Iterative solution of multichannel three-body equations

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A recently proposed method for solving scattering equations is generalized to the case of multichannel scattering equations. In the present work we write the final result of the method for multichannel scattering equations in such a way that it has all the important features of a related method for single channel Lippmann-Schwinger-type equations proposed by Kowalski and Noyes but is more general in practice. The method relies on the introduction of an auxiliary equation containing an arbitrary function. The kernel of the auxiliary equation is in general weaker in nature than the original kernel and hence the auxiliary equation is expected to have a (rapidly) convergent iterative solution. It is suggested that the method could be an efficient method for solving three-body scattering equations. Using the iterative solution of the auxiliary equation, the method is used numerically to compute fully off-shell t matrix elements for the spin doublet and the spin quartet neutron-deuteron scattering in the Amado model. The iterative solution of the auxiliary equation is found to converge significantly faster than the conventional Padé technique–an unexpected result–if the freedom in the choice of the arbitrary function is exploited.

[NUCLEAR REACTIONS Multichannel scattering equations, iterative solution, spin $\frac{1}{2}$ and spin $\frac{3}{2}$ neutron-deuteron scattering, Amado model, off-shell t matrix elements and phase shifts computed.]

I. INTRODUCTION

We reexamine a recently proposed method[1,2] for solving fully off-shell multichannel scattering equations. The method relies on solving an auxiliary equation whose kernel contains an arbitrary flexible function. The solution of the original equation is then related to that of the auxiliary equation. If the freedom in the choice of the arbitrary function is exploited the kernel of the auxiliary equation can be made sufficiently weak in order to have a convergent iterative solution. Another advantage of the method is that certain types of fixed point singularities can be removed from the kernel of the original equation so that the kernel of the auxiliary equation has a less singular or nonsingular structure.

The essence of the method of Ref. 1 (I) and Ref. 2 (II) is the following. A single-channel one-variable partial-wave Lippmann-Schwinger-type equation can be written as

$$t(p,r;E) = V(p,r) + \int dq\, q^2 f(q)\, V(p,q) G_0(q) t(q,r;E), \tag{1.1}$$

with $G_0(q) = (k^2 - q^2 + i\epsilon)^{-1}$, $k^2 = E$ in units $\hbar = 2\mu = 1$, and where p, q, r are momentum variables and μ is the reduced mass. Unless otherwise specified, the integration limits in Eq. (1.1) and in the rest of the paper are from 0 to ∞. The function $f(q)$ is some weight function, which in the case of partial-wave Lippmann-Schwinger equations is a constant λ. Following I and II we introduce the operators A, $\bar{V}$, and H_0 defined by

$$\langle p|A(E)|q\rangle = [\langle p|V|q\rangle - \langle p|V|k\rangle \gamma(k,q)]\, G_0(q), \tag{1.2}$$

$$\langle p|\bar{V}|q\rangle = \langle p|V|k\rangle, \tag{1.3}$$

and

$$\langle p|H_0(E)|q\rangle = \delta(p-q) G_0(q) \gamma(k,q), \tag{1.4}$$

where $\gamma(k,q)$ satisfies

$$\gamma(k,k) = 1. \tag{1.5}$$

Now Eq. (1.1) is rewritten as[1,2]

$$t(E) = V + \bar{V} H_0(E) t(E) + A(E) t(E), \tag{1.6}$$

and we introduce the auxiliary equation

$$\Gamma(E) = V + A(E)\Gamma(E), \tag{1.7}$$

with the kernel $A(E)$ which does not have the fixed point singularity of $G_0(q)$. Then $t(E)$ can be expressed in terms of $\Gamma(E)$ as[1,2]

$$\langle p|t(E)|r\rangle = \langle p|\Gamma(E)|r\rangle + \langle p|\Gamma(E)|k\rangle I(k,r) \tag{1.8a}$$

$$= \left[\frac{\langle p|\Gamma(E)|k\rangle}{\langle k|\Gamma(E)|k\rangle}\, t(k)\, \frac{\langle r|\Gamma(E)|k\rangle}{\langle k|\Gamma(E)|k\rangle}\right] + \left\{\langle p|\Gamma(E)|r\rangle - \frac{\langle p|\Gamma(E)|k\rangle\langle k|\Gamma(E)|r\rangle}{\langle k|\Gamma(E)|k\rangle}\right\}, \tag{1.8b}$$

where

$$I(k,r) = \frac{\int dq\, q^2 f(q) \gamma(k,q) G_0(q) \langle q|\Gamma(E)r\rangle}{1 - \int dq\, q^2 f(q) \gamma(k,q) G_0(q) \langle q|\Gamma(E)|k\rangle} \tag{1.9}$$

and

$$t(k) \equiv \langle k|t(E)|k\rangle = \langle k|\Gamma(E)|k\rangle \left[1 - \int dq\, q^2 f(q)\gamma(k,q) \times G_0(q)\langle q|\Gamma(E)|k\rangle\right]^{-1}. \quad (1.10)$$

Equations (1.8)–(1.10) are the fundamental results of I and II. The form (1.8b) has certain advantages as have been first pointed out by Kowalski and Noyes[3] (KN). The quantity in the square bracket of Eq. (1.8b) is separable and is known in the literature as the KN approximation and is exact for half-on-shell values of momentum variables. The second term in the curly bracket of Eq. (1.8b) is real if V is real and is zero for half-on-shell values of the momentum variables. A multichannel generalization of Eq. (1.8a) has already appeared in I. Here we propose a multichannel generalization of Eq. (1.8b), which has the advantages of the KN method in the single channel case.

It is easily realized that the applicability of the present method is not restricted to the case of the two-body equations if one is interested only in the iterative solution of scattering equations. Of course, in the case of three-body problems, special care is needed to make the kernel of the auxiliary equation nonsingular above the breakup threshold as has been shown in Ref. 4. Otherwise the method can be immediately applied to the case of three-body equations with finite-rank two-body interactions, since in this case the three-body equations are known[5] to reduce to multichannel scattering equations of the form given by Eq. (1.1) in a single momentum variable. The auxiliary equation, which we solve by iteration, is nonsingular below the breakup threshold. But this does not mean that the applicability of the method is limited for energies below the breakup threshold.

