Method for scattering equations. II. Iterative solution

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A recently proposed method for t matrix Lippmann-Schwinger-type equations is reexamined and is investigated numerically. The method relies on the introduction of an auxiliary equation containing an arbitrary function, whose kernel is free from the fixed point singularity of the original equation. In the present work we rewrite the final result of the method in such a way that it has all the important features of a related method by Kowalski and Noyes and is simple to use in practice. With special choices of the arbitrary function the present method can be considered as an off-shell extension of methods of Bolsterli, of Kowalski, and of Sasakawa. The method also readily gives a practical way of calculating K matrix elements. Using the iterative solution of the auxiliary equation the method is tested numerically to compute off-shell t matrix elements for three commonly used nucleon-nucleon potentials for various choices of the arbitrary function.

[NUCLEAR REACTIONS Singular scattering equations, nonsingular auxiliary equations, off-shell t matrix elements and phase shifts computed.]

I. INTRODUCTION

Numerical solutions of Lippmann-Schwinger-type equations are usually performed by approximating the integral equation by a matrix equation of finite dimension. Such approximation needs a special prescription such as contour rotation or delicate treatment of a principal value integral because of the occurrence of a fixed point singularity in the kernel of the equation. Moreover, it is well known that the iterative solution of such equations diverges except for weak potentials or for very high energies. Such divergence is associated with the occurrence of an eigenvalue of the kernel (of the equation) of magnitude greater than one[1] (both attractive or repulsive).

A recently proposed method[2] for solving fully off-shell Lippmann-Schwinger equations for the t matrix is reexamined. The method relies on solving an auxiliary equation whose kernel is free from singularities and is also sufficiently weak to have convergent iterative solutions for a wide class of potentials. The solution of the original equation is then related to that of the auxiliary equation. The auxiliary equation contains an arbitrary flexible function and this makes the method sufficiently general to include various other methods as special cases.

Sasakawa[3] proposed an efficient method for computing phase shifts using a wave function description of scattering. Kowalski[4] has shown that the Sasakawa approach can be reformulated to yield a practical method for computing half-on-shell t matrix elements using the momentum space Lippmann-Schwinger equations. The present method as has been demonstrated in SA is a generalization of the Kowalski-Sasakawa method[4] to the case of fully off-shell t matrix elements.

In the present work we rewrite the result of SA in the form first presented independently by Kowalski and Noyes[5] (KN). Although the kernel of the nonsingular equation of the present method is in general different from that of KN, it has all the essential features of that formulation and is simple to implement in practice. The present method is more general than the usual version of the KN method, because the arbitrary flexible function of the present method can be conveniently chosen in order to build in various desirable features in the formulation.

Kowalski[6] showed that a recently proposed method by Bolsterli[7] for computing phase shifts belongs to the Kowalski-Sasakawa class[3,4] and can be generalized to compute half-on-shell K matrix elements. For a particular choice of the arbitrary function the present method is shown to be a generalization of the Bolsterli-Kowalski method[6,7] for computing the fully off-shell t or K matrix elements.

The kernel of the auxiliary equation apart from being nonsingular is also weaker in nature compared to the original equation. Coester[8] studied the convergence properties of iterative solutions of the auxiliary equation analytically. In the present work we study the convergence properties of the iterative solution numerically for different choices of the arbitrary function for the Yukawa, the Malfliet-Tjon, and the Reid soft-core $^1S_0$ potential.

The plan of the paper is as follows. In Sec. II

we give a brief resume of the method, write it in the KN form,[5] discuss various choices of the arbitrary function and exhibit relation with other methods. In Sec. III we report numerical results for the three above-mentioned potentials, and finally in Sec. IV we give a discussion and some concluding remarks.

II. THE METHOD

For the sake of completeness we give a brief resume of the method following SA. (For a complete review including a discussion of multichannel problems see SA.) The partial wave Lippmann-Schwinger equation (suppressing the partial wave index and in units $\hbar = 2m = 1$) can be written as

$$\langle p|t(E)|r\rangle = \langle p|V|r\rangle + \frac{2}{\pi}\int q^2 dq \langle p|V|q\rangle \times (k^2 - q^2 + i\epsilon)^{-1}\langle q|t(E)|r\rangle , \tag{1}$$

where $k^2 = E$ and the integration limits in Eq. (1) and throughout the rest of the paper are from 0 to ∞. The on-shell t matrix element $t(k)$ is related to the phase shift δ by

