Solvable model for pion-pion S and P waves derived from noncovariant perturbation theory

Walter Ferchländer and Dieter Schütte
Institut für Theoretische Kernphysik, Universität Bonn, Nussallee 14-16, D-5300 Bonn, West Germany
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Motivated by Lagrangian field theory, we construct an effective Hamilton operator for P- and S-wave pion-pion interactions, which can be handled easily by noncovariant perturbation methods. The result is a factorable T-matrix equation with unambiguous off-shell behavior, which includes the effects of intermediate states with $\epsilon(1300)$, $S^*(980)$, and $\rho(770)$ mesons as well as with $K\bar{K}$ and $N\bar{N}$ pairs and describes the influence of inelastic pair effects on elastic scattering. The numerical calculations are in good agreement with the experimental S_{00} and P_{11} amplitudes up to a scattering energy of 2.5 GeV; a range of fundamental importance for nuclear physics.

[NUCLEAR REACTIONS Pion-pion scattering; theory, S and P waves, field theoretical model, noncovariant perturbation theory.]

I. INTRODUCTION

For a long time field theoretic approaches to pion-pion interactions were almost wholly abandoned in favor of the S-matrix approach. The realization that the fields involved were not fundamental entities and that dispersion relations should give the right equations served as a justification for this development. Later, however, Lagrangian field theories came back with a new point of view: They were regarded as heuristic devices, without the claim of fundamental status for the fields, but nonetheless useful in elucidating those properties of pion-pion dynamics which cannot be understood by dispersion relation technics, e.g., questions of renormalization or off-shell processes.

The "classical" Lagrangian approaches to pion-pion interactions are the ϕ^4 theory[1] and the σ and ρ models.[2,3] These single-channel calculations gave some very encouraging results, especially in understanding the fundamental technics. A systematic study of multichannel effects has been made by Iagolnitzer *et al.*[4] His rather successful calculations were based on a Lagrangian which contains the fields associated with the octets of vector and pseudoscalar mesons with SU(3) symmetric masses. In all these models the scattering phase shifts were calculated in Padé approximation. A different procedure was used by Ecker and Honerkamp[5]: Based on a chiral invariant pion-nucleon Lagrangian—which includes four-vertices—they evaluated the pion-pion phase shifts using covariant perturbation theory and superpropagator methods. Their results are in good agreement with the data. In comparison with the above theories we start from a somewhat different point of view. Our fundamental Lagrangian only serves as suitable basis for an effective Hamiltonian which has to fulfill the following criteria:

(1) The T-matrix equation should be soluble up to all orders of perturbation expansion.

(2) The experimental S- and P-wave data, which are very important for nuclear physics, should be well reproduced in the low and intermediate energy region.

(3) The T matrix should show definite off-shell behavior and should be factorable in order that it can, e.g., be used for three-body calculations.

This was achieved by the following procedure. In Sec. II of this paper we present a rather complicated Lagrangian density combining the σ and ρ models with pion-nucleon and pion-kaon four-vertices. From this Lagrangian we extract an effective Hamilton operator by allowing only a certain class of suitable and physically dominant diagrams. Via noncovariant perturbation theory we arrive at a T-matrix integral equation with definite off-shell behavior which can be summed up directly because of the separable structure of the Born term. This will be done in Sec. III resulting in a factorable structure of the T matrix itself. Section IV is devoted to the determination of the coupling constants and to the definition of the form factors. In Sec. V we make a short remark about numerical questions, and Sec. VI presents the numerical results of our calculations.

Before beginning with our model, a short comment about using noncovariant perturbation theory in problems with relativistic particles involved: It is well known that covariant and noncovariant perturbation theories would yield the same results if both series were completely summed. But nature is not kind, and we must neglect most of the possible diagrams in either case in order to get soluble problems. Even if we perform such a restriction under formally equivalent principles (like one-boson exchange for *NN* scattering), the two methods may now lead to different results since there is no one-to-one correspondence between single diagrams or even between classes of diagrams of both perturbation series. It is by no means clear that the reduced covariant problem is *a priori* more relativistic than the reduced noncovariant one. Therefore, the best one can do is to adopt a pragmatic standpoint and decide which method works *a posteriori*. It is in this spirit that we use noncovariant perturbation theory for the subject of this paper.

Another motivation for the use of noncovariant perturbation theory is that only in this way can one hope to get insight into the role of the pion-pion interaction in intermediate energy phenomena.