It is usually difficult to develop an approximate method for three-body equations compared to a similar method for two-body equations because the approximate solution has to reproduce the complicated singularity structure of the amplitude correctly, especially above the breakup threshold. This makes certain three-body methods difficult to implement in practice. We shall not face this problem here because we use iterative solution of the auxiliary equation which will correctly reproduce all the singularity structures. This was known from the analysis of Ref. 6 which shows how the iterative solution of an integral equation preserves the analytic structure of the solution.

Next we test the method numerically in the simple case of elastic neutron-deuteron scattering with separable interaction with Yamaguchi form factors[7]—commonly known as the Amado model[8]—for energies below the breakup threshold. Iterative solutions of these equations are known to diverge except at very high energies.[9] Similar calculation for phase shifts was performed by Whiting and Fuda[10] who used a Gaussian form factor for the separable two-body interactions. The present calculation is more general and efficient than that of Whiting and Fuda for the following reasons. The present method is equally applicable for fully off-shell values of momentum variables whereas that of Whiting and Fuda[10] is limited to half-on-shell values of momentum variables. Another interesting difference in the case of the spin $\frac{1}{2}$ neutron-deuteron scattering is that Whiting and Fuda made a subtraction in the kernel only in the more important nucleon-nucleon spin triplet (deuteron) channel of the problem. This makes the kernel real for energies below the breakup threshold. But over and above this subtraction we make another subtraction in the spin singlet (virtual nucleon-nucleon state) channel. This introduces a zero in the kernel of the auxiliary equation corresponding to the spin singlet nucleon-nucleon channel which is intrinsically coupled to the spin triplet nucleon-nucleon channel in the spin $\frac{1}{2}$ neutron-deuteron Amado model.[8] This makes the kernel of the present formulation much weaker than that considered by Whiting and Fuda[10] for the more difficult spin $\frac{1}{2}$ case. Hence we find a much better convergence for the iterative solution of the auxiliary equation.

We also compare the present convergence rate with that obtained by Brady and Sloan[11] using the conventional Padé technique.[12] We find that the rate of convergence of the present method could be significantly faster than that obtained by using the Padé technique for exactly the same mathematical equations. The present method is also slightly simpler to implement in practice than the Padé technique because it is numerically somewhat more tedious to construct the Padé approximants using the iterative solution than to construct the solution using that of the auxiliary equation in the present method.

In Sec. II we describe the present method for multichannel scattering and write the final result in the form first presented by Kowalski and Noyes.[3] In Sec. III we apply the present method to the case of three-body equations, develop the present method in the simple case of neutron-deuteron scattering in the Amado model,[8] and solve the equations numerically using iterative solution of the auxiliary equation. Finally in Sec. IV we give a brief discussion and concluding remarks.

II. THE METHOD

The multichannel scattering equations have in general the same form as Eq. (1.1), but now the various variables have the channel indices over and above the momentum labels. In explicit notation the multichannel generalization of Eq. (1.1) becomes [see Eq. (22) of I]

$$t_{\beta\alpha}(p_\beta, r_\alpha; E) = V_{\beta\alpha}(p_\beta, r_\alpha) + \sum_\sigma \int dq_\sigma q_\sigma^2 f_\sigma(q_\sigma) V_{\beta\sigma}(p_\beta, q_\sigma) G_\sigma(q_\sigma) t_{\sigma\alpha}(q_\sigma, r_\alpha; E), \tag{2.1}$$

where $G_\sigma(q_\sigma) = (k_\sigma^2 - q_\sigma^2 + i\epsilon)^{-1}$ carries the possible singularities of the σ channel, while $f_\sigma(q_\sigma)$ is a weight function, which in this case is a constant λ, and k_σ is the on-shell value of momentum for channel σ.

Following I we can write the solution of Eq. (2.1) as

$$t_{\beta\alpha}(p_\beta, r_\alpha; E) = \Gamma_{\beta\alpha}(p_\beta, r_\alpha; E) + \sum_\sigma \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E) I_{\sigma\alpha}(k_\sigma, r_\alpha; E), \tag{2.2}$$

where $I_{\sigma\alpha}$ is the solution of

$$I_{\beta\alpha}(k_\beta, r_\alpha; E) = d_{\beta\alpha}(k_\beta, r_\alpha; E) + \sum_\sigma d_{\beta\sigma}(k_\beta, k_\sigma; E) I_{\sigma\alpha}(k_\sigma, r_\alpha; E), \tag{2.3}$$

with

$$d_{\sigma\alpha}(k_\sigma, r_\alpha; E) = \int dq_\sigma q_\sigma^2 f_\sigma(q_\sigma) G_\sigma(q_\sigma) \gamma_\sigma(k_\sigma q_\sigma) \Gamma_{\sigma\alpha}(q_\sigma, r_\alpha; E), \tag{2.4}$$

and where $\Gamma_{\beta\alpha}(p_\beta, r_\alpha; E)$ is the solution of the following auxiliary equation:

$$\Gamma_{\beta\alpha}(p_\beta, r_\alpha; E) = V_{\beta\alpha}(p_\beta, r_\alpha) + \sum_\sigma \int dq_\sigma q_\sigma^2 f_\sigma(q_\sigma) A_{\beta\sigma}(p_\beta, q_\sigma; E) \Gamma_{\sigma\alpha}(q_\sigma, r_\alpha; E). \tag{2.5}$$

