Exact solution of the Faddeev equations for the harmonic oscillator ground state

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The Faddeev equations in N space dimensions for three identical spinless particles in their ground state interacting via harmonic oscillator, two-body potentials is solved analytically. Unlike the well-known Schrödinger solution, the individual Faddeev amplitudes may be negative and contain a long-range component of arbitrary strength. Upon symmetrizing to obtain the full wave function, this component vanishes and the Schrödinger solution results. Three-dimensional plots of the various wave functions are presented.

[NUCLEAR STRUCTURE Faddeev calc., configuration space, harmonic oscillator.]

The use of Faddeev techniques to calculate the bound state wave functions of three-body systems has become increasingly common. Most attempts involve the use of momentum space techniques and partial-wave projected forces. Several attempts to solve the coordinate space version[1] of the Faddeev equations have also been made recently for the case of local potentials.[2-4] Because there is no simple physical interpretation of the individual Faddeev amplitudes, as opposed to the complete, fully symmetrized (Schrödinger) wave function, it is difficult to develop an intuitive feeling for the structure of these solutions. In order to facilitate this and to illustrate the difference between the two functions, we develop here an exact analytic solution for the problem of three identical spinless particles in their ground state interacting via identical harmonic oscillator, two-body potentials in N space dimensions. The complete force in all partial waves will be used. This means that, in principle, a partial differential equation in three variables must be solved. For our problem, symmetry requirements reduce this to an ordinary differential equation in one variable.

The Schrödinger solution of the harmonic oscillator problem is well known. We define our coordinates so that the vector displacement of two specially chosen particles (2 and 3) is $\vec{x}$, while $\vec{y}\,' = \sqrt{3}/2\vec{y}$ is the vector displacement of the third particle from the center of mass of the other two. The Schrödinger equation and its solution ψ in the bound state rest frame is then given by[1-3]

$$\left(E+\frac{\vec{\nabla}_x^{\,2}+\vec{\nabla}_y^{\,2}}{M}\right)\psi=\frac{\beta^2}{M}(\vec{x}^2+\vec{y}^2)\psi\,, \tag{1a}$$

$$\psi=3e^{-\beta(\vec{x}^2+\vec{y}^2)/2}\,, \tag{1b}$$

$$E=N\omega_0=\frac{2\beta N}{M} \tag{1c}$$

with $\beta=\sqrt{Mk}/2$, $\omega_0=(k/M)^{1/2}$, and, in general, $E=(A-1)N\omega_0/2$ for A particles of mass M. The potential between particles 2 and 3 is $k\vec{x}^2/6$, the factors having been chosen so that the potential is $k\vec{r}_i^{\,\prime 2}/2$ for each particle, expressed in terms of its separation from the center of mass, $\vec{r}_i^{\,\prime}$. This leads to the usual $(A-1)$ independent oscillators.

The Faddeev equations (actually, three identical equations) are a specific decomposition of Eq. (1a), in terms of amplitudes $\phi(\vec{x},\vec{y})$, which singles out the interaction between particles 2 and 3: $2\beta^2\vec{x}^2/3M$. The Faddeev decomposition assumption,

$$(1+P)\phi=\psi\,, \tag{2a}$$

with P the sum of cyclic ($P^{(+)}$) and anticyclic ($P^{(-)}$) permutation operators, leads one to three identical equations for ϕ, one of which is

$$\left(E+\frac{\vec{\nabla}_x^{\,2}+\vec{\nabla}_y^{\,2}}{M}\right)\phi=V\psi=\frac{2\beta^2}{3M}x^2\psi\,. \tag{2b}$$

It is easy to demonstrate that (2b) upon symmetrization produces Eq. (1).