II. OUTLINE OF THE FORMALISM

The aim of this section is the construction of a noncovariant T-matrix integral equation for pion-pion S and P waves with unambiguous off-shell behavior. We start from the following interaction Lagrangian which couples pions to scalar-isoscalar and vector-isovector resonance mesons [$\epsilon(1300)$, $S^*(980)$, and $\rho(770)$] and likewise—via four-vertices—to nucleons, antinucleons, kaons, and antikaons:

$$\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_2 + \mathfrak{L}_3$$

with

$$\begin{aligned}\mathfrak{L}_1 &\equiv 2mF_0\sigma(x)\vec{\phi}(x)\cdot\vec{\phi}(x) + F_1\vec{\rho}^{\mu}(x)\cdot\vec{\phi}(x)\times[\partial_\mu\vec{\phi}(x)],\\ \mathfrak{L}_2 &\equiv \frac{\lambda_0^2}{2M}\bar{\psi}(x)\psi(x)\vec{\phi}(x)\cdot\vec{\phi}(x) + \frac{\lambda_1^2}{4M^2}\bar{\psi}(x)\gamma^\mu\vec{\tau}\psi(x)\cdot\vec{\phi}(x)\times[\partial_\mu\vec{\phi}(x)]\\ \mathfrak{L}_3 &\equiv \alpha_0^2\xi^\dagger(x)\xi(x)\vec{\phi}(x)\cdot\vec{\phi}(x) + \frac{\alpha_1^2}{4M_K^2}\xi^\dagger(x)\vec{\tau}\,\overleftrightarrow{\partial}^\mu\xi(x)\cdot\vec{\phi}(x)\times[\partial_\mu\vec{\phi}(x)],\end{aligned} \tag{2.1}$$

with

$$A\overleftrightarrow{\partial}^\mu B \equiv A(\partial^\mu B) - (\partial^\mu A)B$$

and $x=(x_0,\vec{x})$. Here m, M, and M_K are the masses of pions, nucleons, and kaons, and $F_{0/1}$, $\lambda_{0/1}$, and $\alpha_{0/1}$ are the related coupling constants. The field operators are pions

$$\phi_j(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{(2\omega_k)^{1/2}}\times[b_j(\vec{k})e^{-ikx} + b_j^\dagger(\vec{k})e^{ikx}]$$

with

$$\omega_k = (m^2 + \vec{k}^2)^{1/2},$$

scalar mesons

$$\sigma(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\frac{1}{(2\Omega_k)^{1/2}}\times[c(\vec{k})e^{-ikx} + c^\dagger(\vec{k})e^{ikx}]$$

with

$$\Omega_k = (m_s^2 + \vec{k}^2)^{1/2},$$

vector mesons

$$\rho_j^\mu(x) = \frac{1}{(2\pi)^{3/2}}\sum_{s=1}^{3}\int d^3k\,\frac{\epsilon^{\mu s}(\vec{k})}{(2\Omega_k)^{1/2}}\times[c_{js}(\vec{k})e^{-ikx} + c_{js}^\dagger(\vec{k})e^{ikx}]$$

with

$$\Omega_k = (m_v^2 + \vec{k}^2)^{1/2},$$

nucleons

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}}\sum_{r,t=1}^{3}\int d^3p[a_{rt}^\dagger(\vec{p})\bar{u}_r(\vec{p})e^{ipx} + \bar{a}_{rt}(\vec{p})\bar{v}_r(\vec{p})e^{-ipx}]\eta_t^\dagger,$$

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\sum_{r,t=1}^{3}\int d^3p[a_{rt}(\vec{p})u_r(\vec{p})e^{-ipx} + \bar{a}_{rt}^\dagger(\vec{p})v_r(\vec{p})e^{ipx}]\eta_t$$

with

$$\bar{\psi} \equiv \psi^\dagger\gamma^0;$$

and kaons

$$\xi^\dagger(x) = \frac{1}{(2\pi)^{3/2}}\sum_{t=1}^{3}\int\frac{d^3p}{(2\epsilon_p)^{1/2}}[d_t^\dagger(\vec{p})e^{ipx} + \bar{d}_t(\vec{p})e^{-ipx}]\eta_t^\dagger,$$

$$\xi(x) = \frac{1}{(2\pi)^{3/2}}\sum_{t=1}^{3}\int\frac{d^3p}{(2\epsilon_p)^{1/2}}[d_t(\vec{p})e^{-ipx} + \bar{d}_t^\dagger(\vec{p})e^{ipx}]\eta_t$$

with

$$\epsilon_p = (M_K^2 + \vec{p}^2)^{1/2}. \tag{2.2}$$

Here $b^\dagger$, $c^\dagger$, $a^\dagger$, $\bar{a}^\dagger$, $d^\dagger$, and $\bar{d}^\dagger$ are the corresponding creation operators for pions, resonance mesons, nucleons, antinucleons, kaons, and antikaons. η_t are Pauli spinors for isospin, $u_r(\vec{p})$ and $v_r(\vec{p})$ Dirac spinors for the nucleons, and $\epsilon_s^\mu(\vec{k})$ the polarization vector for the vector mesons (see Appendix).