Here

$$A_{\beta\sigma}(p_\beta, q_\sigma; E) = [V_{\beta\sigma}(p_\beta, q_\sigma) - V_{\beta\sigma}(p_\beta, k_\sigma)\gamma_\sigma(k_\sigma, q_\sigma)] G_\sigma(q_\sigma), \tag{2.6}$$

and $\gamma_\sigma(k_\sigma, q_\sigma)$ is a function which satisfies

$$\gamma_\sigma(k_\sigma, k_\sigma) = 1. \tag{2.7}$$

Equations (2.2) to (2.7) are the fundamental equations of the multichannel formulation of I. We write the final result given by Eq. (2.2) in the KN form.[3] The half-on-shell version of Eq. (2.2) can be written as

$$t_{\beta\alpha}(p_\beta, k_\alpha; E) = \sum_\sigma \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E)[\delta_{\sigma\alpha} + I_{\sigma\alpha}(k_\sigma, k_\alpha; E)], \tag{2.8}$$

and the on-shell version is written as

$$t_{\beta\alpha}(k_\beta, k_\alpha; E) = \sum_\sigma \Gamma_{\beta\sigma}(k_\beta, k_\sigma; E)[\delta_{\sigma\alpha} + I_{\sigma\alpha}(k_\sigma, k_\alpha; E)]. \tag{2.9}$$

Eliminating $I_{\sigma\alpha}$ between Eqs. (2.8) and (2.9) we get

$$t_{\beta\alpha}(p_\beta, k_\alpha; E) = \sum_{\sigma,\rho} \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E) \Theta_{\sigma\rho}(k_\sigma, k_\rho; E) t_{\rho\alpha}(k_\rho, k_\alpha; E), \tag{2.10}$$

where Θ is a matrix whose momentum variables only take the on-shell values and satisfies

$$\Theta^{-1}(k_\sigma, k_\rho; E) = \Gamma(k_\sigma, k_\rho; E),$$

or

$$\sum_\sigma \Gamma_{\beta\sigma}(k_\beta, k_\sigma; E) \Theta_{\sigma\rho}(k_\sigma, k_\rho; E) = \sum_\sigma \Theta_{\beta\sigma}(k_\beta, k_\sigma; E) \Gamma_{\sigma\rho}(k_\sigma, k_\rho; E) = \delta_{\beta\rho}. \tag{2.11}$$

Equation (2.2) for $p = k$ can be written as

$$t_{\beta\alpha}(k_\beta, r_\alpha; E) - \Gamma_{\beta\alpha}(k_\beta, r_\alpha; E) = \sum_\sigma \Gamma_{\beta\sigma}(k_\beta, k_\sigma; E) I_{\sigma\alpha}(k_\sigma, r_\alpha; E) \tag{2.12}$$

which, using Eq. (2.11), gives the following formal solution for $I_{\sigma\alpha}$:

$$I_{\sigma\alpha}(k_\sigma, r_\alpha; E) = \sum_\rho \Theta_{\sigma\rho}(k_\sigma, k_\rho; E)[t_{\rho\alpha}(k_\rho, r_\alpha; E) - \Gamma_{\rho\alpha}(k_\rho, r_\alpha; E)]. \tag{2.13}$$

Now recalling that

$$t_{\rho\alpha}(k_\rho, r_\alpha; E) = t_{\alpha\rho}(r_\alpha, k_\rho; E), \tag{2.14}$$

and using Eq. (2.13), Eq. (2.2) can be written as

$$t_{\beta\alpha}(p_\beta, r_\alpha; E) = \left[\Gamma_{\beta\alpha}(p_\beta, r_\alpha; E) - \sum_{\sigma\rho} \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E)\Theta_{\sigma\rho}(k_\sigma, k_\rho; E)\Gamma_{\rho\alpha}(k_\rho, r_\alpha; E)\right] + \sum_{\sigma\rho} \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E)\Theta_{\sigma\rho}(k_\sigma, k_\rho; E)t_{\alpha\rho}(r_\alpha, k_\rho; E). \tag{2.15}$$

Using Eq. (2.10) in Eq. (2.15) we get

$$t_{\beta\alpha}(p_\beta, r_\alpha; E) = \left[\sum_{\sigma\rho\,\delta\mu} \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E)\Theta_{\sigma\rho}(k_\sigma, k_\rho; E)\Gamma_{\alpha\delta}(r_\alpha, k_\delta; E)\Theta_{\delta\mu}(k_\delta, k_\mu; E)t_{\mu\rho}(k_\mu, k_\rho; E)\right] + \left\{\Gamma_{\beta\alpha}(p_\beta, r_\alpha; E) - \sum_{\sigma\rho} \Gamma_{\beta\sigma}(p_\beta, k_\sigma; E)\Theta_{\sigma\rho}(k_\sigma, k_\rho)\Gamma_{\rho\alpha}(k_\rho, r_\alpha; E)\right\}. \tag{2.16}$$

Equation (2.16) is the desired equation and is a multichannel generalization of Eq. (1.8b). The quantity in the square bracket is similar to the KN approximation in the case of multichannel problems and because of Eqs. (2.10), (2.11), and (2.14) is exact half-on-shell. The second term in the curly bracket of Eq. (2.16) is the fully off-shell residual term and because Eq. (2.11) is zero half-on-shell. The residual term is real for usual momentum space multichannel Lippmann-Schwinger equations. But the applicability of the present equations is easily extended to the case of Faddeev type equations[5,13,14] with finite-rank two-body interactions as we shall see in Sec. III.

III. THREE-BODY METHOD

A. Finite-rank potential

We consider the Alt, Grassberger, and Sandhas equation[14] for the three-body scattering amplitude U.