$$t(k) \equiv \langle k|t(E)|k\rangle = -k^{-1}e^{i\delta}\sin\delta . \tag{2}$$

Next we introduce a function $\gamma(k, q)$ such that

$$\gamma(k, k) = 1 . \tag{3}$$

We also introduce three operators A, $\overline{V}$, and H_0 defined by[2]

$$\langle p|A(E)|q\rangle = [\langle p|V|q\rangle - \langle p|V|k\rangle\gamma(k, q)] \times (k^2 - q^2 + i\epsilon)^{-1} , \tag{4}$$

$$\langle p|\overline{V}|q\rangle = \langle p|V|k\rangle , \tag{5}$$

and

$$\langle p|H_0(E)|q\rangle = \delta(p - q)(k^2 - q^2 + i\epsilon)^{-1}\gamma(k, q) . \tag{6}$$

Then the original Lippmann-Schwinger equation can be rewritten as

$$t(E) = V + \overline{V}(E)H_0(E)\,t(E) + A(E)\,t(E) . \tag{7}$$

Next we introduce the auxiliary equations

$$\Gamma(E) = V + A(E)\Gamma(E) , \tag{8}$$

with nonsingular kernel $A(E)$. Then $t(E)$ is expressed in terms of the solutions $\Gamma(E)$ of nonsingular Eq. (8) as

$$\langle p|t(E)|r\rangle = \langle p|\Gamma(E)|r\rangle + \langle p|\Gamma(E)|k\rangle I(k, r) , \tag{9}$$

with

$$I(k, r) = \frac{2/\pi\int q^2 dq(k^2 - q^2 + i\epsilon)^{-1}\gamma(k, q)\langle q|\Gamma(E)|r\rangle}{1 - 2/\pi\int q^2 dq(k^2 - q^2 + i\epsilon)^{-1}\gamma(k, q)\langle q|\Gamma(E)|k\rangle} . \tag{10}$$

Equations (4), (8), (9), and (10) are the fundamental results of SA. The kernel $A(E)$ of Eq. (8) is nonsingular. This method can be readily used for K matrix elements also. The only modification we need to make is to replace the t matrix t in Eq. (9) by the K matrix K and the integrals in Eq. (10) by their principal values.

Next we show that our final result given by Eq. (9) can be written in a simpler form which was originally presented independently by Kowalski and Noyes.[5] In order to achieve this we note from Eq. (9) that for $r = k$ the half-on-shell t matrix elements become[2]

$$\langle p|t(E)|k\rangle = \frac{\langle p|\Gamma(E)|k\rangle}{\langle k|\Gamma(E)|k\rangle}t(k) , \tag{11}$$

with

$$t(k) = \langle k|\Gamma(E)|k\rangle\left[1 - \frac{2}{\pi}\int q^2 dq(k^2 - q^2 + i\epsilon)^{-1}\gamma(k, q)\langle q|\Gamma(E)|k\rangle\right]^{-1} . \tag{12}$$

Similarly for $p = k$ Eq. (9) becomes

$$\langle k|t(E)|r\rangle = \langle k|\Gamma(E)|r\rangle + \langle k|\Gamma(E)|k\rangle I(k, r) . \tag{13}$$

Using Eq. (13), Eq. (9) can be rewritten as

$$\langle p|t(E)|r\rangle = \langle p|\Gamma(E)|r\rangle + \frac{\langle p|\Gamma(E)|k\rangle}{\langle k|\Gamma(E)|k\rangle}[\langle k|t(E)|r\rangle - \langle k|\Gamma(E)|r\rangle] . \tag{14}$$

Using Eq. (11) and remembering that $\langle p|t(E)|k\rangle = \langle k|t(E)|p\rangle$, Eq. (14) becomes[9]

$$\langle p|t(E)|r\rangle = \left[\frac{\langle p|\Gamma(E)|k\rangle}{\langle k|\Gamma(E)|k\rangle}t(k)\,\frac{\langle r|\Gamma(E)|k\rangle}{\langle k|\Gamma(E)|k\rangle}\right] + \left\{\langle p|\Gamma(E)|r\rangle - \frac{\langle p|\Gamma(E)|k\rangle\langle k|\Gamma(E)|r\rangle}{\langle k|\Gamma(E)|k\rangle}\right\} . \tag{15}$$

