PHYSICAL REVIEW C VOLUME 22, NUMBER 6 DECEMBER 1980

Effective interactions and the coupled reaction channel formalism

R. Goldflam* and K. L. Kowalski
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
(Received 2 June 1980)

Many-body scattering theory is used to derive coupled reaction channel-type equations. With the choice of the Alt, Grassberger, and Sandhas off-shell extension for the rearrangement transition operators, the effective interactions which enter these equations are shown to be free of all relevant two-cluster channel unitarity cuts and satisfy well-behaved dynamical equations which possess an explicit multiple scattering structure. Several approximations to these equations are presented and the importance of various nonorthogonality effects is discussed. Symmetrized forms of these equations which take into account the effects of particle identity are found. This permits the inclusion of the Pauli principle in coupled reaction channel calculations in a simple and practical manner.

[NUCLEAR REACTIONS Generalized coupled reaction channel and resonating group formalisms. Dynamical connected-kernel integral equations for effective potentials. Pauli principle and channel nonorthogonality effects.]

I. INTRODUCTION

Over the past few decades two-cluster descriptions of nuclear reactions, such as the distorted wave Born approximation[1] (DWBA) or the coupled reaction channel method[2-3] (CRC), have provided the basis for the analysis of a large variety of physical phenomena. These methods, though often very successful in fitting the experimental data and in the determination of various features of nuclear structure, suffer from some serious limitations. In particular, they do not include a consistent prescription for assessing their accuracy or for calculating systematic improvements upon them.

A proper formulation of such approximations can be obtained in many-body scattering theory,[4,5] and several studies in this direction have been reported recently.[6-13] Approximate many-body equations based on the truncation of the asymptotic channel space and various coupling schemes are developed in the Refs. 6–9. The multiple scattering structure of the exact and approximate many-body equations is discussed in Refs. 10 and 11, respectively. A reformulation of the many-body equations which includes explicitly the effects of distortions is presented in Ref. 12. Recently, Greben and Levin have reported calculations[13] comparing the DWBA and CRC approximations[1-3] in various forms and the bound state approximation (BSA) to the N-body equations of Baer, Kouri, Levin, and Tobocman (BKLT)[14] for a number of systems and reactions. The primary difference between the CRC and BKLT equations in the BSA is the absence of so-called nonorthogonality terms in the latter.[13] The differences due to the neglect of these terms can be considerable. However, due to the lack of an exact model of comparison no definitive conclusions about which set of equations is preferable are reached in Ref. 13.

In this work we present the derivation of coupled-reaction-channel-type equations using the many-body equations of Ref. 10 as the starting point.[15,16] The effective interactions which enter into these equations satisfy a set of coupled connected-kernel integral equations. Our discussion of these interactions contains two important new features not contained in earlier discussions of this problem.[5] We demonstrate that the Alt-Grassberger-Sandhas choice[17] of the transition operator leads to a definition of effective interactions which are continuous across the elastic unitarity cuts corresponding to the channels included in the reaction model. Also, our equations for these effective interactions have an explicit multiple scattering structure which allows a systematic consideration of various low-density approximations. We find that some of these lead to approximate equations which are similar to those of the CRC method.[2,3] Finally, our equations are shown to be label transforming[16,18] and are therefore readily symmetrized. This allows for a straightforward inclusion of the Pauli principle in the theory.

An interesting aspect of these considerations is that in the lowest order in the interaction our approximate equations contain nonorthogonality terms. This appears relevant to questions concerning the occurrence of such terms in various approximate theories.[13,19] We also note that, since our equations are exact and well defined, it is possible to generate from them, at least in principle, solvable approximate equations to any desired level of accuracy.

This paper is organized as follows. Section II contains our notation and the many-body scattering integral equations.[4] In Sec. III we define the ef-

fective interactions with the aid of the idea of the reaction mechanism.[6] This approach allows a compact description of the problem as well as generalizations to include excited states in a rather simple manner. The integral equations for the effective interactions are derived in Sec. IV. The antisymmetrized forms of these equations are obtained in Sec. V. The wave function equations which result from the present treatment are found in Sec. VI. Some approximations based on the low-density expansion of our equations are considered in Sec. VII. Our results are summarized in Sec. VIII. The details of some of the derivations have been placed in several Appendices.