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha}G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma}t_\gamma G_0 U_{\gamma\alpha}, \tag{3.1}$$

where α, β, γ denotes a pair; $\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$; t_γ is the usual two-body t matrix for pair γ. Here $G_0 = (s - H_0)^{-1}$ with H_0 the three particle kinetic energy and s the complex energy parameter defined by $s = E + i\epsilon$, where E is the three-body center of mass energy. Multiplying Eq. (3.1) from both sides by G_0 we have

$$G_0 U_{\beta\alpha} G_0 = \bar{\delta}_{\beta\alpha}G_0 + \sum_\gamma \bar{\delta}_{\beta\gamma}G_0 t_\gamma (G_0 U_{\gamma\alpha} G_0), \tag{3.2}$$

which has the same structure as the Lippmann-Schwinger equation. Let us consider a finite rank expansion for t_γ given by

$$t_\gamma = \sum_{n'n} |\gamma n'\rangle \tau_{\gamma n' n} \langle \gamma n|, \tag{3.3}$$

with

$$\langle \vec{p}'_\gamma | \tau_{\gamma n' n} | \vec{p}_\gamma \rangle = \delta(\vec{p}'_\gamma - \vec{p}_\gamma) F_{\gamma n' n}(s - p_\gamma^2/2M_\gamma), \tag{3.4}$$

where M_γ is the relative mass between the pair γ and the remaining particle. $\vec{p}_\gamma$ and $\vec{p}'_\gamma$ refer to the momentum of the spectator particle in the center of mass frame. Defining

$$X_{\beta n', \alpha n}(\vec{p}_\beta, \vec{p}_\alpha) = \langle \vec{p}_\beta | \langle \beta n' | G_0 U_{\beta\alpha} G_0 | \alpha n \rangle | \vec{p}_\alpha \rangle, \tag{3.5}$$

$$Z_{\beta n', \alpha n}(\vec{p}_\beta, \vec{p}_\alpha) = \langle \vec{p}_\beta | \langle \beta n' | \bar{\delta}_{\beta\alpha} G_0 | \alpha n \rangle | \vec{p}_\alpha \rangle, \tag{3.6}$$

where $|\alpha n\rangle$'s refer to the form factor of the two particles in the pair α and taking the matrix elements of Eq. (3.2) between appropriate momentum states, we have

$$X_{\beta n', \alpha n}(\vec{p}_\beta, \vec{p}_\alpha) = Z_{\beta n', \alpha n}(\vec{p}_\beta, \vec{p}_\alpha) + \sum_{\gamma m m'} \int d\vec{p}_\gamma Z_{\beta n', \gamma m}(\vec{p}_\beta, \vec{p}_\gamma) \times F_{\gamma m m'}(s - \vec{p}_\gamma^{\,2}/2M_\gamma) \times X_{\gamma m', \alpha n}(\vec{p}_\gamma, \vec{p}_\alpha). \tag{3.7}$$

We see that Eq. (3.7) is an integral equation in one vector variable and the unknown X's of this equation are related to the physical scattering amplitudes. Moreover, below the breakup threshold Z's are real, and after partial wave projection Eq. (3.7) reduces to a system of coupled equations in one variable and has the same structure[5] as the multichannel scattering equations we considered in Sec. II. Hence we can apply the method we considered in Sec. II to solve Eq.

(3.7). In the next subsection we shall consider the problem of neutron-deuteron scattering in the simplified Amado model where the nucleon-nucleon interaction is taken to be a sum of two S-wave separable terms—one for the spin one state and another for the spin zero state—and demonstrate analytically and numerically the applicability of the present method.

B. Amado model

In this subsection we write a simple model for the three nucleon system with the two nucleon interaction taken to be of spin-dependent S-wave separable form. This model was first proposed by Amado[8] from a field theoretic consideration, was studied by Aaron, Amado, and Yam,[15] by Aaron and Amado,[16] and later was shown by Lovelace[5] to be identical to the Faddeev equations[13] with separable two-body interactions.

The model we consider is of three identical nucleons of mass m interacting through a spin and isospin dependent S-wave separable two-body potential of the form

$$\langle \vec{q}' | V_\mu | \vec{q} \rangle = -\sum_{n=0}^{1} \lambda_n g_n(q') g_n(q) P_n , \tag{3.8}$$

where P_0 is the spin-isospin projection operator for the deuteron and P_1 is the corresponding operator for the singlet state. The form factors $g_n(q)$ are of the Yamaguchi[7] form

$$g_n(q) = N_n (q^2 + \beta_n^2)^{-1} , \tag{3.9}$$

where β_n is a range parameter, N_0 is chosen to normalize the deuteron wave function, and N_1 is arbitrary and eventually drops out of the physical scattering problem. However, if we define N_1 like N_0 in our numerical calculation we get[9]

$$N_n = \frac{1}{\pi} [\,|\alpha_n| \beta_n (\alpha_n + \beta_n)^3\,]^{1/2} , \quad n = 0, 1 . \tag{3.10}$$

λ_0 and λ_1 are fixed to give the correct energies for the triplet and singlet states such that[9]

$$\lambda_n = [\,|\alpha_n| (\alpha_n + \beta_n)\,]^{-1} . \tag{3.11}$$

We take[15] $\alpha_0 = .231\,713\ \mathrm{fm}^{-1}$, $\alpha_1 = -.039\,925\ \mathrm{fm}^{-1}$, $\beta_0 = 1.405\,52\ \mathrm{fm}^{-1}$, and $\beta_1 = 1.177\,12\ \mathrm{fm}^{-1}$. α_0^2 is the deuteron binding energy and α_1^2 is the energy of the singlet state. In this work we use units in which $\hbar = m \equiv 1$.

The two-body t matrix t_γ has the form

$$\langle \vec{p}_\gamma, \vec{q}_\gamma | t_\gamma(s) | \vec{p}'_\gamma, \vec{q}'_\gamma \rangle = \sum_n g_n(q_\gamma) \delta(\vec{p}_\gamma - \vec{p}'_\gamma) \times F_n(s - \tfrac{3}{4} p_\gamma^2) g_n(q'_\gamma) , \tag{3.12}$$

where $\vec{q}_\gamma$ is the relative momentum of two particles in the pair and $\vec{p}_\gamma$ has the same meaning as in Sec. III A. F_n is defined by

$$F_n^{-1}(z) = -\frac{1}{\lambda_n} + 4\pi \int_0^\infty q^2 dq \, \frac{g_n^2(q)}{q^2 - z} . \tag{3.13}$$