We can either solve the symmetrized version of Eq. (2b) [i.e., Eq. (1a)] or assume that we know the solution ψ. This gives

$$(2\beta N+\vec{\nabla}_x^{\,2}+\vec{\nabla}_y^{\,2})\phi=2\beta^2\vec{x}^2e^{-\beta(\vec{x}^2+\vec{y}^2)/2}\,. \tag{3}$$

It is easy to demonstrate that ϕ will be a function of $\beta\vec{x}^2$ and $\beta\vec{y}^2$ only. Furthermore, we can replace these two quantities by $z=\beta(\vec{x}^2+\vec{y}^2)$ and $\beta(\vec{x}^2-\vec{y}^2)$. The quantity z is invariant under the operators $P^{(+)}$ and $P^{(-)}$, while $\beta(\vec{x}^2-\vec{y}^2)$ is not. One finds, in fact,

$$P^{(\pm)}(\vec{x}^2-\vec{y}^2)=\frac{\vec{y}^2-\vec{x}^2}{2}\pm\sqrt{3}\,\vec{x}\cdot\vec{y}\,. \tag{4}$$

If one writes ϕ in terms of ψ and an auxiliary function ξ,

$$\phi=e^{-\beta(\vec{x}^2+\vec{y}^2)/2}+\xi\,, \tag{5a}$$

one finds that

$$(2\beta N+\vec{\nabla}_x^{\,2}+\vec{\nabla}_y^{\,2})\xi(\vec{x}^2,\vec{y}^2)=\beta^2(\vec{x}^2-\vec{y}^2)e^{-\beta(\vec{x}^2+\vec{y}^2)/2}\,, \tag{5b}$$

where obviously $(1+P)\xi\equiv 0$, which is consistent with Eqs. (4) and (5). Constructing an approximate power series solution for ξ and imposing Eq. (2b) results in the restriction

$$\xi=\beta/2(\vec{x}^2-\vec{y}^2)g[\beta(\vec{x}^2+\vec{y}^2)]\,. \tag{5c}$$

The partial differential equation (5b) then leads to

$$2zg''(z)+(2N+4)g'+Ng=e^{-z/2} \tag{5d}$$

which must be solved in order to complete ϕ. Changing variables to $r^2=z$ and writing $g=r^{-(N+1)}h(r)$ produces

$$r^2h''(r)+rh'(r)+[2Nr^2-(N+1)]h(r)=2r^{N+3}e^{-r^2/2} \tag{6}$$

which is Bessel's differential equation with an inhomogeneous term. The solution is elementary and we find

$$\phi=e^{-r^2/2}+\frac{\beta(\vec{x}^2-\vec{y}^2)}{2r^{N+1}}\left[\lambda J_{N+1}(\sqrt{2N}r)+\pi Y_{N+1}(\sqrt{2N}r)\int_0^r dt\,t^{N+2}e^{-t^2/2}J_{N+1}(\sqrt{2N}t)-\pi J_{N+1}(\sqrt{2N}r)\int_0^r dt\,t^{N+2}e^{-t^2/2}Y_{N+1}(\sqrt{2N}t)\right], \tag{7}$$

where $r^2=\beta(\vec{x}^2+\vec{y}^2)$ and λ is an *arbitrary* constant.

The complete solution possesses some unusual features. The first is the relative complexity of those terms in ϕ which do not contribute to ψ. The second feature is the completely "spurious" component of g (or ϕ) which can be present with an arbitrary strength $(\lambda J_{N+1}/r^{N+1})$. The latter component is well-behaved for small r and oscillates as r increases, with an envelope $\lambda(2/\pi)^{1/2}/r^{N+3/2}$. If we further write $\vec{x}^2-\vec{y}^2\equiv r^2\chi$, we find that the spurious component falls off as $1/r^{N-1/2}$ or $r^{-5/2}$ for $N=3$, a well-known result.[5] The remaining part of g has the same asymptotic behavior and vanishes for small r.

For r large compared to 2, ξ dominates ϕ. Whereas ψ is positive definite, ϕ is not. In order to make these remarks as graphic as possible, we have plotted the various components of ϕ as a function of x and y for $N=3$ and $\beta=1$. The latter choice is no restriction, since ϕ is a function of $\sqrt{\beta}\,x$ and $\sqrt{\beta}\,y$. Figure 1 illustrates the smooth behavior of $e^{-r^2/2}$, while Fig. 2 depicts the spurious component with an arbitrary normalization. Figure 3 shows ϕ with $\lambda=0$. Note the $x=y$ plane, which corresponds to $e^{-r^2/2}$, and the negative regions which are not present in the Schrödinger solution.

