$	1.793	$\nu d_{5/2}$	−18.45	−2.79	−2.91	−2.60		±(4.2 ± 0.9) [f]
^{27}Al $\frac{5}{2}^+$	1.804	$(\pi d_{5/2})^{-1}$	32.14	14.56	12.63	12.99		±(15.9 ± 0.4) [g]
^{33}S $\frac{3}{2}^+$	1.881	$\nu d_{3/2}$	2.03	2.38	2.34	2.22		
^{35}Cl $\frac{3}{2}^+$	1.921	$\pi d_{3/2}$	−0.87	−1.90	−1.24	−1.36	−1.6 ± 0.3 [h]	
^{37}Cl $\frac{3}{2}^+$	1.921	$\pi d_{3/2}$	−0.87	−1.22	−0.61	−0.72	−1.3 ± 0.3 [h]	
^{39}K $\frac{3}{2}^+$	1.950	$(\pi d_{3/2})^{-1}$	−0.90	−0.90	−0.36	−0.46		±(0.32) [i] or ±(6.6) [j]

[a] The nomenclature of the column headings is consistent with that of Tables IV, V.
[b] $g_p^s=5.585$, $g_n^s=3.826$, $g_p^l=1.0$, $g_n^l=0.0$.
[c] $\tilde{g}_p^s=4.626$, $\tilde{g}_n^s=-3.687$, $\tilde{g}_p^l=1.0$, $\tilde{g}_n^l=0.0$.
[d] $\tilde{g}_p^s=4.801$, $\tilde{g}_n^s=-3.512$, $\tilde{g}_p^l=1.0$, $g_n^l=0.0$.
[e] Obtained from $\alpha_3(\exp)\times\Omega(\text{e.s.p.})$, where $\alpha_3(\exp)\simeq 0.33$ (see text).
[f] Reference 24.
[g] Reference 28.
[h] Reference 7; uncertainties in the Sternheimer corrections have not been included.
[i] Obtained from the g^l and g^s values quoted in Ref. 24.
[j] Reference 23.

effective g factors.

Empirical $E4$ effective charges have recently been determined from a comparison of the sd-shell calculations with experimental $B(E4)$ values.[14] Again, only the isoscalar effective charge can be determined, the result being $\tilde{e}_p(L=4)+\tilde{e}_n(L=4)\simeq 2.0e$. We will assume the free-space value for the isovector quantity, $\tilde{e}_p(L=4)-\tilde{e}_n(L=4)=1.0e$. Hence we will use $\tilde{e}_p(L=4)=1.5e$ and $\tilde{e}_n(L=4)=0.5e$. Nothing is previously known about the $M5$ effective g factors. If the $4\hbar\omega$ contributions are ignored in first-order perturbation theory, Eqs. (18) are valid for the relationship between $\delta^s(L=5)$ and $\delta(L=4)$. We will use effective g factors for the $M5$ operator based on these relations together with the $E4$ effective charges of $\tilde{e}_p=1.5e$ and $\tilde{e}_n=0.5e$.

III. COMPARISON WITH EXPERIMENT

The calculated multipole moments are compared with experimental values in Table IV–VIII. We distinguish between two methods of determining the experimental moments, one corresponding to the electromagnetic interaction in the long wavelength limit ($q=0$), the other corresponding to

TABLE VII. Electric hexadecapole moments Q_4.

	b (fm)	ψ(e.s.p.) [a]	Theory Q_4(e.s.p.) [b] ($e$fm⁴)	Q_4(sd) [b] ($e$fm⁴)	Q_4(sd) [c] ($e$fm⁴)	Experiment $Q_4(e,e)$ ($e$fm⁴)
^{17}O $\frac{5}{2}^+$	1.763	$\nu d_{5/2}$	0	0	3.62	
^{25}Mg $\frac{5}{2}^+$	1.793	$\nu d_{5/2}$	0	−1.60	−2.93	$\pm(15.3^{+2.3}_{-10.0})$ [d]
^{27}Al $\frac{5}{2}^+$	1.804	$(\pi d_{5/2})^{-1}$	−3.97	−5.05	−8.52	±(30 ± 3) [e]; ±6 [f]

[a] The nomenclature in the column headings is consistent with that of Tables IV–VI.
[b] $e_p=1$, $e_n=0$.
[c] $\tilde{e}_p=1.5$, $\tilde{e}_n=0.5$.
[d] Reference 16.
[e] Reference 21.
[f] Reference 29.