The canonical transition from the Lagrangian density (2.1) to the Hamiltonian density $\mathcal{H}$ is given by[6]

$$\mathcal{H} = -\mathfrak{L} + \text{contact terms}. \tag{2.3}$$

It is surely not possible to handle this equation in a general way. Therefore, in the spirit of the Lee model,[7] we define a Hamiltonian density by retaining only those terms which lead to vertices with different incoming and outgoing particle lines:

$$\mathcal{H}_{\text{eff}} = -\mathcal{L}\big|_{\text{only Lee terms}} \quad . \tag{2.4}$$

Setting

$$H = \int d^3x\,{:}\mathcal{H}{:}\big|_{x_0=0} \tag{2.5}$$

and using the explicit expansion of the field operators (2.2), we arrive at a Hamilton operator which is suitable to noncovariant perturbation theory:

$$H = T_0{}^0 + T_0{}^1 + T_0{}^2 + T_0{}^3 + W^1 + W^2 + W^3$$
$$\equiv H^0 + W$$

with

$$T_0{}^0 \equiv \sum_k \omega_k^0 b_k^\dagger b_k \,,$$

$$T_0{}^1 \equiv \sum_{n,q} \Omega_q^{n0} c_q{}^{n\dagger} c_q^n \,,$$

$$T_0{}^2 \equiv \sum_\alpha E_\alpha^0 [a_\alpha^\dagger a_\alpha + \bar{a}_\alpha^\dagger \bar{a}_\alpha] \,,$$

$$T_0{}^3 \equiv \sum_\beta \epsilon_\beta^0 [d_\beta^\dagger d_\beta + \bar{d}_\beta^\dagger d_\beta] \,, \tag{2.6}$$

$$W^1 = \tfrac{1}{2} \sum_{n,klq} [V_{klq}^{1n} b_k^\dagger b_l^\dagger c_q^n + \text{H.c.}] \,,$$

$$W^2 = \tfrac{1}{2} \sum_{n,kl\alpha\beta} [W_{kl\beta\alpha}^{2n} b_k^\dagger b_l^\dagger \bar{a}_\alpha a_\beta + \text{H.c.}] \,,$$

$$W^3 = \tfrac{1}{2} \sum_{n,kl\alpha\beta} [W_{kl\beta\alpha}^{3n} b_k^\dagger b_l^\dagger \bar{d}_\alpha d_\beta + \text{H.c.}] \,.$$

In these equations the index n runs over different resonance mesons and different couplings. The other indices represent momentum, spin, and isospin of the corresponding particle, e.g., $q = (\vec{q}, s_q, t_q)$. Since we are using continuum normalization for the field operators, the summation also contains all momentum integrations. Here ω_k^0, Ω_q^{n0}, E_α^0, and ϵ_α^0 are the free and undressed energies of pions, resonance mesons, nucleons, and kaons. The matrix elements V_{klq}^{1n}, $W_{kl\beta\alpha}^{2n}$, and $W_{kl\beta\alpha}^{3n}$ are explicitly given in the Appendix.

Because of the special structure of our Hamilton operator only the energies of the resonance mesons must be renormalized; the other free energies can be identified with the physical ones, e.g., $E_\alpha^0 \equiv E_\alpha = (M^2 + p_\alpha^2)^{1/2}$ for the nucleons. The calculation of the self-energy corrections for the resonance mesons is a very difficult task, because there are, in principle, many intermediate states. If we confine ourselves to the main process, i.e., the decay of the meson into two pions, the self-energy is simply

$$\delta\Omega_q^n = \Omega_q^n - \Omega_q^{n0} = \left\langle 0 \,\middle|\, c_q^n W^1 \mathcal{P} \frac{1}{\Omega_q^n - H^0} W^1 c_q^{n\dagger} \,\middle|\, 0 \right\rangle . \tag{2.7}$$

Here $\mathcal{P}$ means Cauchy's principal value, because real decays are involved. This approximation is very good, because each resonance meson governs the whole scattering amplitude in the region of its pole.

Application of noncovariant perturbation theory to Hamiltonian (2.6) yields the T-matrix operator equation

$$T(z) = W + W \frac{1}{z - H^0 + i\epsilon} T(z) \ , \tag{2.8}$$

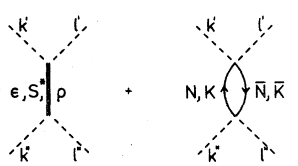
where z is the scattering energy. For elastic pion-pion scattering this leads to the integral equation

$$T_{k'l'kl} = U_{k'l'kl}^{kl} + \tfrac{1}{2} \sum_{k''l''} \frac{U_{k'l'k''l''}^{kl} T_{k''l''kl}}{\omega_k + \omega_l - \omega_{k''} - \omega_{l''} + i\epsilon} \tag{2.9}$$

whose Born term defines an energy dependent and complex momentum space potential with unambiguous off-shell behavior:

$$U_{k'l'k''l''}^{kl} = \sum_{n,p} \frac{V_{k'l'p}^{1n} V_{k''l''p}^{1n*}}{\omega_k + \omega_l - \Omega_p^{n0} + i\epsilon} + \sum_{mn,\alpha\beta} \frac{W_{k'l'\alpha\beta}^{2m} W_{k''l''\alpha\beta}^{2n*}}{\omega_k + \omega_l - E_\alpha - E_\beta + i\epsilon} + \sum_{mn,\alpha\beta} \frac{W_{k'l'\alpha\beta}^{3m} W_{k''l''\alpha\beta}^{3n*}}{\omega_k + \omega_l - \epsilon_\alpha - \epsilon_\beta + i\epsilon} \tag{2.10}$$

or graphically

$U_{k'l'k''l''}^{kl} =$  (2.11)

The matrix elements of this potential in the center-of-mass system of the pions are given in the Appendix.