The on-shell three-body momentum k is defined by

$$E = \tfrac{3}{4} k^2 - \alpha_0^2 . \tag{3.14}$$

With this definition, $F_n(s - \frac{3}{4}p^2)$ has the form

$$F_n(s - \tfrac{3}{4}p^2) = -\frac{2\pi}{3} f_n(p) G_n(p) , \tag{3.15}$$

where

$$G_0(p) = (k^2 - p^2 + i\epsilon)^{-1} , \tag{3.16}$$

$$G_1(p) = -\tfrac{3}{4} \{\,|\alpha_1| + [\,\alpha_0^2 + \tfrac{3}{4}(p^2 - k^2)\,]^{1/2}\}^{-2} , \tag{3.17}$$

and

$$f_n(p) = -\frac{2}{\pi} \frac{\{\beta_n + [\,\alpha_0^2 + \frac{3}{4}(p^2 - k^2)\,]^{1/2}\}^2 \{\,|\alpha_n| + [\,\alpha_0^2 + \frac{3}{4}(p^2 - k^2)\,]^{1/2}\}}{|\alpha_n| (\alpha_n + \beta_n) \{2\beta_n + \alpha_n + [\,\alpha_0^2 + \frac{3}{4}(p^2 - k^2)\,]^{1/2}\}} , \tag{3.18}$$

$n = 0, 1$.

The integral equation (3.7) for this case, after antisymmetrization and partial wave analysis, takes the explicit form[5,8,9,14]

$$\langle p | X_{nn'}^{L,S} | p' \rangle = \langle p | Z_{nn'}^{L,S} | p' \rangle + \sum_m \int_0^\infty q^2 dq \, \langle p | Z_{nm}^{L,S} | q \rangle f_m(q) G_m(q) \langle q | X_{mn'}^{L,S} | p' \rangle , \tag{3.19}$$

with

$$\langle p | Z_{nn'}^{L,S} | p' \rangle = \frac{8\pi^2}{3} J_{nn'}^S \int_{-1}^{1} dx \, \frac{P_L(x) g_n\left(\left|\frac{\vec{p}}{2} + \vec{p}'\right|\right) g_{n'}\left(\left|\frac{\vec{p}'}{2} + \vec{p}\right|\right)}{p^2 + p'^2 + pp'x - s} , \tag{3.20}$$

where x is the cosine of the angle between $\vec{p}$ and $\vec{p}'$. The values of the parameters α_1, β_1, α_0 and β_0 are taken from Ref. 16. The nonzero spin-isospin overlap factors are $J_{00}^{3/2} = -\frac{1}{2}$, $J_{00}^{1/2} = J_{11}^{1/2} = \frac{1}{4}$, and $J_{01}^{1/2} = J_{10}^{1/2} = -\frac{3}{4}$. The physical elastic scattering amplitude is related to elastic scattering phase shift by

$$\langle k | X_{00}^{L,S} | k \rangle = e^{i\delta_{L,S}} \sin\delta_{L,S} / k \tag{3.21}$$

so that the elastic neutron-deuteron scattering

length a_S is defined by

$$a_S = -\langle 0|X_{00}^{L,S}(s=-\alpha_0^2)|0\rangle . \tag{3.22}$$

Because of the spin-isospin factors, Eq. (3.19) reduces to a single equation for the spin quartet case and to a set of two coupled equations for the spin doublet case.

Suppressing the L, S labels, Eq. (3.19) now has the same form as Eq. (2.1) and hence the technique of Sec. II is directly applicable. Now we introduce the equation for Γ in the following way:

$$\langle p|\Gamma_{nn'}^{L,S}|p'\rangle = \langle p|Z_{nn'}^{L,S}|p'\rangle + \sum_m \int dq\, q^2 \langle p|A_{nm}^{L,S}|q\rangle \times \langle q|\Gamma_{mn'}^{L,S}|p'\rangle , \tag{3.23}$$

with

$$\langle p|A_{nm}^{L,S}|q\rangle = [\langle p|Z_{nm}^{L,S}|q\rangle - \langle p|Z_{nm}^{L,S}|k'\rangle\gamma_m(k',q)] \times f_m(q)G_m(q) . \tag{3.24}$$

This completes the definition of the basic equations of the present method.

The method of Whiting and Fuda[10] is a special case of the equations presented in this section with $\gamma_1 = 0$ and for half-on-shell t matrix elements. The present method is equally good for on-shell, half-on-shell, and off-shell t matrix elements, whereas the method of Ref. 10 is only good for half-on-shell t matrix elements. In Ref. 10 the authors introduced only a subtraction in the more important nucleon-nucleon spin 1 channel and, consequently, $\gamma_1 = 0$ always. But here we shall introduce a $\gamma_1 \neq 0$ over and above a $\gamma_0 \neq 0$ and this will improve the rate of convergence of the iterative solution of Eq. (3.23). In Sec. II, G_n always had a fixed point singularity but in Eq. (3.19) only G_0 has such a singularity and G_1 does not have this singularity. Still we can make a subtraction for $n = 1$ in order to improve the rate of convergence of the iterative solution.

C. Numerical results

Now we are left with the problem of choosing the functions γ_0 and γ_1. We have no fundamental theory about how to choose these functions. The function γ_0 which is related to the deuteron pole is more important for achieving convergence. This has also been found by Whiting and Fuda[10] who found a convergent result by varying only γ_0. First we shall choose $\gamma_1 = 0$ and vary γ_0 in order to obtain the best convergence. Then we shall use this particular γ_0 and choose a nonzero γ_1 in order to improve the rate of convergence. Before choosing γ's we rewrite the Eq. (3.24) for $L = 0$ with A_{nm} in the following way:

$$\langle p|A_{nm}^{0,S}|q\rangle = \left[\langle p|Z_{nm}^{0,S}|q\rangle f_m(q) - \langle p|Z_{nm}^{0,S}|k'\rangle f_m(k') \times \left\{\gamma_m(k',q)\frac{f_m(q)}{f_m(k')}\right\}\right] G_m(q) , \tag{3.25}$$

which helps in the choice of γ_m. As in Ref. 10, we choose a simple algebraic function for the expression in the curly brackets, which should have the property of approximating the q dependence of $\langle p|Z_{00}^{0,S}|q\rangle f_0(q)$ such that the difference in the square brackets of Eq. (3.25) has the lowest operator norm. Then the operator A will have all the eigenvalues less than one in magnitude and the Neumann series of Eq. (3.23) is expected to converge.