An obvious question which arises is whether our

$\psi = (1 + P)\phi$

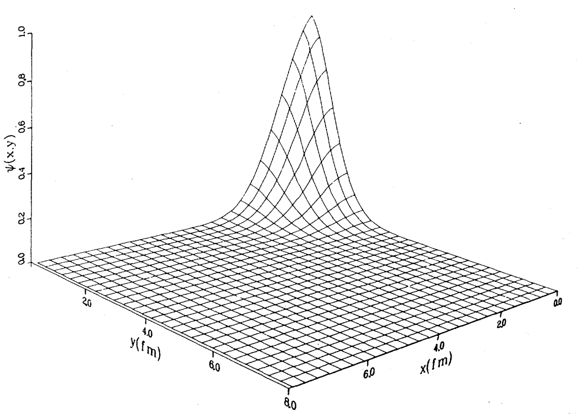

FIG. 1. Schrödinger wave function ψ for the three-body harmonic oscillator problem. The coordinates x and y are defined in the text.

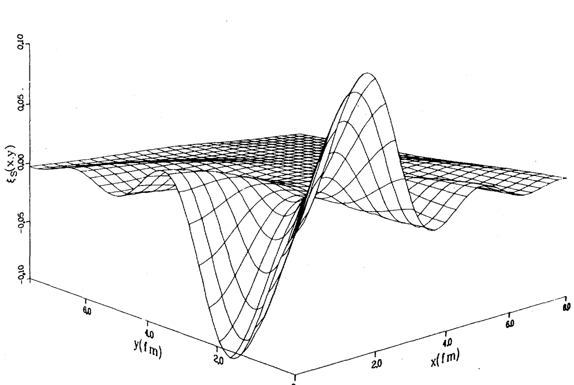

FIG. 2. Spurious component of Faddeev amplitude ϕ, ξ_s, with an arbitrary normalization.

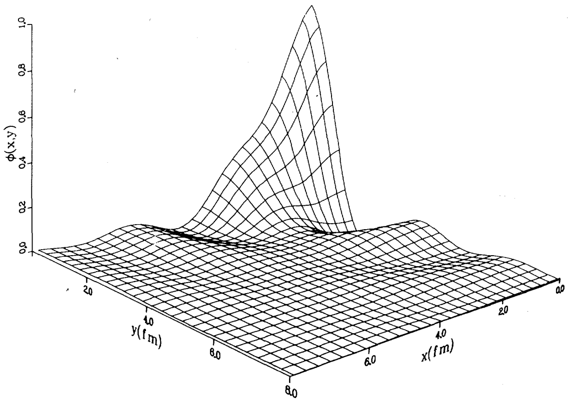

FIG. 3. Complete Faddeev amplitude, ϕ, without a spurious component.

solution has any features which would be typical of a more realistic force model. In particular, is the spurious component to be expected in such a calculation? While we cannot answer this question entirely, we note that the spurious (Bessel) component arises as a solution of the homogeneous part of Eq. (5), and satisfies the boundary conditions only because the harmonic oscillator has *positive* energies for all bound states. For negative energies, which apply to physical bound state problems, no such solution exists. The oscillatory nature of the former amplitude is also quite likely an artifact of the model we have solved. The Faddeev amplitude can nevertheless be negative in physical cases, even when the Schrödinger wave function is positive definite.[3]

Several previous attempts have been made to solve analytically the Faddeev coordinate space equations.[6,7] These efforts use two-body forces which have been truncated to exist only in s waves, unlike our force. Nevertheless, an interesting peculiarity of the harmonic oscillator ground state is that the total wave function ψ does not depend on the angles of $\vec{x}$ and $\vec{y}$, but only on the magnitudes x and y. Therefore, if we assume only s-wave forces, the s-wave projection operation on ψ returns ψ intact. Furthermore, the s-wave kinetic energy operator is the same as Eq. (2b) with $\vec{\nabla}_x^2$ replaced by $x^{1-N}\partial/\partial x\, x^{N-1}\partial/\partial x$, etc. Thus, our solution for the complete harmonic oscillator force is also the solution for the s-wave force problem. This condition will not necessarily hold for other states of the harmonic oscillator, or for other force models.

In summary, we have developed a solution for the Faddeev equations for three identical spinless particles in their ground state interacting via harmonic oscillator two-body forces in N space dimensions. The solution has a spurious component of arbitrary magnitude and an oscillatory nature which dominates for large r. Unlike the Schrödinger wave function, the Faddeev amplitudes are not positive definite.

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