III. SOLUTION OF THE SCATTERING EQUATION

Turning to an angular momentum representation we get the following one-dimensional integral equation:

$$T^{LT}_{\omega'\omega}(z)=U^{LT}_{\omega'\omega}(z)+\tfrac{1}{8}\int_{2m}^{\infty} dE\, kE\, \frac{U^{LT}_{\omega' E}(z)T^{LT}_{E\omega}(z)}{z-E+i\epsilon} \quad . \tag{3.1}$$

Here L and T are angular momentum and total isospin; ω', ω, and E are the total energies of the two pions, and $z=\omega$ for scattering. The individual terms of the pseudopotential are

$$\begin{aligned}
U^{100}_{\omega'\omega}(z)&=\left[\sum_s \frac{F_s^{\,2}}{4\pi}\,\frac{1}{m_s(z-m_s^0)}\right]\frac{48m^2}{\pi}\,\frac{1}{\omega'\omega}\,F^{10}(\omega',\omega)\ ,\\
U^{111}_{\omega'\omega}(z)&=\left[\sum_v \frac{F_v^{\,2}}{4\pi}\,\frac{1}{m_v(z-m_v^0)}\right]\frac{8}{3\pi}\,\frac{k'k}{\omega'\omega}\,F^{11}(\omega',\omega)\ ,\\
U^{200}_{\omega'\omega}(z)&=\left(\frac{\lambda_0^{\,2}}{4\pi}\right)^2\frac{48}{\pi^2M^2}\,\frac{1}{\omega'\omega}\left[\int_{2M}^{\infty}\frac{dE}{E(z-E)}\,(p^3-p_z^3)F^{20}(\omega',\omega,E)+\frac{p_z^{\,3}}{z}\right.\\
&\qquad\left.\times\left(\ln\left|\frac{z}{2M}-1\right|-i\pi\theta(z-2M)\right)\right]F^{20}(\omega',\omega,z),\\
U^{211}_{\omega'\omega}(z)&=\left(\frac{\lambda_1^{\,2}}{4\pi}\right)^2\frac{8}{3\pi^2M^4}\,\frac{k'k}{\omega'\omega}\left\{\int_{2M}^{\infty}\frac{dE}{E(z-E)}\,[(p-p_z)M^2+\tfrac{2}{3}(p^3-p_z^{\,3})]\,F^{21}(\omega',\omega,E)\right.\\
&\qquad\left.+\frac{1}{z}\,(p_zM^2+\tfrac{2}{3}p_z^{\,3})\left[\ln\left|\frac{z}{2M}-1\right|-i\pi\theta(z-2M)\right]F^{21}(\omega',\omega,z)\right\},\\
U^{300}_{\omega'\omega}(z)&=\left(\frac{\alpha_0^{\,2}}{4\pi}\right)^2\frac{24}{\pi^2}\,\frac{1}{\omega'\omega}\left\{\int_{2M_K}^{\infty}\frac{d\epsilon}{\epsilon(z-\epsilon)}\,(p-p_z)F^{30}(\omega',\omega,\epsilon)+\frac{p_z}{z}\left[\ln\left|\frac{z}{2M_K}-1\right|-i\pi\theta(z-2M_K)\right]F^{30}(\omega',\omega,z)\right\},\\
U^{311}_{\omega'\omega}(z)&=\left(\frac{\alpha_1^{\,2}}{4\pi}\right)^2\frac{4}{9\pi^2M_K^{\,4}}\,\frac{k'k}{\omega'\omega}\left\{\int_{2M_K}^{\infty}\frac{d\epsilon}{\epsilon(z-\epsilon)}\,(p^3-p_z^{\,3})+F^{31}(\omega',\omega,\epsilon)\right.\\
&\qquad\left.+\frac{p_z^{\,3}}{z}\left[\ln\left|\frac{z}{2M_K}-1\right|-i\pi\theta(z-2M_K)\right]F^{31}(\omega',\omega,z)\right\}.
\end{aligned} \tag{3.2}$$

In these equations the first subscript refers to the different intermediate states: 1 for resonance mesons, 2 for $N\bar{N}$, and 3 for $K\bar{K}$; the other two subscripts refer to L and T.