TABLE VIII. Magnetic triakontadupole ($M5$) moments Γ.

	b (fm)	ψ(e.s.p.) [a]	Theory Γ(e.s.p.) [b] ($\mu_N$fm^4)	$\Gamma(sd)$ [b] ($\mu_N$fm^4)	$\Gamma(sd)$ [c] ($\mu_N$fm^4)	Experiment $\Gamma(e,e)$ ($\mu_N$fm^4)
^{17}O $\frac{5}{2}^+$	1.763	$\nu d_{5/2}$	−69.3	−69.3	−65.7	
^{25}Mg $\frac{5}{2}^+$	1.793	$\nu d_{5/2}$	−74.1	−47.0	−44.9	$\pm(40\pm3)$ [a]
^{27}Al $\frac{5}{2}^+$	1.804	$(\pi d_{5/2})^{-1}$	110.9	80.2	60.5	$\pm(67\pm4)$ [e]

[a]The nomenclature of the column headings is consistent with that of Tables IV–VII.
[b]$g_p^s=5.585$, $g_n^s=-3.826$, $g_p^l=1.0$, $g_n^l=0.0$.
[c]$\tilde{g}_p^s=4.214$, $\tilde{g}_n^s=-3.628$, $\tilde{g}_p^l=1.0$, $\tilde{g}_n^l=0.0$.
[d]Obtained from $\alpha_5(\exp)\times\Gamma$(e.s.p.), where $\alpha_5(\exp)=0.50\pm0.08$ (Ref. 16).
[e]Reference 28.

electron scattering results obtained at finite momentum transfer q and then extrapolated to $q=0$ using some model for the shape of the form factor $F(q)$. Standard techniques[7] have been used to obtain experimental values at $q=0$ for all magnetic dipole and electric quadrupole moments and two magnetic octupole moments of the nuclei we consider.

The elastic electron scattering cross sections can be expressed in terms of the moments Q_L and M_L using a multipole expansion in the plane wave Born approximation by[5]

$$\frac{d\sigma}{d\Omega}=\sigma_M\left[\sum_{L,\text{even}} Q_L^2[F_L^C(q)]^2 q^{2L}\frac{(2L+1)}{[(2L+1)!!]^2(2J+1)}\begin{pmatrix} J & L & J\\ J & 0 & -J\end{pmatrix}^{-2} +\left[\tfrac{1}{2}+\tan^2(\theta/2)\right]\sum_{L,\text{odd}} M_L^2[F_L^M(q)]^2 q^{2L}\frac{(L+1)(2L+1)}{L[(2L+1)!!]^2(2J+1)}\begin{pmatrix} J & L & J\\ J & 0 & -J\end{pmatrix}^{-2}\right], \quad (20)$$

where $F_L^C(q)$ and $F_L^M(q)$ are the Coulomb and magnetic form factors normalized so that $F(q=0)=1$ and σ_M is the Mott cross section.

The experimental values of the moments Q_L and M_L are determined as the normalization factors by which the theoretical cross sections of Eq. (20) are matched to the experimental cross sections. Typical analyses of such data use simple assumptions about the shapes of the form factors $F(q)$ such as the extreme single-particle model for magnetic form factors[16-18] and, for Coulomb form factors,[5,16] the Tassie[15] model,

$$F_L^C(q)\sim\int\rho_L(r)j_L(qr)r^2dr\,,$$

where

$$\rho_L(r)\sim r^{L-1}\frac{d\rho(r)}{dr} \quad (21)$$

and $\rho(r)$ is the ground state charge density. In the present work we compare our predictions to such values. In cases where the cross section over some range of q-transfer values is dominated by the form factor of a single multipole, the moment extracted for that multipole with the simple form factors should be fairly reliable. The $E4$ moments are the most difficult to extract because the contribution from the $E4$ form factor is small compared with the $E0$ and $E2$ contributions at all values of q; therefore, errors in the detailed shape of the $E0$ and $E2$ form factors can easily lead to an error in the extraction of the $E4$ strength. It will be interesting to eventually make the analysis more internally consistent by using the same shell-model values of Table I which yield the moment values to also generate the form factors. This expanded analysis would yield additional insight into the nuclear structure via the sensitivity of the detailed shape to components of the transition density which do not enter into the moments themselves.