II. N-BODY EQUATIONS

In this section we review the notation of many-body scattering theory and present a particular form of the scattering integral equations for the transition operators. The detailed discussion of this notation and the relevant combinatorial mathematics can be found elsewhere.[6,9-11,20,21]

We define a *partition* a as the grouping of N distinguishable particles into n_a distinct clusters. We use the Latin letters $a, b, c, \ldots$ to denote the partitions of the system and the Greek letters $\alpha, \beta, \gamma \ldots$ to designate the two-cluster partitions. The unique partitions for which $n_a = 1$ and $n_a = N$ will be denoted as 1 and 0. We say that a partition b is *contained in* another partition a, $b \subset a$, if b can be obtained from a by subdividing one or more of its clusters. We write $b \subseteq a$ to include the possibility of equality. The alternative possibility, where b is *not contained in or equal to* a, is denoted by $b \not\subseteq a$. These relations are conveniently expressed in terms of the matrices Δ and $\bar{\Delta}$ with the elements.[20,21]

$$\Delta_{a,b} = 1, \quad b \subseteq a$$
$$= 0, \quad \text{otherwise}, \tag{2.1}$$

and

$$\bar{\Delta}_{a,b} = 1 - \Delta_{a,b} \ . \tag{2.2}$$

The N-particle Hamiltonian H is taken to be the sum of the kinetic energy H_0 and the potential V:

$$H = H_0 + V \ . \tag{2.3}$$

We assume that V can be written as the sum of two-particle interactions $V_{i'}$, i.e.,

$$V = \sum_{i'} V_{i'} \ , \tag{2.4}$$

where i' denotes a particle pair [i.e., an $(N-1)$-cluster partition].

The partition Hamiltonian H_a is defined by

$$H_a = H_0 + V_a \ , \tag{2.5}$$

where

$$V_a \equiv \sum_{i'} \Delta_{a,i'} V_{i'} \tag{2.6}$$

is the interaction internal to partition a. The interaction external to partition a is defined as

$$V^a \equiv H - H_a = \sum_{i'} \bar{\Delta}_{a,i'} V_{i'} \ . \tag{2.7}$$

In what follows we also require the idea of an interaction which is both external to partition a and internal to partition b, viz.,

$$V_b^a \equiv \sum_{i'} \bar{\Delta}_{a,i'} V_{i'} \Delta_{b,i'} \tag{2.8}$$

and we write $V^{a,b} = V^a - V_b^a$.

The transition operator corresponding to the reaction $b \to a$ is given by

$$T_{a,b}^{(+)} = V^a + V^a \, G \, V^b \ , \tag{2.9}$$

where

$$G = (z - H)^{-1} \tag{2.10}$$

if the full N-particle Green's function and z is a complex parametric energy, the dependence upon which we suppress unless necessary. In writing (2.9) it is important to realize that the choice of the transition operator is not uniquely determined by the values of the on-shell matrix elements $\langle \phi_a(\nu_a)\vec{k}_a | \, T_{a,b}^{(+)} \, | \phi_b(\nu_b)\vec{k}_a \rangle$. Here $| \phi_a(\nu_a)\vec{k}_a \rangle$ denotes the eigenstate of H_a with eigenvalue $E(\nu_a, \vec{k}_a)$ in which all the clusters of the partition a are bound. The collection ν_a of the internal quantum numbers defines a *channel*, and $\vec{k}_a$ are the momenta associated with the relative motion of the n_a clusters. For example, an alternative on-shell equivalent definition of the transition operator has been introduced by Alt, Grassberger, and Sandhas (AGS),[17] viz.,

$$T_{a,b} = \bar{\delta}_{a,b} G_b^{-1} + T_{a,b}^{(+)} \ , \tag{2.11}$$

where

$$G_a = (z - H_a)^{-1}$$

and

$$\bar{\delta}_{a,b} = 1 - \delta_{a,b} \ .$$

The operators (2.11) possess a symmetrical relationship to the total Green's function.[19] This property has been found to be crucial for the inclusion of the Pauli principle into the theory of the optical potential.[16] The choice (2.11) is also essential in generalizing the optical potential idea

to rearrangement collisions. We discuss this point in more detail in the next section.