The unrenormalized masses of the scalar (s) and vector (v) mesons are

$$m^0_{s/v}=m_{s/v}-\delta m_{s/v}$$

with

$$\begin{aligned}
\delta m_s&=\frac{F_s^{\,2}}{4\pi}\,\frac{6m^2}{\pi m_s}\left[\int_{2m}^{\infty}d\omega\,\frac{k-k_{m_s}}{\omega(m_s-\omega)}\,F^{10}(\omega,\omega)\right.\\
&\qquad\left.+\frac{k_{m_s}}{m_s}\ln\left(\frac{m_s}{2m}-1\right)\right]F^{10}(m_s,m_s)\ ,\\
\delta m_v&=\frac{F_v^{\,2}}{4\pi}\,\frac{1}{3\pi m_v}\left[\int_{2m}^{\infty}d\omega\,\frac{k^3-k^3_{m_v}}{\omega(m_v-\omega)}\,F^{11}(\omega,\omega)\right.\\
&\qquad\left.+\frac{k^3_{m_v}}{m_v}\ln\left(\frac{m_v}{2m}-1\right)\right]F^{11}(m_v,m_v)\ .
\end{aligned} \tag{3.3}$$

In these equations E and ϵ are the total intermediate energies of the $K\bar{K}$ and $N\bar{N}$ pairs, $\theta(x)$ is the usual Heaviside unit step function, and the F^{ij} functions are form factors which are necessary to guarantee convergence of the T-matrix equation (see Sec. IV). p_z means the momentum corresponding to energy z.

Integral equations like (3.1) can numerically be solved by the well-known method of matrix inversion. But if we choose separable form factors, the total potential remains factorable:

$$U^{LT}_{\omega'\omega}(z)=g^{LT}(z)f^{LT}(\omega')f^{LT}(\omega)\ . \tag{3.4}$$

This structure allows the algebraic summation of the T matrix.[8] The result is also factorable:

$$T^{LT}_{\omega'\omega}(z)=\frac{f^{LT}(\omega')f^{LT}(\omega)}{\dfrac{1}{g^{LT}(z)}-\tfrac{1}{8}\int_{2m}^{\infty}dE\,k_E E\,\dfrac{f^{LT}(E)^2}{z-E+i\epsilon}}\ . \tag{3.5}$$

Turning to on-shell values and multiplying with the kinematic factor $-\pi k_\omega \omega/8$, we get the usual normalized scattering amplitude

$$A^{LT}(\omega)=\frac{\Gamma^{LT}(\omega)}{\dfrac{1}{g^{LT}(\omega)}+\dfrac{1}{\pi}\int_{2m}^{\infty}dE\,\dfrac{\Gamma^{LT}(E)}{\omega-E+i\epsilon}} \tag{3.6}$$

with

$$\Gamma^{LT}(\omega)=-\frac{\pi}{8}\,k_\omega\,\omega f^{LT}(\omega)^2\,.$$

IV. COUPLING CONSTANTS AND FORM FACTORS

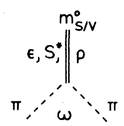
Because there are vector mesons and pair terms in our theory, we cannot hope to get well-defined scattering amplitudes without form factors even with renormalized coupling constants. Therefore, we prefer to regard the coupling constants of the pair terms as free parameters and to relate the others to the experimental widths. This relation is easily established since the experimental amplitudes show typical Breit-Wigner behavior in the regions of the resonances. This means for our theory that it should be a good approximation to determine the coupling constants from the Breit-Wigner equation for the corresponding resonance meson alone. The procedure is as follows. If we write Eq. (3.6) for a single resonance meson, it shows—after some simple algebraic manipulations—a sort of Breit-Wigner form:

$$A^{LT}(\omega)=\frac{-\frac{1}{2}\gamma^{LT}}{\omega-m_{s/v}+\delta m_{s/v}-\Delta^{LT}(\omega)+\dfrac{i}{2}\gamma^{LT}(\omega)}\,, \tag{4.1}$$

where $\gamma^{LT}(m_{s/v})$ is the width and proportional to the coupling constant.

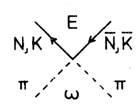
For the three- and four-vertices we use the following phenomenological and factorable form factors:

$$\frac{\Lambda_1{}^2+m_{s/v}{}^2}{\Lambda_1{}^2+\omega^2}\left(\frac{\omega}{m_{s/v}}\right)^{1-L}\quad\text{for}$$



and

$$\frac{\Lambda_1{}^2+(2M_{N/K})^2}{\Lambda_1{}^2+\omega^2}\left(\frac{\omega}{2M_{N/K}}\right)^{1-L}\left(\frac{\Lambda_2{}^2+(2M_{N/K})^2}{\Lambda_2{}^2+E^2}\right)^2\quad\text{for}$$



The cutoff masses Λ_1 and Λ_2 have to be fitted to the scattering data.

V. NUMERICAL REMARK

The numerical integrations were performed by taking 48 Gauss points E_1 and weights dE_i according to

$$E_i=500\tan(\tfrac{1}{2}\pi E_i')+2M\,,$$
$$dE_i=500\,\frac{\frac{1}{2}\pi dE_i'}{\cos^2(\frac{1}{2}\pi E_i')}\,, \tag{5.1}$$

where E_i' and dE_i' are taken from the interval (0, 1); $2M$ is the threshold of the corresponding particle pair.

VI. RESULTS

For our parameter fit we tried to fix the input data as close as possible to experimental values contained in the particle tables.[11] The best fit parameters and their experimental equivalents are given in Tables I and II. Figures 1 and 2 show our S- and P-wave amplitudes compared to the results of the partial wave analysis of Froggatt and Petersen.[9] We see that our results are in good agreement with the experimental data.