First we choose a particular γ as has been done intuitively by Whiting and Fuda[10] for the case $m = 0$. In particular we use

$$\gamma_m(k',p)\frac{f_m(p)}{f_m(k')} = \left(\frac{k'^2+\alpha}{p^2+\alpha}\right)^M \tag{3.26}$$

or

$$\gamma_m(k',p) = \frac{f_m(k')}{f_m(p)}\left(\frac{k'^2+\alpha}{p^2+\alpha}\right)^M , \tag{3.27}$$

where M and α are two arbitrary parameters to be found out from numerical experimentation. Although these equations are written in a slightly different way from those in Ref. 10, the choice for γ_0 given by Eq. (3.27) has the same functional form as the choice (16) of Ref. 10.

Next we tried to choose a γ based on the simple idea that we would like to make some power of the difference

$$[\langle p|Z_{nm}^{L,S}|q\rangle - \langle p|Z_{nm}^{L,S}|k'\rangle\gamma_m(k',q)] \tag{3.28}$$

as small as possible for all values of n and p. We considered the square of this expression and, as $\gamma_m(k',q)$ is not a function depending on n and p, we sum and integrate over these variables and claim

$$\delta \sum_n \int \omega_m(p)dp\,[\langle p|Z_{nm}^{L,S}|q\rangle - \langle p|Z_{nm}^{L,S}|k'\rangle\gamma_m(k',q)]^2 = 0 , \tag{3.29}$$

where δ denotes small variations of γ_m and $\omega_m(p)$ is a weight function. The Eq. (3.29) yields the following functional form for γ_m:

$$\gamma_m(k',q) = \frac{\sum_n \int \omega_m(p)dp\langle p|Z_{nm}^{L,S}|k'\rangle\langle p|Z_{nm}^{L,S}|q\rangle}{\sum_n \int \omega_m(p)dp\langle p|Z_{nm}^{L,S}|k'\rangle\langle p|Z_{nm}^{L,S}|k'\rangle} . \tag{3.30}$$

[Note that in Eq. (3.30) the numerical value of γ_m depends on the spin state.] We shall mainly use these two choices of γ given by Eqs. (3.27) and

(3.30) in the present work.

From an analysis of the multiple series by Sloan[9] it is known that Eq. (3.19) for $L \neq 0$ converges smoothly and hence in this work we concentrate on the case of $L = 0$ only. In order to see how the choice (3.27) works in practice first we solved Eq. (3.23) for the neutron-deuteron system. We know from the work of Ref. 11 that Eq. (3.19) for the spin $\frac{3}{2}$ state diverges at zero incident neutron energy, the limit of the ratio of the successive terms in the multiple scattering series u being -2.68, whereas in the spin $\frac{1}{2}$ case the corresponding value of u is 2.45. But we expect that Eq. (3.23) will have a convergent iterative solution. Next we solve the Eq. (3.23) numerically at incident neutron laboratory energies $E_{\text{lab}} = 0$ and 2.45 MeV by mapping the integral in it to the interval -1 to $+1$ and approximating it by a 32-point gaussian quadrature. We solve Eq. (3.23) with $\gamma_1 = 0$ and $k' = k$, because $k' = k$ has the added advantage of removing the fixed point singularity from the propagator G_0 of Eq. (3.24), and hence making the kernel nonsingular below the breakup threshold.

First we consider γ_0 of Eq. (3.27) and solve Eq. (3.23) by iteration for the spin quartet state. We obtain a good convergence rate for $M = 1, 2, 3$, and for a reasonably wide range of values of α. In the case of the spin doublet state the good convergence of the Neumann series of Eq. (3.23) was more difficult to obtain. After some experimentation we find that $M = 2$ and $\alpha = 4.0$ fm^{-2} give the best convergence in this case. We can somewhat justify the value of $M = 2$ in Eq. (3.27) because with this value of M the asymptotic form of the two terms in the square brackets of Eq. (3.25) for $q \to \infty$ are the same. Equation (3.23) also gave very good convergence for the spin quartet state with this γ_0.

Next we fix γ_0 to be that given by Eq. (3.27) with $M = 2$ and $\alpha = 4.0$ fm^{-2}, and consider γ_1 to be given by Eq. (3.27) in order to improve the convergence of Eq. (3.23) for the spin doublet case. The kernel corresponding to the channel $m = 1$ does not have a pole as in the case $m = 0$ channel and hence we have no advantage in choosing $k' = k$ for this channel. Such subtraction in the kernel will introduce a zero in the kernel and we choose k' to be the point where $\langle p|Z_{nm}|q\rangle$ has a maximum. It was always at $q = 0$ and so we choose $k' = 0.0003$ fm^{-1} in this case. Again we experiment for α and M for $m = 1$, and we find that $\alpha = 10.0$ fm^{-2} and $M = 2$ give the best convergence, with γ_0 defined with $\alpha = 4.0$ fm^{-2} and $M = 2$. The result for the spin quartet state is independent of γ_1 and stays unchanged.

We tried a second form for γ_1. We realized that $\langle p|Z^{L,S}_{nm}|q\rangle$ for various n and m are roughly proportional apart from an overall scaling factor. So if γ_0 defined by Eq. (3.27) gave good convergence, such a function will also give good convergence if used in place of γ_1. So we choose

$$\gamma_1(k', p) = \frac{f_0(k')}{f_0(p)}\left(\frac{k'^2 + \alpha}{p^2 + \alpha}\right)^M . \tag{3.31}$$

The difference between this γ_1 and the one defined by Eq. (3.27) is in the first term on the right hand side of Eq. (3.31). Equation (3.31) has the first factor as $f_0(k')/f_0(p)$ whereas Eq. (3.27), as in our first choice of γ_1, has the factor $f_1(k')/f_1(p)$. Without varying our previous γ_0 defined by Eq. (3.27) with $\alpha = 4.0$ fm^{-2} and $M = 2$, we experiment with γ_1 defined by Eq. (3.31) and we find that we obtain the best convergence with $\alpha = 5.2$ fm^{-2}, $M = 2$, and $k' = 0.0003$ fm^{-1}. In this connection it is to be remembered that γ_1 does not enter into the equations for the spin quartet case.