In order to obtain integral equations for the transition operators which possess an explicit multiple scattering structure, it is useful to introduce the operators[10]

$$\tau^{a,b} = V^{a,b} + V^a G V^b \ , \tag{2.12}$$

which are related to the AGS transition operators by

$$T_{a,b} = \bar{\delta}_{a,b} G_b^{-1} + V_b^a + \tau^{a,b} \ . \tag{2.13}$$

In the absence of many-body forces the two-cluster partition-labeled operators $\tau^{\alpha,\beta}$ satisfy the coupled integral equations[10]

$$\tau^{\alpha,\beta} = W_{\rm MS}^{\alpha,\beta} + \sum_\gamma W^{\alpha,0}(\gamma) G_0 \tau^{\gamma,\beta} \ . \tag{2.14}$$

The remaining operators $\tau^{a,b}$ with n_a, $n_b > 2$ can be obtained from $\tau^{\alpha,\beta}$ by quadrature. Scattering equations which couple only two-cluster partition labeled operators are called minimally coupled.[4]

In (2.14), $W^{a,b}(c)$ denotes the c-connected part of the operator $\tau^{a,b}$. Also

$$W_{\rm MS}^{\alpha,\beta} = \sum_a{}' W^{\alpha,\beta}(a) \ , \tag{2.15}$$

where the prime indicates that the $a=1$ partition has been omitted from the sum. We recall that an operator $\mathcal{O}$ is said to be *c connected* if its momentum-space matrix elements have structure[4,6,10,11,21]

$$\langle \vec{p}_1 \ldots \vec{p}_N | \mathcal{O} | \vec{p}_1' \ldots \vec{p}_N' \rangle = \left[\prod_{n=1}^{n_c} \delta(\vec{P}_n - \vec{P}_n') \right] \times \mathcal{O}_c(\vec{p}_1 \ldots \vec{p}_N | \vec{p}_1' \ldots \vec{p}_N') \ . \tag{2.16}$$

Here $\vec{p}_i$ refers to the momenta of the individual particles, $\vec{P}_n$ denotes the momenta associated with the relative motion of the clusters of c, and $\mathcal{O}_c(\vec{p}_1 \ldots \vec{p}_N | \vec{p}_1' \ldots \vec{p}_N')$ does not contain any δ-function singularities. The least connected contributions to (2.15) are the single-scattering terms

$$W^{\alpha,\beta}(i') = \bar{\Delta}_{\alpha,i'} t_{i'} \bar{\Delta}_{\beta,i'} \ , \tag{2.17}$$

where

$$t_{i'} = V_{i'} + V_{i'} G_0 t_{i'} \tag{2.18}$$

is the ordinary two-particle transition operator on the N-particle Hilbert space. We note that the free Green's function G_0 contains the kinetic energies of all the particles but no interactions.

Using the relation[10]

$$W_{\rm MS}^{\alpha,\beta} = \sum_b{}' W^{\alpha,0}(b) G_0 G_\beta^{-1} + \sum_\gamma W^{\alpha,0}(\gamma) G_0 V_\beta^\gamma - V_\beta^\alpha \tag{2.19}$$

and Eqs. (2.13) and (2.14) we also obtain integral equations for the AGS operators

$$T_{\alpha,\beta} = \bar{\delta}_{\alpha,\beta} G_\beta^{-1} + \sum_a{}' W^{\alpha,0}(a) G_0 G_\beta^{-1} \bar{\delta}_{n_a,2} + W^{\alpha,0}(\beta) G_0 G_\beta^{-1} + \sum_\gamma W^{\alpha,0}(\gamma) G_0 T_{\gamma,\beta} \ . \tag{2.20}$$

Equations (2.20) are used in the next section as the starting point for the derivation of the effective potentials for rearrangement collisions.

III. EFFECTIVE INTERACTIONS

The definition of effective interactions for rearrangement collisions is facilitated by the introduction of the idea of a *reaction mechanism* (RM)[6,7,9,11] as a set A of physically important channels. We consider, henceforth, RM's which include only two-cluster channels; the generalization to include breakup channels in A is nontrivial and will be considered elsewhere. The structure of such a RM is conveniently represented by the subset $\mathfrak{B}$ of the set $\mathfrak{B}_0$ of all two-cluster partitions[9,11]:

$$\mathfrak{B} \equiv \{ \alpha \in \mathfrak{B}_0 | \nu_\alpha \in A \} \ . \tag{3.1}$$

We now introduce the RM projector[6,7,9,11]

$$\mathcal{P}_\alpha(A) = \sum_{\nu_\alpha \in A} \int d\vec{k} | \phi_\alpha(\nu_\alpha)\vec{k} \rangle \langle \phi_\alpha(\nu_\alpha)\vec{k} | \ . \tag{3.2}$$