Equation (15) is the desired equation. The quantity in the square bracket is the KN approximation[5] to the t matrix and because of Eq. (11) is exact half-on-shell. The second term in the curly bracket of Eq. (15) is the fully off-shell residual term and is zero half-on-shell. Equation (15) is simpler to use than Eq. (9) and is more general than the KN method. In Eq. (9) we need to calculate the function $I(k,r)$, whereas in Eq. (15) we need to calculate $t(k)$ given by Eq. (12). A glance at Eqs. (9) and (12) tells that the calculation of $t(k)$ is simpler than that of $I(k,r)$ for arbitrary r. Moreover, if the iterative solution for Γ is employed Eq. (15) yields a symmetric half-on-shell t matrix whereas Eq. (9) does not. The original version of the KN method can be considered as a special case of the present method containing an arbitrary function γ. It is to be noted that the kernel of the present nonsingular equation is in general different from that of the KN method, but they can be made equal for a special choice of γ as we shall see in the following. Next we consider various choices of γ and their consequences.

A. Sasakawa choice

The original description of Sasakawa theory uses[3]

$$\gamma(k,q)=\frac{k}{q} \tag{16}$$

for $L=0$. Sasakawa demonstrates that this choice gives rapid convergence of iterative solution of Γ for Yukawa, square-well, and some other potentials.

B. Kowalski-Noyes choice

If we take

$$\gamma(k,q)=\frac{\langle k|V|q\rangle}{\langle k|V|k\rangle}, \tag{17}$$

the kernel A becomes the kernel of the KN method. It is obvious from Eqs. (4) and (8) that with this choice of γ

$$\langle k|\Gamma(E)|r\rangle=\langle k|V|r\rangle, \tag{18}$$

and hence Eq. (15) becomes

$$\langle p|t(E)|r\rangle=\left[\frac{\langle p|\Gamma(E)|k\rangle}{\langle k|V|k\rangle}t(k)\frac{\langle r|\Gamma(E)|k\rangle}{\langle k|V|k\rangle}\right]+\left\{\langle p|\Gamma(E)|r\rangle-\langle p|\Gamma(E)|k\rangle\frac{\langle k|V|r\rangle}{\langle k|V|k\rangle}\right\}. \tag{19}$$

$t(k)$ of Eq. (19) is defined by Eq. (12) with γ given by Eq. (17).

C. Blasczac-Fuda choice

Blasczac and Fuda[10] recommend using

$$\gamma(k,q)=\left(\frac{q}{k}\right)^{L} \tag{20}$$

for the Lth partial wave and claim following Coester[8] that such a choice for γ will give a converging iterative solution for Γ for a wide class of potentials.

D. Bolsterli-Kowalski choice

Kowalski[6] demonstrated that the method proposed by Bolsterli[7] corresponds to taking

$$\gamma(k,q)=\frac{\phi(q)-B(q,k)}{1-B(k,k)}, \tag{21}$$

with $\phi(k)=1$ (in the K matrix formulation of the present method), where

$$B(q,k)=P\frac{2}{\pi}\int\phi(p)p^2dp(k^2-p^2+i\epsilon)^{-1}\langle p|V|q\rangle. \tag{22}$$

P in Eq. (22) denotes the principal value prescription for the integral. This choice of γ has certain advantages in using Eqs. (9) and (15). In the implementation of Eq. (9) or (15) we need to evaluate integrals of the type we encounter in Eq. (10). Principal values of such integrals are written as

$$\tau(r)=P\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\times\langle q|\Gamma(E)|r\rangle. \tag{23}$$

The explicit momentum space matrix element of (8) can be written as

$$\langle q|\Gamma(E)|r\rangle=\langle q|V|r\rangle+P\frac{2}{\pi}\int p^2dp\langle q|V|p\rangle(k^2-p^2+i\epsilon)^{-1}\times\langle p|\Gamma(E)|r\rangle-\langle q|V|k\rangle\tau(r). \tag{24}$$

With the help of Eqs. (21), (22), and (24), Eq. (23) becomes

$$\tau(r)=\frac{1}{1-B(k,k)}[B(r,k)-B(k,k)\tau(r)] \tag{25}$$

or

$$\tau(r)=B(r,k). \tag{26}$$

Hence $I(k,r)$ given by Eq. (10) in the t matrix formulation takes the simple form

$$I(k,r)=\frac{B(r,k)-ik\langle k|\Gamma(E)|r\rangle}{1-B(k,k)+ik\langle k|\Gamma(E)|k\rangle}. \tag{27}$$