(1) S_{00} wave. The main structure of the S_{00} amplitude up to an energy of about 1400 MeV is dominated by the ϵ meson with a mass of 1300 MeV and a resonance width of 300 MeV. The S^* meson with a mass of 980 MeV and a width of 40 MeV causes a characteristic cut at nearly 1000 MeV, which is strongly damped above $K\bar{K}$ threshold by the creation of real kaon pairs. Above 1400 MeV the real and imaginary parts of the amplitude are rapidly raised by intermediate $N\bar{N}$ pairs, resulting in a resonance-like structure at 2200 MeV, which is too high in comparison with experiment. The influence of virtual $K\bar{K}$ and $N\bar{N}$ pairs can also be seen in the low energy region, especially in the scattering length

$$a_{00}=\lim_{p\to 0}\ \mathrm{Re}A^{00}(\omega)\,\frac{m}{p}=0.26,\quad \omega=2(m^2+p^2)^{1/2} \tag{6.1}$$

TABLE I. Parameters of particles. $I^G(J^P)$: Isospin$^{G\text{ parity}}$ (Spin$^{\text{parity}}$).

Particle	$I^G(J^P)$	Mass (MeV) Exp. (Ref. 11)	Mass (MeV) Theory	Width (MeV) Exp. (Ref. 11)	Width (MeV) Theory
p	$\frac{1}{2}(\frac{1}{2}^+)$	938.28	938.9		
n	$\frac{1}{2}(\frac{1}{2}^+)$	939.57	938.9		
$K^\pm$	$\frac{1}{2}(0^-)$	493.67	497.7		
K^0	$\frac{1}{2}(0^-)$	497.67	497.7		
$\pi^\pm$	$1^-(0^-)$	139.57	138	0.0	
π^0	$1^-(0^-)$	134.96	138	(7.95 ± 0.55) eV	
ρ	$1^+(1^-)$	776 ± 3	776	155 ± 3	155
S^*	$0^+(0^+)$	980 ± 10	980	40 ± 10	40
ϵ	$0^+(0^+)$	$\simeq1300$	1300	$200-400$	300

which corresponds the experimental result at 0.26 ± 0.05.[9,10]

(2) P_{11} wave. The P_{11} amplitude is governed by the ρ meson with a mass of 776 MeV and a width of 155 MeV. $K\bar{K}$ and $N\bar{N}$ pairs cause some structure in the ρ' region, but cannot reproduce the whole experimental resonance at about 1600 MeV. The introduction of a particular ρ'-resonance meson would lead to difficulties in our theory. The experimental ρ' decays into four pions with a probability of about 75% and into two pions with only a 25% probability,[11] from which one can estimate an absorption parameter of roughly 0.25 for elastic $\pi\pi$ scattering.

This inelasticity cannot be reproduced by our model, because we do not incorporate 4π decays. The result would be a nearly undamped ρ' resonance at 1600 MeV, and therefore we prefer to forget about it and be content with calculating that small part of the ρ' structure which stems from the decay into kaon pairs. The scattering length in this partial wave is

$$a_{11}=\lim_{p\to0}\mathrm{Re}A^{11}(\omega)\left(\frac{m}{p}\right)^3=0.038 \qquad (6.2)$$

which agrees with experiment: 0.038 ± 0.002.[10]

TABLE II. Coupling constants and cutoff masses.

Vertex	Coupling constant	Cutoff masses (MeV)
$\pi\pi N\bar{N}$	$\lambda_0^2/4\pi=3.10$	1200/1400
	$\lambda_1^2/4\pi=1.00$	1800/1200
$\pi\pi K\bar{K}$	$\alpha_0^2/4\pi=1.45$	1200/1400
	$\alpha_1^2/4\pi=8.00$	1800/1200
$\pi\pi S$ [a]	$F_0^2/4\pi=0.36$ [a]	1200
$\pi\pi\epsilon$	$F_0^2/4\pi=3.49$ [a]	1200
$\pi\pi\rho$	$F_1^2/4\pi=2.94$ [a]	1800

[a] From experimental width.

VII. CONCLUSION

Motivated by Lagrangian field theory, we have derived an effective Hamilton operator for pion-pion S and P waves which, we believe, contains only physically relevant processes. We showed that this Hamilton operator can be handled easily within noncovariant perturbation theory in spite of relativistic particles being involved. The free parameters could be fixed by fitting the experimental data without difficulty. The result is a factorable T matrix with unambiguous off-shell behavior, which reproduces S- and P-wave scattering well, and which can now be incorporated