We note that $\mathcal{P}_\alpha(A) \equiv 0$ unless $\alpha \in \mathfrak{B}$. For any off-shell extension $\bar{T}_{\alpha,\beta}$ of the transition operator, effective interactions $\mathfrak{U}_{\alpha,\beta}(A)$ can be introduced via

$$\bar{T}_{\alpha,\beta} = \mathfrak{U}_{\alpha,\beta}(A) + \sum_{\gamma \in \mathfrak{B}} \mathfrak{U}_{\alpha,\gamma}(A) g_\gamma(A) \bar{T}_{\gamma,\beta} \tag{3.3a}$$

$$= \mathfrak{U}_{\alpha,\beta}(A) + \sum_{\gamma \in \mathfrak{B}} \bar{T}_{\alpha,\gamma}(A) g_\gamma(A) \mathfrak{U}_{\gamma,\beta}(A) \ , \tag{3.3b}$$

where

$$g_\gamma(A) = \mathcal{P}_\gamma(A) G_\gamma \ . \tag{3.4}$$

Evidently Eqs. (3.3) yield effective two-body equations of the coupled-reaction-channel type[3,13] in which only the channels included in the RM are coupled. These equations are discussed in more detail in Sec. V. When the RM is restricted to a

single (elastic) channel ν_β, then $\mathfrak{U}_{\alpha,\beta}(A) = \delta_{\alpha,\beta}\mathfrak{U}_{\beta,\beta}$, where $\mathfrak{U}_{\beta,\beta}$ is the usual optical potential for elastic scattering.[10, 22]

As the generalization of the theory of the optical potential[10, 22] we require the operators $\mathfrak{U}_{\alpha,\beta}(A)$ to be continuous across all the unitarity cuts associated with the (two-cluster) reaction channels from A. This requirement is motivated by the unitarity constraints satisfied by the AGS transition operators and yields a clean separation of the effects of the channels included in A from those which are excluded.

From the definition of the AGS transition operator (2.11) we find the unitarity relation[16]

$$T_{\alpha,\beta}(z) - T^\dagger_{\alpha,\beta}(z) = -2\pi i \sum_c{}' \; T_{\alpha,c}(z)\hat{\delta}(E - H_c) \times T^\dagger_{c,\beta}(z) , \tag{3.5}$$

where $z = E + i0$ and we have defined

$$\hat{\delta}(E - H_c) \equiv \sum_{\nu_c} \int d\vec{k}\, |\phi_c(\nu_c)\vec{k}\rangle \times \langle \phi_c(\nu_c)\vec{k}|\, \delta[E - E(\nu_c, \vec{k})] . \tag{3.6}$$

From (3.3) and (3.5) we find[23]

$$\mathfrak{U}_{\alpha,\beta}(A) - \mathfrak{U}^\dagger_{\alpha,\beta}(A) = -2\pi i \sum_c \mathfrak{U}^\dagger_{\alpha,c}(A)\, \mathfrak{Q}_c(A)\delta(E - H_c)\mathfrak{U}_{c,\beta}(A) , \tag{3.7}$$

where

$$\mathfrak{Q}_c(A) = 1 - \mathcal{P}_c(A) . \tag{3.8}$$

In writing (3.7) we have ignored the contributions of any possible pole singularities in $\mathfrak{U}_{\alpha,\beta}(A)$ to the right side. Equations (3.6)–(3.8) show that the effective interactions $\mathfrak{U}_{\alpha,\beta}(A)$ defined using the AGS scattering operators are continuous across all the unitarity cuts corresponding to the channels $\nu_\gamma \in A$, all $\gamma \in \mathfrak{B}$.

It is important to realize that both the on-shell and off-shell matrix elements of $\mathfrak{U}_{\alpha,\beta}(A)$ enter into (3.3). Thus different off-shell extensions of the transition operators $\overline{T}_{\alpha,\beta}$ in (3.3) yield different effective interactions $\mathfrak{U}_{\alpha,\beta}(A)$ which may not be free of all A-type discontinuities.[24] This is the case for the conventional off-shell extension (2.9). For this reason we confine ourselves henceforth only to the potentials $\mathfrak{U}_{\alpha,\beta}(A)$ defined using the AGS transition operators (2.11), i.e., we set $\overline{T}_{\alpha\beta} = T_{\alpha,\beta}$ in (3.3).

From the preceding discussion it is evident that the use of the AGS transition operators in our development represents the essential ingredient required to obtain effective interactions free of all A-type discontinuities. This generalizes the work of Ref. 16, where it has been shown that the AGS operator can be used to obtain a well-defined theory of the optical potential which not only includes all the effects of the Pauli principle but also possesses no discontinuities across any of the two-cluster elastic unitarity cuts which are physically equivalent due to particle identity. These results suggest that the AGS off-shell extension should be used as the standard definition of the rearrangement scattering transition operator.