Historically the extreme-single-particle model has been useful as a qualitative guideline for relating observed properties and nuclear structure and such values are included in Tables IV–VIII along with the experimental and full shell-model values. In all cases except ^{21}Ne and ^{23}Na the extreme-single-particle model provides a relevant comparison since the one-body transition densities are dominated by the appropriate term (see Table I). However, it would be inappropriate to attach any significance to the extreme-single-particle $d_{3/2}$ model for ^{21}Ne and ^{23}Na since the one-body transition densities are in fact dominated by the $d_{5/2}$ contributions in these cases.

A. Magnetic dipole moments

The theoretical and experimental magnetic dipole moments are given in Table IV. These results are representative samples from a comparison which has recently been made for all dipole moments in the *sd* shell.[2] Only the experimental dipole moments at $q=0$, as obtained, for example, by nuclear magnetic resonance, have been measured. For the $J^\pi=\frac{3}{2}^+$ and $\frac{5}{2}^+$ nuclei the magnetic scattering is dominated by the $M3$ and $M5$ form factors except at the lowest momentum transfer. The data in this low q region are consistent with an extreme-single-particle form factor together with the $q=0$ moment value.[16-19] Experiments on individual nuclei with $J^\pi=\frac{1}{2}^+$ ground states (^{19}F and ^{31}P) are needed in order to measure the q dependence of the purely dipole magnetic form factors.

The dipole moments calculated by using free-space g factors with the *sd*-shell wave functions are in much better agreement with experiment than are the extreme-single-particle (Schmidt) predictions. This agreement can be improved somewhat further, especially in the upper *sd* shell, by using empirical single-particle matrix elements obtained from a least-squares fit to experimental dipole moments (see the column labeled $\mu(sd)^{\rm d}$ in Table IV). The remaining discrepancies between experiment and theory indicate the level of inadequacies inherent in the present set of wave functions and in the assumption of a purely one-body effective moment operator. At the beginning or end of the *sd* shell the moments may be sensitive to 2p-2h and 4p-4h core excited components which are not included in the model space, and in the middle of the *sd* shell the moments may be sensitive to as yet undetermined aspects of the model-space one- and two-body interactions.

B. Electric quadrupole moments

The theoretical and experimental electric quadrupole moments are given in Table V. The $q=0$ moments have been extracted from measurements of the hyperfine splittings in atomic and molecular beams.[7] The largest uncertainties in the experimental values often come from the uncertainty in evaluating the atomic wave function corrections arising from the polarization of the core electrons by the nuclear quadrupole moment (the Sternheimer correction[20]). We have quoted values and errors from the compilations and have made no attempt at a critical evaluation. It would be valuable to have a new systematic evaluation of the Sternheimer corrections. $E2$ and $E4$ moments are difficult to extract from the electron scattering data since the Coulomb form factors are dominated by the $E0$ contribution. However, the $E2$ moments which have been extracted using the Tassie model[16,21,22] are in fair agreement with the $q=0$ values, as can be seen in Table V.

Both the extreme-single-particle and the full *sd*-shell-model wave functions yield quadrupole moments in poor agreement with experiment if the free-space values of the proton and neutron charges are used. The introduction of an effective charge operator in the form of a single mass-independent parameter, an isoscalar effective charge of $\tilde{e}_p+\tilde{e}_n=1.7e$ ($\tilde{e}_p=1.35e$ and $\tilde{e}_n=0.35e$), suffices to make the agreement with experiment very good in all cases except ^{39}K. This agreement can be slightly improved further by modifying the isovector effective charge to be $\tilde{e}_p-\tilde{e}_n=0.9e$ ($\tilde{e}_p=1.3e$ and $\tilde{e}_n=0.4e$).

The experimental quadrupole moment of ^{39}K is consistent with the value $\tilde{e}_p=1.0e$ for the proton effective charge. This fact has previously been cited as evidence for a large quenching of the isovector effective charge[11] the values $\tilde{e}_p=1.0e$ and $\tilde{e}_p+\tilde{e}_n=1.7e$ require $\tilde{e}_p-\tilde{e}_n=0.3e$). However, such a small value for $\tilde{e}_p$ is inconsistent with comparisons of Q moments and $B(E2)$ values (Refs. 3, 12) in the region $A=20$–36, and the anomalously small value in ^{39}K may be due to 3p-1h and 5p-3h components in the wave function or to some systematic error in the experimental value.