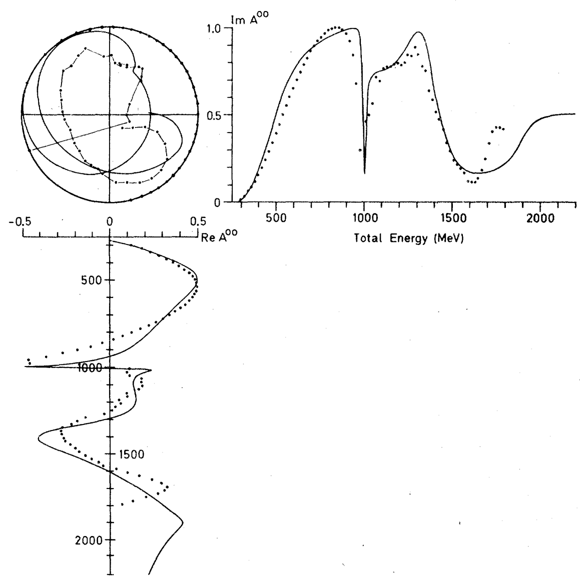


FIG. 1. S_{00} amplitude for elastic pion-pion scattering (solid lines) compared to the phase-shift analysis (point lines) of Froggatt and Petersen (Ref. 9).

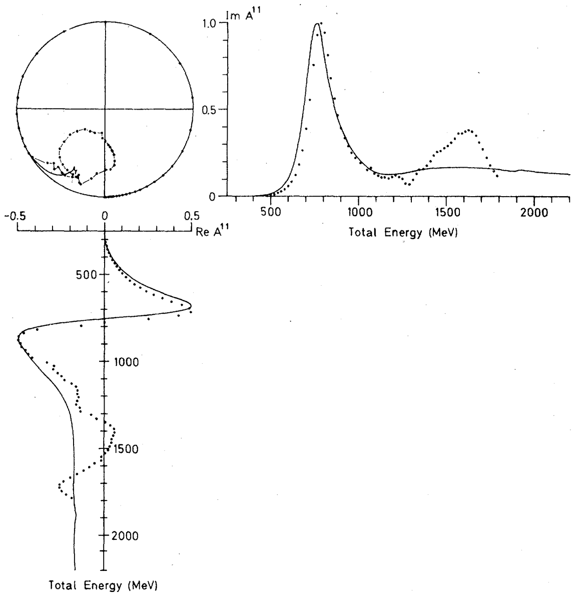


FIG. 2. P_{11} amplitude for elastic pion-pion scattering (solid lines) compared to the phase-shift analysis (point lines) of Froggatt and Petersen (Ref. 9).

in problems of nuclear physics, e.g., in nuclear matter calculations or three-body reactions.

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APPENDIX

For our calculations we use the following representation of the Dirac spinors and the polarization vector:

$$u_r(\vec{p}) = \left(\frac{E_p + M}{2E_p}\right)^{1/2} \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p + M}\end{pmatrix} \eta_r ,$$

$$v_r(\vec{p}) = \gamma^5 u_r(\vec{p}) , \qquad \text{(A1)}$$

$$\epsilon^{is}(\vec{k}) = -g^{is} + \frac{k^i k^s}{m_v(m_v + \Omega_k)} ,$$

$$\epsilon^{0s}(\vec{k}) = k^s/m_v ,$$

where η_r is the usual Pauli spinor.

The vertex functions of our effective interaction Hamiltonian operator (2.6) are

$$V^{10}_{klq} = V^{10*}_{klq} = -\frac{F_0}{\sqrt{4\pi}} \frac{m}{\pi(\omega_k \omega_l \Omega_q)^{1/2}} \delta(\vec{k} + \vec{l} - \vec{q})\delta_{j_k j_l} ,$$

$$V^{11}_{klq} = -V^{11*}_{klq}$$

$$= \frac{F_1}{\sqrt{4\pi}} \frac{\epsilon^{\mu s}(\vec{q})(k_\mu - l_\mu)}{4\pi(\omega_k \omega_l \Omega_q)^{1/2}} \delta(\vec{k} + \vec{l} - \vec{q}) i\epsilon_{j_k j_l j_q} ,$$

$$W^{20}_{kl\beta\alpha} = -\frac{\lambda_0^2}{4\pi} \frac{\bar{v}_{r\alpha}(\vec{p}_\alpha) u_{r\beta}(\vec{p}_\beta)}{4\pi^2 M(\omega_k\omega_l)^{1/2}} \delta(\vec{p}_\alpha + \vec{p}_\beta - \vec{k} - \vec{l})$$

$$\times \eta^\dagger_{t\alpha} \eta_{t\beta} \delta_{j_k j_l} ,$$

$$W^{20*}_{kl\alpha\beta} = -\frac{\lambda_0^2}{4\pi} \frac{\bar{u}_{r\alpha}(\vec{p}_\alpha) v_{r\beta}(\vec{p}_\beta)}{4\pi^2 M(\omega_k\omega_l)^{1/2}} \delta(\vec{p}_\alpha + \vec{p}_\beta - \vec{k} - \vec{l})$$

$$\times \eta^\dagger_{t\alpha} \eta_{t\beta} \delta_{j_k j_l} ,$$

$$W^{21}_{kl\beta\alpha} = \frac{\lambda_1^2}{4\pi} \frac{\bar{v}_{r\alpha}(\vec{p}_\alpha) i\gamma^\mu u_{r\beta}(\vec{p}_\beta)(k_\mu - l_\mu)}{16\pi^2 M^2(\omega_k\omega_l)^{1/2}} \qquad \text{(A2)}$$