Next we used γ given by Eq. (3.30) in our numerical calculation. For $\omega_m(p)$ we took simple functions independent of n, such as $\omega_m(p) = p^l$, where l is an integer. First we took $\gamma_1 = 0$ and found that γ_0, given by Eq. (3.30), gave the best convergence for $\omega_0(p) = p^2$ for both the spin doublet and quartet states. Then we took this particular γ_0 and we found that for γ_1 given by Eq. (3.30) we get the best convergence for the spin doublet equations for $\omega_1(p) = p^3$.

In this paper we only consider two energies, $E_{\text{lab}} = 0$ and $E_{\text{lab}} = 2.45$ MeV, and the auxiliary equation (3.23) we solve for this purpose is nonsingular, whereas at energies above the breakup threshold Eq. (3.23) will have complicated logarithmic singularities in the kernel and in the Born term. The present method can be easily extended for energies above the breakup threshold but as we have complicated singularities for the real momentum variables we shall need special prescription such as contour rotation[15-17] for this purpose. Otherwise the rate of convergence will be faster at higher energies. In order to show the convergence of the Neumann series for Eq. (3.23) we choose not to consider energies above the breakup threshold which will only complicate the method numerically.

Next we exhibit the numerical results. Table I exhibits spin doublet $S = \frac{1}{2}$ and quartet $S = \frac{3}{2}$ neutron-deuteron scattering lengths (results at $E_{\text{lab}} = 0$) and Table II exhibits the elastic scattering phase shifts at $E_{\text{lab}} = 2.45$ MeV for the following choices of γ:

(A) γ_0 defined by Eq. (3.27) with $k' = k$, $\alpha = 4.0$ fm^{-2}, and $M = 2$; $\gamma_1 = 0$.

(B) γ_0 as above and γ_1 given by Eq. (3.27) with $k' = 0.0003$ fm^{-1}, $\alpha = 10.0$ fm^{-2}, and $M = 2$.

TABLE I. Neutron-deuteron elastic scattering lengths for spin doublet and quartet states for different N and various choices of γ. $N=0$ corresponds to no iterations and refer to $\Gamma=V$ in Eq. (1.7). The row labeled AAY is the exact result taken from Ref. 15. The column labeled Padé gives the results of diagonal Padé approximants taken from Ref. 11.

	Spin doublet state					Spin quartet state		
N	(A)	(B)	(C)	(D)	Padé	(A)	(D)	Padé
0	14.0264	6.7165	6.5390	6.7096		4.2334	4.3360	
1	4.8221	1.9019	1.2462	2.3633		6.1471	6.2763	
2	2.0324	−0.1782	−1.1249	−2.3825	7.08	6.3317	6.3246	6.642
3	0.1560	−0.8227	−1.0261	−1.6926		6.3160	6.3164	
4	−0.6205	−1.0160	−1.0341	−1.1052	2.55	6.3185	6.3184	6.321
5	−0.9232	−1.0402	−1.0331	−1.0299		6.3178	6.3179	
6	−1.0109	−1.0408	−1.0345	−1.0336	−0.68	6.3180	6.3178	6.317
7	−1.0326	−1.0380	−1.0351	−1.0357		6.3179	6.3179	
8	−1.0363	−1.0364	−1.0353	−1.0355	−1.01	6.3179	6.3179	6.317
9	−1.0362	−1.0357	−1.0353	−1.0354		6.3179	6.3179	
10	−1.0358	−1.0355	−1.0354	−1.0354	−1.04	6.3179	6.3179	
AAY			−1.04				6.32	

(C) γ_0 as in choice (A) and γ_1 defined by Eq. (3.31) with $k'=0.0003$ fm^{-1}, $\alpha=5.2$ fm^{-2}, and $M=2$.

(D) γ_0 and γ_1 defined by Eq. (3.30) with $\omega_0(p)=p^2$, $\omega_1(p)=p^3$, and k' as above.

We also show the results for the exact solution of Eq. (3.19) taken from Aaron, Amado, and Yam[15] for $E_{\rm lab}=0$. The entry in the column labeled Padé in Table I is the solution of Eq. (3.19) calculated by the technique of Padé approximants.[11] The result for the diagonal Padé approximants labeled $[m,m]$ in Ref. 11 corresponds to $2m$ iterations and is exhibited for $N=2m$. $N=0$ corresponds to taking $\Gamma=Z$ in Eq. (3.23).

In Figs. 1 and 2 we show some fully off-shell t matrix elements, for the spin doublet and quartet states, respectively, for $E_{\rm lab}=0$. In Fig. 1 we show $X_{00}^{0,1/2}(p,0.65)$ and $X_{00}^{0,1/2}(0.65,p)$ for various iterations, which were calculated using Eq. (2.2).

TABLE II. Neutron-deuteron elastic scattering phase shifts at $E_{\rm lab}=2.45$ MeV for spin doublet and quartet states for different N and various choices of γ. N has the same meaning as in Table I.

	Spin doublet state				Spin quartet state	
N	(A)	(B)	(C)	(D)	(A)	(D)
0	4.3267	1.6213	1.6650	1.6379	2.4349	2.4208
1	2.1062	2.3611	2.4758	2.3912	2.0294	2.0099
2	2.4395	2.6691	2.8107	2.8350	1.9996	2.0022
3	2.7415	2.7925	2.8407	2.8564	2.0030	2.0030
4	2.8630	2.8317	2.8402	2.8419	2.0029	2.0029
5	2.8742	2.8405	2.8394	2.8397	2.0029	2.0029
6	2.8554	2.8410	2.8393	2.8394	2.0029	2.0029
7	2.8423	2.8403	2.8393	2.8394	2.0029	2.0029
8	2.8382	2.8397	2.8393	2.8394	2.0029	2.0029
9	2.8381	2.8395	2.8394	2.8394	2.0029	2.0029
10	2.8389	2.8394	2.8394	2.8394	2.0029	2.0029

In Fig. 2(a) we show results for $X_{00}^{0,3/2}(0.07,p)$ and $X_{00}^{0,3/2}(p,0.07)$ calculated using the more symmetric equation (2.16). These matrix elements are equal for graphical purposes and hence only one set of curves is shown. In Fig. 2(b) we show the same matrix elements as in Fig. 2(a) but calculated using the nonsymmetric formulation (2.2). The final convergence is good in both Figs. 2(a) and 2(b) but the result for small values of N is clearly better in Fig. 2(a).