$t(k)$ of Eq. (12) also takes the simple form

$$t(k)=\langle k|\Gamma(E)|k\rangle[1-B(k,k)+i\gamma\langle k|\Gamma(E)|k\rangle]^{-1}. \tag{28}$$

Hence once B and γ are calculated we do not need to evaluate any integrals to calculate the fully off-shell t matrix elements. We may use Eqs. (9) and (15) with Eqs. (27) and (28). The original version of the Bolsterli-Kowalski method[6,7] was for half-on-shell K matrix elements. Here we have generalized the method to the case of fully off-shell K or t matrix elements. In the case of the K matrix elements the imaginary quantities in Eqs. (27) and (28) should be set equal to zero.

E. Coester choice

Coester[8] suggested that the choice

$$\gamma(k,q)=\frac{\langle k|t(E)|q\rangle}{\langle k|t(E)|k\rangle} \tag{29}$$

has certain advantages. We show that this choice makes the half-on-shell t matrix elements $\langle k|t_N(E)|r\rangle$ given by Eq. (13) exact for each order of the iterative solution of Eq. (8), where the suffix N here and in the following denotes the order of iteration. From Eq. (13) we have

$$\langle k|t_N(E)|r\rangle=\langle k|\Gamma_N(E)|r\rangle + \langle k|\Gamma_N(E)|k\rangle I_N(k,r), \tag{30}$$

where by Eqs. (10) and (29) we have

$$I_N(k,r)=\frac{\langle k|\Delta_N(E)|r\rangle}{t(k)-\langle k|\Delta_N(E)|k\rangle}, \tag{31}$$

where

$$\Delta_N(E)=t(E)\,G_0(E)\Gamma_N(E),$$

with

$$\langle q|G_0(E)|p\rangle=\delta(p-q)(k^2-p^2+i\epsilon)^{-1}.$$

In order to prove our claim we make use of the following identity

$$\langle k|t(E)|r\rangle=\langle k|\Gamma_N(E)|r\rangle+\langle k|\Delta_N(E)|r\rangle. \tag{32}$$

A proof of this identity is given in the Appendix. Using Eqs. (30), (31), and (32) it is trivial to check that $\langle k|t_N(E)|r\rangle=\langle k|t(E)|r\rangle$ for any N. But this proof does not assume a convergent iterative solution for Γ. In other words, Eq. (29) for γ does not necessarily guarantee a convergent iterative solution for Γ. But even with such divergent Γ_N the half-on-shell t matrix given by Eq. (30) will be exact for each order of iteration.

III. NUMERICAL CALCULATIONS

To see how the method works in practice we carried out numerical calculations with the Yukawa potential, the Malfliet-Tjon potential, and the Reid soft-core $^1S_0$ potential. Of these the last two are particularly important because they have highly repulsive core at short distances, and hence the sign of the phase-shift changes at high energies as in the realistic nucleon-nucleon potential. We calculate numerical results using the iterative solution of Eq. (8) for Γ. Already there exist numerical results[10] employing an iterative solution for Γ in the case of half-on-shell t matrix elements for the Reid soft-core $^1S_0$ potential for $\gamma(k,q)=1$. The present numerical calculation is more general than that of Ref. 10 in that here we use various choices of γ and calculate fully off-shell t matrix elements using both Eqs. (9) and (15). Of course, if the exact solution of Γ is employed both these forms will lead to the same result. But if an approximate iterative solution of Γ is used one may lead to better convergence properties than the other. It is intuitively expected that the manifestly more symmetric form (15), which is symmetric for half-on-shell t matrix elements if an iterative solution for Γ is used, will lead to better convergence properties. Here we present results for the S wave case only because the convergence for higher partial waves is expected to be more rapid than that for the S wave.