The practical usefulness of Eq. (3.3) is limited unless one can derive well-behaved (i.e., solvable in principle) equations for the effective interactions $\mathfrak{U}_{\alpha,\beta}(A)$ or determine them by some other methods (e.g., phenomenologically).[25] In the next section we develop such dynamical equations which possess an explicit multiple scattering structure. These equations permit the systematic development of approximations for the description of rearrangement collisions. This provides a significant improvement over the traditional developments of the CRC[2, 3] and optical potential[22, 26] formalisms.

IV. DYNAMICAL EQUATIONS FOR $\mathfrak{U}_{\alpha,\beta}(A)$

We now derive coupled integral equations for the effective interaction operators $\mathfrak{U}_{\alpha,\beta}(A)$ with kernels which become connected upon iteration. These integral equations should be manifestly free of all A-type discontinuities. We begin our derivation by introducing the operators[10, 16]

$$\Lambda_{\alpha,\beta}(A) = \delta_{\alpha,\beta} + \sum_\gamma [\, W^{\alpha,0}(\gamma)G_0 - V^\alpha_\gamma g_\gamma(A)]\Lambda_{\gamma,\beta}(A) . \tag{4.1}$$

Here $V^\alpha_\gamma g_\gamma(A)$ is the A-discontinuous[24] part of $W^{\alpha,0}(\gamma)\, G_0$ (see Appendix A). Thus the kernels of (4.1), and hence the operators $\Lambda_{\alpha,\beta}(A)$, are free of all A-type discontinuities. We also note that

$$\Lambda_{\alpha,\beta}(A) = \delta_{\alpha,\beta} + \sum_\gamma \Lambda_{\alpha,\gamma}(A)[\, W^{\gamma,0}(\beta)\, G_0 - V^\gamma_\beta g_\beta(A)] . \tag{4.2}$$

The arguments of Ref. 16 are easily extended to show that Eqs. (4.1) and (4.2) have connected kernels after one interation.

From (4.1) and (2.20) we find[16]

$$T_{\alpha,\beta}=I_{\alpha,\beta}(A)+\sum_{\gamma,\lambda}\Lambda_{\alpha,\gamma}(A)V_\lambda^\gamma g_\lambda(A)T_{\lambda,\beta}\ , \tag{4.3}$$

where the inhomogeneous term is given by

$$I_{\alpha,\beta}(A)=\sum_\gamma\Lambda_{\alpha,\gamma}(A)\left\{\bar{\delta}_{\gamma,\beta}+\left[W^{\gamma,0}(\beta)+{\sum_b}'W^{\gamma,0}(b)\bar{\delta}_{n_b,2}\right]G_0\right\}G_\beta^{-1}\ . \tag{4.4}$$

The only A-type discontinuity which could arise in (4.4) comes from the discontinuous part, $V_\beta^\gamma g_\beta$, of $W^{\gamma,0}(\beta)\,G_0$. This term is multiplied by G_β^{-1} and the product is shown in Appendix A to possess no A-type discontinuities. Several alternate forms of $I_{\alpha,\beta}(A)$ exist[16] and some of these are contained in Appendices B and D.

Equation (3.3b) can be rewritten in the form

$$\mathfrak{U}_{\alpha,\beta}(A)=\sum_\gamma T_{\alpha,\gamma}[\delta_{\gamma,\beta}-g_\gamma(A)\mathfrak{U}_{\gamma,\beta}(A)]\ . \tag{4.5}$$

This allows us to convert (4.3) into a set of integral equations for the effective interactions:

$$\mathfrak{U}_{\alpha,\beta}(A)=I_{\alpha,\beta}(A)+\sum_{\gamma\in\mathfrak{B}}K_{\alpha,\gamma}(A)\,\mathfrak{U}_{\gamma,\beta}(A)\ , \tag{4.6}$$

Here $K_{\alpha,\beta}(A)$ is (cf. Appendix B):

$$K_{\alpha,\beta}(A)=\left\{\delta_{\alpha,\beta}-\sum_\lambda\Lambda_{\alpha,\lambda}(A)\times\left[1+{\sum_b}'W^{\lambda,0}(b)G_0\bar{\delta}_{n_b,2}\right]\right\}\mathcal{P}_\beta(A)\ . \tag{4.7}$$

Alternative forms of $K_{\alpha,\beta}(A)$ are given in Appendix B. It is also shown there that Eqs. (4.6) are coupled connected-kernel integral equations for the operators $\mathfrak{U}_{\alpha,\beta}(A)$, $\alpha,\beta\in\mathfrak{B}$, with kernel and inhomogeneous terms which are manifestly free of all A-type discontinuities. This implies that the $\mathfrak{U}_{\alpha,\beta}(A)$ are also free of all such singularities and remain so, even when approximated.