$$\times \delta(\vec{p}_\alpha + \vec{p}_\beta - \vec{k} - \vec{l}) \eta^\dagger_{t\alpha} \tau^j \eta_{t\beta} \epsilon_{j j_k j_l} ,$$

$$W^{21*}_{kl\alpha\beta} = -\frac{\lambda_1^2}{4\pi} \frac{\bar{u}_{r\alpha}(\vec{p}_\alpha) i\gamma^\mu v_{r\beta}(\vec{p}_\beta)(k_\mu - l_\mu)}{16\pi^2 M^2(\omega_k\omega_l)^{1/2}}$$

$$\times \delta(\vec{p}_\alpha + \vec{p}_\beta - \vec{k} - \vec{l}) \eta^\dagger_{t\alpha} \tau^j \eta_{t\beta} \epsilon_{j j_k j_l} ,$$

$$W^{30}_{kl\beta\alpha} = W^{30*}_{kl\alpha\beta} = -\frac{\alpha_0^2}{4\pi} \frac{1}{4\pi^2(\omega_k\omega_l\epsilon_\alpha\epsilon_\beta)^{1/2}}$$

$$\times \delta(\vec{p}_\alpha + \vec{p}_\beta - \vec{k} - \vec{l}) \eta^\dagger_{t\alpha} \eta_{t\beta} \delta_{j_k j_l} ,$$

$$W^{31}_{kl\beta\alpha} = W^{31*}_{kl\alpha\beta} = -\frac{\alpha_1^2}{4\pi} \frac{(k_\mu - l_\mu)(p^\mu_\alpha - p^\mu_\beta)}{32\pi^2 M_K^2(\omega_k\omega_l\epsilon_\alpha\epsilon_\beta)^{1/2}}$$

$$\times \delta(\vec{p}_\alpha + \vec{p}_\beta - \vec{k} - \vec{l}) \eta^\dagger_{t\alpha} \tau^j \eta_{t\beta} \epsilon_{j j_k j_l} .$$

In the center-of-mass system of the two pions we get six potential terms:

$$U^{10}_{\vec{k}'\vec{k}}(z) = \frac{F_0^2}{4\pi} \frac{m^2}{\pi^2 m_s} \frac{1}{z - m_s^0} \frac{1}{\omega_{k'}\omega_k} \delta_{j'l'} \delta_{jl} F^{10}(k', k) ,$$

$$U^{11}_{\vec{k}'\vec{k}}(z) = \frac{F_1^2}{4\pi} \frac{1}{4\pi^2 m_v} \frac{1}{z - m_v^0} \frac{k'k\cos\theta}{\omega_{k'}\omega_k} (\delta_{j'j}\delta_{l'l} - \delta_{j'l}\delta_{l'j}) F^{11}(k', k, \theta) ,$$

$$U^{20}_{\vec{k}'\vec{k}}(z) = \left(\frac{\lambda_0^2}{4\pi}\right)^2 \frac{1}{\pi^3 M^2} \frac{1}{\omega_{k'}\omega_k} \int_0^\infty dp \frac{p^4}{E_p^2(z - 2E_p + i\epsilon)} \delta_{j'l'} \delta_{jl} F^{20}(k', k, p) , \qquad \text{(A3)}$$

$$U^{21}_{\vec{k}'\vec{k}}(z) = \left(\frac{\lambda_1^2}{4\pi}\right)^2 \frac{1}{4\pi^3 M^4} \frac{k'k\cos\theta}{\omega_{k'}\omega_k} \int_0^\infty dp \frac{p^4}{E_p^2(z - 2E_p + i\epsilon)} (M^2/p^2 + \tfrac{2}{3})(\delta_{j'j}\delta_{l'l} - \delta_{j'l}\delta_{l'j}) F^{21}(k', k, p, \theta) ,$$

$$U^{30}_{\vec{k}'\vec{k}}(z) = \left(\frac{\alpha_0^2}{4\pi}\right)^2 \frac{1}{\pi^3 M_K^2} \frac{1}{\omega_{k'}\omega_k} \int_0^\infty dp \frac{p^4}{\epsilon_p^2(z - 2\epsilon_p + i\epsilon)} \frac{M_K^2}{2p^2} \delta_{j'l'} \delta_{jl} F^{30}(k', k, p, \theta) ,$$

$$U^{31}_{\vec{k}'\vec{k}}(z) = \left(\frac{\alpha_1^2}{4\pi}\right)^2 \frac{1}{4\pi^3 M_K^4} \frac{k'k\cos\theta}{\omega_{k'}\omega_k} \int_0^\infty dp \frac{p^4}{\epsilon_p^2(z - 2\epsilon_p + i\epsilon)} \times \tfrac{1}{6}(\delta_{j'j}\delta_{l'l} - \delta_{j'l}\delta_{l'j}) F^{31}(k', k, p, \theta) .$$

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