The Yukawa potential we use is defined by[11]

$$V(r)=-V_0 r^{-1}e^{-\mu r}, \tag{33}$$

where $V_0=65.246$ MeV fm and $\mu=0.6329$ fm^{-1}, which has a single bound state at an energy $E=-2.240$ MeV. We also use the two term soft-core potential of Malfliet and Tjon, defined by[11]

$$V(r)=-V_A r^{-1}e^{-\mu_A r}+V_R r^{-1}e^{-\mu_R r}, \tag{34}$$

where $V_A=181.5422\pi$ MeV fm, $V_R=457.8828\pi$ MeV fm, $\mu_A=1.55$ fm^{-1}, $\mu_R=3.11$ fm^{-1}, which has a single bound state at an energy $E=-0.35$ MeV, and finally the Reid soft-core $^1S_0$ potential, defined by[12]

$$V(r)=[V_1e^{-\beta_1 r}+V_2e^{-\beta_2 r}+V_3e^{-\beta_3 r}](\beta_1 r)^{-1}, \tag{35}$$

where $\beta_1=0.7$ fm^{-1}, $\beta_2=4\beta_1$, $\beta_3=7\beta_1$, $V_1=-10.463$ MeV, $V_2=-1650.6$ MeV, and $V_3=6484.2$ MeV.

As in Ref. 10 first we convert the integrals from 0 to ∞ to integrals from -1 to 1 and use Gauss quadratures to change the integral equation for Γ to a matrix equation. The number of Gauss quadratures used for this purpose was typically of the order of 32 except in the case of the Reid soft-core $^1S_0$ potential where we needed more points in general in order to achieve the same numerical accuracy. Finally, the resultant matrix equation for Γ is solved by the method of iteration in double precision in an IBM computer.

It was easiest to achieve convergence in the case

of the Yukawa potential and was most difficult in the case of the Reid soft-core potential. The highly repulsive core of the Reid soft-core potential is probably responsible for slow convergence of the iterative solution. In the case of the Yukawa potential there is no repulsive core and hence no difficulty with convergence.

First we report the results for the Reid soft-core $^1S_0$ potential. We tried all the γ's of Sec. II in this case and found that the equation for Γ converges at all energies only for $\gamma=1$. In Fig. 1 we present the results for the fully off-shell t matrix elements given by Eq. (9) using iterative solution Γ_N of Γ, where, as before, N refers to the number of iterations. In the lowest order of iteration $N=1$ and $\Gamma_1=V$. In particular, Fig. 1(a) shows the real parts of $t(p,q)$ and $t(q,p)$ for $q=0.72$ fm^{-1} and $E_{\text{c.m.}}=72$ MeV. Figure 1(b) exhibits the imaginary parts of the corresponding t matrix elements. In Fig. 2 we show the same off-shell t matrix elements obtained by using the more symmetric form given by Eq. (15). As before Fig. 2(a) represents the real parts and Fig. 2(b) represents the imaginary parts of the t matrix. Equation (15) always yields a manifestly symmetric imaginary part and hence we show only one diagram for the imaginary part. Looking at Figs. 1 and 2 we find that the convergence is in general better in the case of Eq. (15) than in the case of Eq. (9). The convergence is particularly bad for large values of off-shell momentum in one sense in the case of Eq. (9). This general conclusion about convergence is true at other energies also. Hence we study the more symmetric form given by Eq. (15) for the two other potentials for various choices of γ.

Next we consider the potential of Malfliet and Tjon. In this case we find that we have convergence at all energies for $\gamma=1$ and also for the Bolsterli-Kowalski choice given by Eq. (21). In Figs. 3(a) and 3(b) we show the convergence of

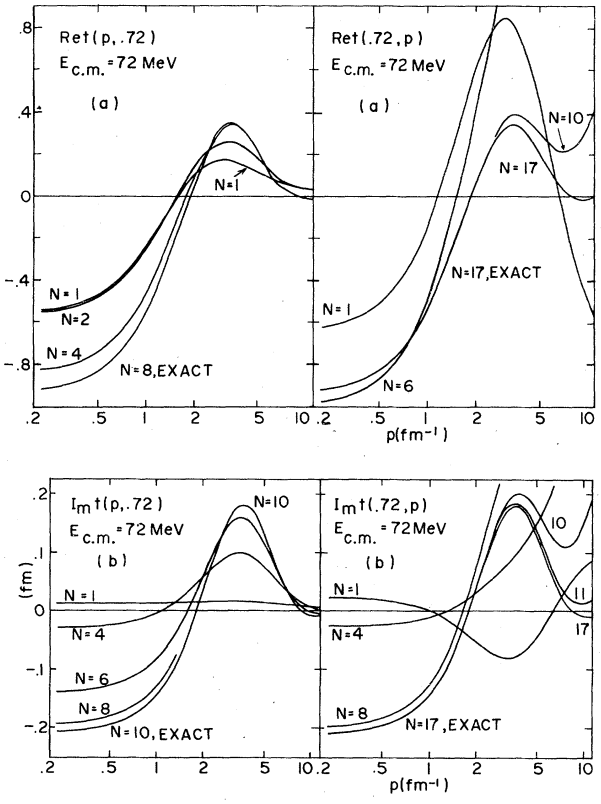

FIG. 1. (a) The real and (b) the imaginary parts of the off-shell t matrix elements for the Reid soft-core $^1S_0$ potential calculated using Eq. (9) for different N and for $\gamma=1$.