The connected kernel property is a necessary condition for compactness. Integral equations with compact kernels are essentially Fredholm equations and as such they can be solved either exactly or approximately in a well-defined and systematic fashion. Thus dynamical equations such as (4.6) represent a significant advantage over alternative, conventional formulations[2,3,27] of coupled reaction channel problems in that they provide a realistic opportunity for calculating systematic corrections to low-order approximations. Further aspects of these equations and their utilization are taken up in the next few sections.

V. ANTISYMMETRIZATION

Up to this point we have assumed that all the particles are distinguishable. When this is not the case, all channels which are physically equivalent because of the particle identity must be treated simultaneously. This can be accomplished in a rather simple manner if we can show that the dynamical equations of the previous two sections are label transforming.[16,18] This concept is defined in Appendix C where we also introduce the notation and review the relevant results of Refs. 16 and 18. For the sake of simplicity, we shall restrict ourselves to the case where all particles are identical (such as N nucleons). The more general case including several particle species represents a straightforward generalization of the present results (see also Refs. 16 and 18).

We define the antisymmetrization procedure by choosing the RM A such that it contains all channels related by permutations, i.e., for any channel $\nu_\alpha\in A$, the channels $\nu_{P(\alpha)}$, $P\in S$ are also included in A. With such a choice of the RM the reaction set $\mathfrak{B}$ is closed under permutations; i.e., for any partition $\gamma\in\mathfrak{B}$, $P(\gamma)\in\mathfrak{B}$. We denote by $\hat{\mathfrak{B}}$ the set of equivalence classes of partitions included in $\mathfrak{B}$.

We now introduce the antisymmetrized effective interaction operators

$$\tilde{\mathfrak{U}}_{\hat{\alpha},\hat{\beta}}(A)=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathfrak{K}_\alpha\mathfrak{U}_{\alpha,\bar{\beta}}(A)\ . \tag{5.1}$$

One can verify[16] that the operators $I_{\alpha,\beta}(A)$ and $K_{\alpha,\beta}(A)$ are label transforming; (4.6) then implies that the $\mathfrak{U}_{\alpha,\beta}(A)$ are also label-transforming operators. The antisymmetrized form of (4.6) is then (cf. Appendix C)

$$\tilde{\mathfrak{U}}_{\hat{\alpha},\hat{\beta}}(A)=\tilde{I}_{\hat{\alpha},\hat{\beta}}(A)+\sum_{\hat{\gamma}\in\hat{\mathfrak{B}}}\tilde{K}_{\hat{\alpha},\hat{\gamma}}(A)\mathcal{P}_{\bar{\gamma}}(A)\,\tilde{\mathfrak{U}}_{\hat{\gamma},\hat{\beta}}(A)\ , \tag{5.2}$$

where

$$\tilde{I}_{\hat{\alpha},\hat{\beta}}(A)=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathfrak{K}_\alpha I_{\alpha,\bar{\beta}}(A)\ , \tag{5.3}$$

$$\tilde{K}_{\hat{\alpha},\hat{\beta}}(A)=\bar{N}_{\hat{\alpha},\hat{\beta}}\sum_{\alpha\in\hat{\alpha}}\mathfrak{K}_\alpha K_{\alpha,\bar{\beta}}(A)\ . \tag{5.4}$$

The antisymmetrized AGS transition operator is

$$\tilde{T}_{\hat{\alpha},\hat{\beta}} = \bar{N}_{\hat{\alpha},\hat{\beta}} \sum_{\alpha \in \hat{\alpha}} \mathcal{R}_\alpha T_{\alpha,\bar{\beta}}(A) \ . \tag{5.5}$$