A glance at Tables I and II and Figs. 1 and 2 show that for all the choices of γ the iterative solution of the auxiliary equation converges quite well and the converged result agrees with the exact result of Aaron, Amado, and Yam[15] and also with the Padé technique at $E_{\rm lab}=0$. The rates of convergence in Tables I and II are also much faster than those obtained by Whiting and Fuda,[10] but the equations they solve are different because

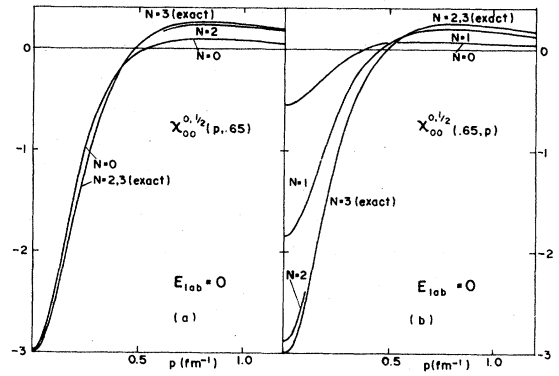

FIG. 1. Off-shell spin doublet t matrix elements (a) $X_{00}^{0,1/2}(p,0.65)$ and (b) $X_{00}^{0,1/2}(0.65,p)$ at $E_{\rm lab}=0$, calculated using Eq. (2.2) for choice (C) of γ for various N. $N=0$ refers to $\Gamma=V$ in Eq. (1.7).

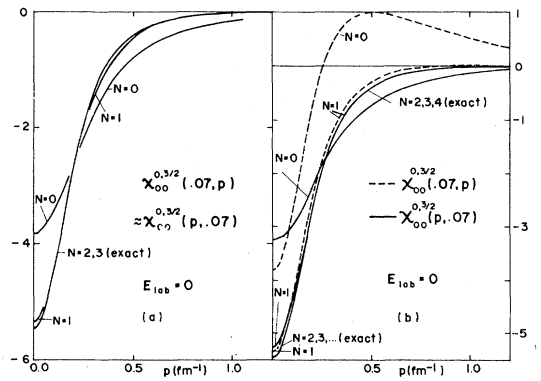

FIG. 2. Same as in Fig. 1 for the spin quartet state for choice (A) of γ. (a) $X_{00}^{0,3/2}(0.07,p) \approx X_{00}^{0,3/2}(p,0.07)$ calculated using Eq. (2.16). Only one set of curves are shown because both these elements are practically the same. (b) The dashed line shows $X_{00}^{0,3/2}(0.07,p)$ and the full line shows $X_{00}^{0,3/2}(p,0.07)$ calculated using Eq. (2.2).

they use exponential form factors in the two-body separable potential in place of the more commonly used Yamaguchi form factors. In Table I we also show the result of Padé approximants taken from the work of Brady and Sloan.[11] For all the choices of γ the convergence is faster than that of Padé approximants for both spin doublet and quartet cases. The difference between the rate of convergence of the present method and Padé approximants is particularly striking in the spin doublet case. In the spin doublet case the best convergence is obtained with choice (C) of γ. In this case the converged value of the spin doublet scattering length (0.25% error) is achieved with *four* iterations whereas in the case of Padé approximants the same convergence is achieved with *ten* iterations. This good trend of convergence is maintained for the spin quartet case and also for $E_{\text{lab}} = 2.45$ MeV.

IV. SUMMARY AND CONCLUSION

Here we critically analyze a recently proposed method for scattering integral equations both analytically and numerically. The method uses the solution of an auxiliary equation whose kernel is weaker than that of the original equation. In particular we consider the multichannel generalization of this formulation and write the final solution in a form which enjoys all the advantages of a method for single channel Lippmann-Schwinger equation proposed by Kowalski and Noyes.[3] The method presented in Sec. II of this paper should be considered as a multichannel generalization of the method presented by us in Ref. 2. We apply the multichannel method of this paper to the solution of three-body equations with finite rank interactions. In Sec. III we illustrate the method numerically in the case of the three-body equations with separable two-body interactions, commonly known as the Amado model.[8] We use an iterative solution of the auxiliary equation and find that with a proper choice of the arbitrary function the auxiliary equation has a rapidly convergent Neumann series solution. The choice of the arbitrary function γ is crucial in order to obtain a good convergence. Here we consider several choices of γ, the choices (A), (B), and (C) being somewhat arbitrary. The simple idea, which led to choice (D), on the other hand, gives a definite criterion to choose the function γ and still remain valid when arbitrary attempts like choices (A), (B), and (C) do not give good convergence. Here we conclude that the present method should be considered as a viable alternative to the Padé technique for solving multichannel scattering integral equations. In particular, as we show in Table I, the present method gives results with much higher precision as compared to the Padé technique using the same number of iterations.

The applicability of the present iterative method can be extended to the case of three-body problems with two-body local interactions. It is well known that the three-body equation in the case of general local two-body interactions is an equation in two vector variables which can be reduced to an equation in two scalar variables after partial wave projection. Still in this case we can write an auxiliary equation in two variables whose kernel is weaker than that of the original equation and which will have a convergent iterative solution. This will be a problem of future interest.

Note added in proof: In this calculation we used $\alpha_0 = 0.231\,713$ fm^{-1}, $\alpha_1 = -0.039\,925$ fm^{-1}, $\beta_0 = 1.405\,52$ fm^{-1}, $\beta_1 = 1.177\,12$ fm^{-1}, and $\hbar^2/m = 41.47$ MeV fm^2.

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