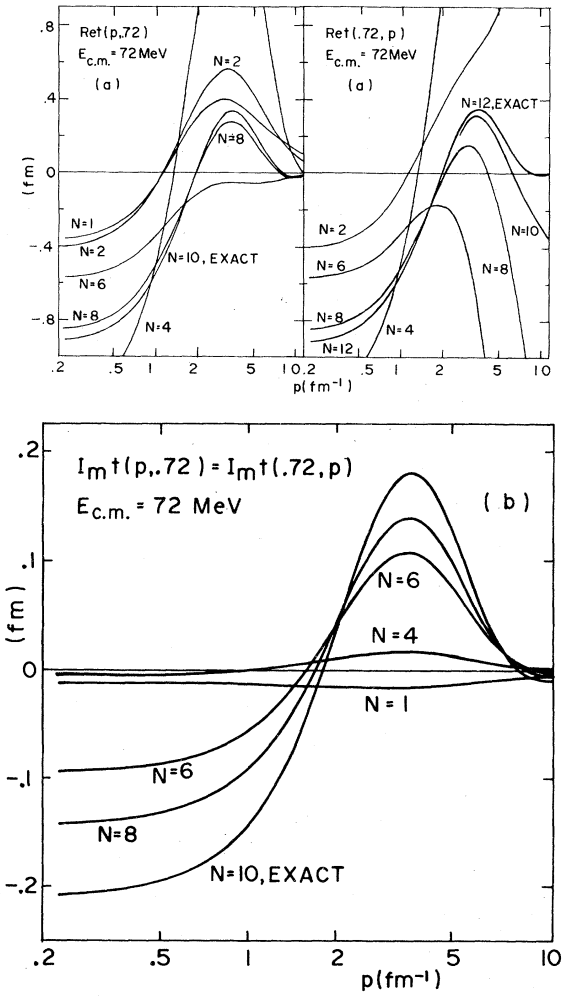

FIG. 2. Same as in Fig. 1 but calculated using Eq. (15). The imaginary part is symmetric and hence we have only one diagram in this case.

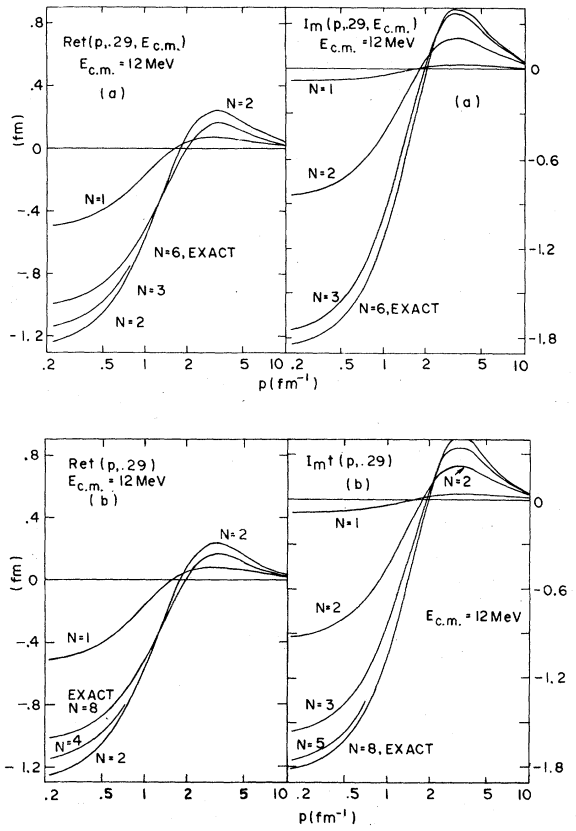

FIG. 3. The real and the imaginary parts of the off-shell t matrix elements for the Malfliet-Tjon potential calculated using Eq. (15) for (a) $\gamma=1$, and for (b) the Kowalski-Bolsterli choice.