Since both $\mathcal{U}_{\alpha,\beta}$ and $T_{\alpha,\beta}$ are label transforming, the antisymmetrized forms of Eqs. (3.3) are then

$$\tilde{T}_{\hat{\alpha},\hat{\beta}} = \tilde{\mathcal{U}}_{\hat{\alpha},\hat{\beta}}(A) + \sum_{\hat{\gamma} \in \hat{\mathcal{B}}} \tilde{\mathcal{U}}_{\hat{\alpha},\hat{\gamma}}(A) g_{\bar{\gamma}}(A) \tilde{T}_{\hat{\gamma},\hat{\beta}} \tag{5.6a}$$

$$= \tilde{\mathcal{U}}_{\hat{\alpha},\hat{\beta}}(A) + \sum_{\hat{\gamma} \in \hat{\mathcal{B}}} \tilde{T}_{\hat{\alpha},\hat{\gamma}} g_{\bar{\gamma}}(A) \tilde{\mathcal{U}}_{\hat{\gamma},\hat{\beta}}(A) \ . \tag{5.6b}$$

We infer from (5.6) that the $\tilde{\mathcal{U}}_{\hat{\alpha},\hat{\beta}}(A)$ represent the effective interactions appropriate to a CRC-type description of rearrangement collisions involving identical particles. Furthermore, the singularity structure of $\mathcal{U}_{\alpha,\beta}(A)$ implies that $\tilde{\mathcal{U}}_{\hat{\alpha},\hat{\beta}}(A)$ is free of all A-type discontinuities. Equations (5.1)–(5.6), together with the definitions of $\tilde{I}_{\hat{\alpha},\hat{\beta}}(A)$ and $\tilde{K}_{\hat{\alpha},\hat{\beta}}(A)$ given in the previous section, represent the generalization of the results of Ref. 16 to encompass rearrangement collisions.

Equations similar to (5.6) have been found for identical particle scattering in Ref. 18. These equations, however, involve all the two-cluster channels and an off-shell extension for the transition operators which does not lend itself to the formulation of multiple-scattering approximations.

VI. GENERALIZED CRC METHOD

In this section we rewrite Eqs. (3.3) and (5.6) in wave function form to demonstrate the relationship of the present formalism to the standard CRC approaches.[2,3] We start by noting that for all partitions α, γ we have

$$(\delta_{\alpha,\gamma} + G_\alpha T_{\alpha,\gamma}) | \phi_\gamma(\nu_\gamma)\vec{k}_\gamma\rangle = | \psi_\gamma(\nu_\gamma)\vec{k}_\gamma\rangle \ . \tag{6.1}$$

Here $|\psi_\gamma(\nu_\gamma)\vec{k}_\gamma\rangle$ is the full scattering many-body wave function which evolved from the asymptotic state $|\phi_\gamma(\nu_\gamma)\vec{k}_\gamma\rangle$ in the infinite past. From (6.1) and (6.3) it is thus straightforward to obtain the projected wave function equations

$$\sum_{\gamma \in \mathcal{B}} [(E - H_\gamma)\delta_{\lambda,\gamma} - \mathcal{P}_\lambda(A)\mathcal{U}_{\lambda,\gamma}(A)] \times \mathcal{P}_\gamma(A) | \psi_\beta(\nu_\beta)\vec{k}_\beta\rangle = 0 \ . \tag{6.2}$$

Equations (6.2) are exact and represent the multichannel generalization of the Feshbach optical potential formalism.

Let us define the wave functions

$$\chi_{\gamma,\beta}(\nu_\gamma\vec{k}_\gamma | \nu_\beta\vec{k}_\beta) = \langle \phi_\gamma(\nu_\gamma)\vec{k}_\gamma | \psi_\beta(\nu_\beta)\vec{k}_\beta\rangle \ , \tag{6.3}$$

which asymptotically describe the relative motion of the clusters of the γ partition. We also define the complex, energy-dependent cluster-cluster potentials ($\gamma, \beta \in \mathcal{B}$)

$$\mathcal{V}_{\gamma,\beta}(\nu_\gamma\vec{k}_\gamma | \nu_\beta\vec{k}_\beta) = \langle \phi_\gamma(\nu_\gamma)\vec{k}_\gamma | \mathcal{U}_{\gamma,\beta}(A) | \phi_\beta(\nu_\beta)\vec{k}_\beta\rangle \tag{6.4}$$

where the dependence on A has been suppressed. Equation (6.2) can then be rewritten as ($\nu_\lambda, \nu_\beta \in A$):

$$[E - E(\nu_\lambda, \vec{k}_\lambda)]\chi_{\lambda,\beta}(\nu_\lambda\vec{k}_\lambda | \nu_\beta\vec{k}_\beta) = \sum_{\gamma \in \mathcal{B}} \sum_{\nu_\gamma \in A} \int d\vec{k}'_\gamma \, \mathcal{V}_{\lambda,\gamma}(\nu_\lambda\vec{k}_\lambda | \nu_\gamma\vec{k}'_\gamma) \times \chi_{\gamma,\beta}(\nu_\gamma\vec{k}'_\gamma | \nu_\beta\vec{k}_\beta) \ . \tag{6.5}$$