C. Magnetic octupole moments

The octupole moments are given in Table VI. Only two $q=0$ moments have been experimentally determined in this case, by the atomic beam method,[7] and the comments made in Sec. III B about the Sternheimer corrections apply. The $M3$ form factors can be cleanly observed in the high q electron scattering from $J^\pi=\frac{3}{2}^+$ nuclei. Only results for ^{39}K have been reported.[23,24] Both $M3$ and $M5$ moments have been extracted for the $\frac{5}{2}^+$ states of ^{25}Mg and ^{27}Al by fitting the data with harmonic-oscillator and Woods-Saxon single-particle $d_{5/2}$ radial wave functions[16-18] with the $M3$ and $M5$ moments as normalization parameters. The ^{17}O data[19] have not been analyzed in this way, but the data are observed to be about a factor of 3 lower than the extreme-single-particle (e.s.p.) $d_{5/2}$ prediction based upon the free-space g factors (see Figs. 1 and 2 in Ref. 19). Thus $\Omega(e,e)\simeq(\frac{1}{3})^{1/2}\Omega$ $\times$(e.s.p.) for ^{17}O.

The theoretical octupole moments have been calculated from the *sd*-shell wave functions using three sets of *g* factors: the free-space values and two sets based on the two sets of $E2$ effective charges as discussed in Sec. II. Only the ^{35}Cl, ^{37}Cl, and ^{39}K moments are very sensitive to these variations in the *g* factors.

The comparison with experiment yields mixed conclusions. The calculations for the ^{17}O moment are not in agreement with experiment and this may be due to (1) an anomaly in the ^{17}O radial wave function, (2) a q dependence of the $M3$ effective *g* factors, (3) the importance of 3p-2h and 5p-4h states, or (4) mesonic exchange corrections. The predictions for ^{25}Mg and ^{27}Al lie just outside the experimental errors. The ^{35}Cl and ^{37}Cl predictions with the free-space *g* factors are in better agreement with the $q=0$ experiments than the predictions using effective *g* factors, but the opposite is true for ^{39}K, where the large empirical reduction relative to the single particle $d_{3/2}$ value is reproduced by the calculations with effective *g* factors.

D. Electric hexadecapole moments

The Q_4 moments for the three nuclei with $\frac{5}{2}^+$ ground states are given in Table VII. The Coulomb form factors are not very sensitive to the Q_4 moments, as is illustrated by the large error on the ^{25}Mg Q_4 value extracted using the Tassie model.[16] The calculated value is within the large error in this case. For ^{27}Al an experimental Q_4 moment with a small error has been quoted[21] which is a factor of 3 larger then the calculated value. This discrepancy may be due to an underestimate in the experimental uncertainty.

E. Magnetic triakontadupole moments

The $M5$ moments for the three nuclei with $\frac{5}{2}^+$ ground states are given in Table VIII. In contrast to the difficulties in extracting $M3$ and $E4$ form factors in these nuclei, the $M5$ form factors are easily measured by high q electron scattering. The *sd*-shell results obtained with free-space *g* factors are too large compared with the ^{25}Mg and ^{27}Al experimental results.[15,16] The introduction of effective *g* factors (based on the $E4$ effective charges as discussed in Sec. II) brings the calculated moments into fair agreement with experiment. In ^{17}O the experimental and calculated single-particle form factors do not have the same shape[19] and thus it is impossible to extract a moment from this comparison. The considerations mentioned above in connection with the $M3$ form factor for ^{17}O are also important in understanding the $M5$ form factor.

SUMMARY

Overall, the comparison of the shell-model predictions for the higher multipole moments of *sd*-shell states with the existing experimental values does not result in simple conclusions, contrary to the results obtained for lower multipolarities. In the case of $M1$ and $E2$ moments, all members of the rather large data sets are reproduced by the analogous shell-model predictions to within deviations which are both relatively small and consistent over the entire range of states. As noted in the foregoing, some of the (fewer) experimentally assigned higher multipolarity moments are reproduced theoretically with the same degree of accuracy achieved for the $M1$ and $E2$ moments. On the other hand, other experimental values differ very significantly from the presently predicted values. In some instances, e.g., the $E4$ moment of ^{27}Al, the discrepancies may result from unrealistic assignments of experimental uncertainties. However, in other cases, e.g., the $M3$ moment of ^{17}O, the discrepancies cannot be thus explained away and may constitute symptoms of fundamental defects in the present approach to nuclear structure.

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