the real and imaginary parts of off-shell t matrix elements $t(p,q)$ for $q=0.29$ fm^{-1} and $E_{c.m.}=12$ MeV for $\gamma=1$ and for the Bolsterli-Kowalski choice (21) of γ with $\phi(p)=1$, respectively. The approximate t matrix elements are practically symmetric in the case of the Malfliet-Tjon potential, and hence we do not show the transposed t matrix elements $t(p,q)$ in this case. The convergence is much better in this case than in the case of the Reid soft-core potential. Of the two γ's considered $\gamma=1$ gave better convergence properties in this case.

Finally we consider the Yukawa potential. In this case we find that the iterative solution for Γ converges at all energies for all the choices of the γ's mentioned in Sec. II except for the Bolsterli-Kowalski choice. The convergence is the fastest in this case because the potential has no soft core.

We show in Table I the elastic scattering phase shifts for the Yukawa potential for various choices of γ; compare with the exact results and study the convergence properties. Table II gives the same quantities for the Malfliet-Tjon potential. We do not present a similar table for the Reid soft-core potential which has already appeared in Ref. 10. (We repeated the calculations in this case and agree with the results of Ref. 10.)

The final results in Figs. 1–3 and Tables I and II show that both the off-shell t matrix elements and the on-shell elastic scattering phase shifts converge rapidly and uniformly, except in the case of the Reid soft-core potential where the convergence is slower because of the strong repulsive core this particular potential possesses. For other choices of γ where the method does not converge at all energies it converges as the energy is increased. Usually in such cases the iterative solution converges at a much lower energy compared to the original Lippmann-Schwinger equation.

IV. SUMMARY AND DISCUSSION

Here we critically analyze a recently proposed method for Lippmann-Schwinger equations both analytically and numerically. The method uses the solution of an auxiliary nonsingular equation whose kernel is free from the fixed point singularity of the original equation. We write the final solution in a form which enjoys all the advantages of the method proposed independently by Kowalski and Noyes which is simple to implement in practice. We test the accuracy of the method numerically by employing an iterative solution for the auxiliary nonsingular equation, and find that the method gives good convergence. When the iterative solution of the auxiliary equation does not converge, of course, we have to use some exact method for solving this equation. The method manifestly satisfies constraints of on-shell unitarity and hence yields real phase shifts corresponding to each order of iteration and can be used to make unitary approximations to the t matrix. Similar ideas can be used to make unitary approximations to the three-body t matrix.

There is, of course, an arbitrariness in the choice of the function γ in the present method. However, this arbitrariness can be turned to good advantage, as has been done in this paper, by varying the function so as to obtain the best convergence. With appropriate choice for γ the present method will be an efficient method for solving Lippmann-Schwinger equations.

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TABLE I. The on-shell phase shifts for the Yukawa potential for different N and the exact phase shifts at different energies. The exact results are taken from Ref. 11.

$E_{c.m.}$ (MeV)	Exact	γ	N = 1	2	4	6	8	10	12
12	1.5053	Choice A	0.9736	1.3264	1.5003	1.5087	1.5060	1.5054	1.5053
		Choice B	1.1313	1.3710	1.4820	1.5011	1.5046	1.5052	1.5053
		Choice C	2.0527	1.2186	1.5110	1.5054	1.5053	1.5053	1.5053
24	1.2818	Choice A	0.9037	1.2069	1.2929	1.2833	1.2819	1.2819	1.2819
		Choice B	0.9929	1.1975	1.2704	1.2803	1.2817	1.2819	1.2819
		Choice C	1.7864	0.8860	1.2881	1.2819	1.2819	1.2819	1.2819
48	1.0803	Choice A	0.8200	1.0694	1.0880	1.0803	1.0803	1.0803	1.0803
		Choice B	0.8689	1.0345	1.0755	1.0798	1.0803	1.0803	1.0803
		Choice C	1.5054	0.9714	1.0851	1.0804	1.0803	1.0803	1.0803
72	0.9725	Choice A	0.7675	0.9822	0.9768	0.9723	0.9725	0.9725	0.9725
		Choice B	0.8014	0.9426	0.9698	0.9723	0.9725	0.9725	0.9725
		Choice C	1.3413	0.9227	0.9761	0.9725	0.9725	0.9725	0.9725
104	0.8810	Choice A	0.7185	0.9007	0.8832	0.8809	0.8810	0.8810	0.8810
		Choice B	0.7426	0.8616	0.8795	0.8809	0.8810	0.8810	0.8810
		Choice C	1.1969	0.8588	0.8836	0.8810	0.8810	0.8810	0.8810
152	0.7928	Choice A	0.6670	0.8163	0.7936	0.7927	0.7928	0.7928	0.7928
		Choice B	0.6837	0.7808	0.7920	0.7928	0.7928	0.7928	0.7928
		Choice C	1.0557	0.7854	0.7945	0.7928	0.7928	0.7928	0.7928
176	0.7604	Choice A	0.6469	0.7842	0.7609	0.7604	0.7604	0.7604	0.7604
		Choice B	0.6614	0.7506	0.7598	0.7604	0.7604	0.7604	0.7604
		Choice C	1.0038	0.7565	0.7618	0.7604	0.7604	0.7604	0.7604

APPENDIX

Here we present a proof of Eq. (32). Using the iterative solution for Γ, the right-hand side (RHS) of Eq. (32) can be written as

$$\begin{aligned} \text{RHS} &= \langle k | [1 + t(E) G_0(E)][V + A(E) V + A^2(E) V + \cdots + A^N(E) V] | r \rangle \\ &= \langle k | t(E) | r \rangle + \langle k | [A(E) + t(E) G_0(E) A(E)] \Gamma_{N-1} | r \rangle . \end{aligned} \tag{A1}$$

TABLE II. Same as in Table I but for the Malfliet-Tjon potential. The exact results are taken from Ref. 11.

$E_{c.m.}$ (MeV)	Exact	γ	N = 1	2	4	6	8	10	12
12	1.0997	Choice C	0.1719	0.6055	1.0921	1.1005	1.0998	1.0997	1.0997
		Choice D	0.1840	0.6507	0.9799	1.0600	1.0851	1.0941	1.0976
24	0.8370	Choice C	0.1376	0.4682	0.8286	0.8371	0.8370	0.8370	0.8370
		Choice D	0.1570	0.5830	0.7997	0.8335	0.8367	0.8369	0.8370
48	0.5501	Choice C	0.0593	0.2794	0.5359	0.5500	0.5501	0.5501	0.5501
		Choice D	0.0712	0.4086	0.5334	0.5538	0.5504	0.5499	0.5501
72	0.3730	Choice C	−0.0034	0.1526	0.3577	0.3730	0.3730	0.3730	0.3730
		Choice D	−0.0042	0.2648	0.3605	0.3784	0.3729	0.3728	0.3731
104	0.2083	Choice C	−0.0673	0.0319	0.1928	0.2082	0.2083	0.2083	0.2083
		Choice D	−0.0834	0.1153	0.2009	0.2140	0.2076	0.2081	0.2084
152	0.0358	Choice C	−0.1366	−0.0949	0.0211	0.0356	0.0358	0.0358	0.0358
		Choice D	−0.1703	−0.0504	0.0363	0.0403	0.0347	0.0358	0.0358
176	−0.0311	Choice C	−0.1635	−0.1438	−0.0452	−0.0313	−0.0311	−0.0311	−0.0311
		Choice D	−0.2041	−0.1157	−0.0272	−0.0275	−0.0323	−0.0310	−0.0311

Next we show that with choice (29) for γ

$$\langle k|[A(E)+t(E)G_0(E)A(E)]|q\rangle=0\,, \tag{A2}$$

and the second term on the right-hand side of (A1) vanishes, and hence the right-hand side of Eq. (32) becomes equal to its left-hand side. In order to prove Eq. (A2) it is easy to see using Eqs. (1) and (4) that its left-hand side can be written as

$$\left[\langle k|V|q\rangle-\langle k|V|k\rangle\,\frac{\langle k|t(E)|q\rangle}{\langle k|t(E)|k\rangle}\right](k^2-q^2+i\epsilon)^{-1}+\left[\{\langle k|t(E)|q\rangle-\langle k|V|q\rangle\}\right.$$
$$\left.-\{\langle k|t(E)|k\rangle-\langle k|V|k\rangle\}\,\frac{\langle k|t(E)|q\rangle}{\langle k|t(E)|k\rangle}\right](k^2-q^2+i\epsilon)^{-1}=0\,. \tag{A3}$$

This completes the proof of Eq. (32).

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