In coordinate space, (6.5) are exact, one-vector-variable integrodifferential equations for the wave functions $\chi_{\gamma,\beta}$. Given the potentials $\mathcal{V}_{\lambda,\gamma}$, (6.5) can be solved by standard CRC methods.[2,3] Usually the $\mathcal{V}_{\lambda,\gamma}$ are approximated, e.g., by phenomenological cluster-cluster potentials.[2,3] In the next section we discuss some possible approximations to $\mathcal{V}_{\lambda,\gamma}$ which follow from the multiple-scattering structure of the operators $\mathcal{U}_{\lambda,\gamma}(A)$.

In practical applications the description of nuclear reactions is further complicated by the effects of particle identity. We now show, using the results of Sec. V and Appendix C, how these effects can be incorporated in (6.1)–(6.5). As before, we restrict ourselves to a system composed of single species of identical particles, such as nucleons.

The antisymmetrization of (6.2) can be carried out straightforwardly since $(E - H_\gamma)\,\delta_{\lambda,\gamma}$ and $\mathcal{U}_{\lambda,\gamma}(A)$ are both label transforming. Using the methods of Appendix C, one finds that

$$\sum_{\hat{\gamma} \in \hat{\beta}} (N_{\hat{\gamma}})^{1/2} [(E - H_{\bar{\gamma}})\delta_{\hat{\lambda},\hat{\gamma}} - \mathcal{P}_{\bar{\lambda}}(A)\tilde{\mathcal{U}}_{\hat{\lambda},\hat{\gamma}}(A)\mathcal{P}_{\bar{\gamma}}(A)] | \tilde{\psi}_{\hat{\beta}}(\nu_{\bar{\beta}})\vec{k}_{\bar{\beta}}\rangle = 0, \tag{6.6}$$

where

$$| \tilde{\psi}_{\hat{\beta}}(\nu_{\bar{\beta}})\vec{k}_{\bar{\beta}}\rangle = R | \psi_{\bar{\beta}}(\nu_{\bar{\beta}})\vec{k}_{\bar{\beta}}\rangle \tag{6.7}$$

is the normalized fully antisymmetrized many-body scattering wave function. In deriving (6.6) we

have assumed that the channel states $|\phi_\gamma(\nu_\gamma)\vec{k}\rangle$ are properly antisymmetrized with regard to their internal structure.

We now introduce the antisymmetrized projected wave functions

$$\tilde{\chi}_{\hat{\gamma},\beta}(\nu_{\bar{\gamma}}\vec{k}_{\bar{\gamma}}|\nu_{\bar{\beta}}\vec{k}_{\bar{\beta}}) = \langle \phi_{\bar{\gamma}}(\nu_{\bar{\gamma}})\vec{k}_{\bar{\gamma}}|\tilde{\psi}_{\bar{\beta}}(\nu_{\bar{\beta}})\vec{k}_{\bar{\beta}}\rangle . \quad (6.8)$$

Equation (6.6) then can be rewritten in the form

$$[E - E(\nu_{\bar{\lambda}},\vec{k}_{\bar{\lambda}})]\tilde{\chi}_{\hat{\lambda},\beta}(\nu_{\bar{\lambda}}\vec{k}_{\bar{\lambda}}|\nu_{\bar{\beta}}\vec{k}_{\bar{\beta}}) = \sum_{\hat{\gamma}\in\hat{\mathfrak{B}}}\sum_{\nu_{\bar{\gamma}}\in A}\int d\vec{k}'_{\bar{\gamma}}\,\tilde{v}_{\hat{\lambda},\hat{\gamma}}(\nu_{\bar{\lambda}}\vec{k}_{\bar{\lambda}}|\nu_{\bar{\gamma}}\vec{k}'_{\bar{\gamma}}) \times \bar{N}_{\hat{\lambda},\hat{\gamma}}\tilde{\chi}_{\hat{\gamma},\beta}(\nu_{\bar{\gamma}}\vec{k}'_{\bar{\gamma}}|\nu_{\bar{\beta}}\vec{k}_{\bar{\beta}}) \quad (6.9)$$

Here $\tilde{v}_{\hat{\lambda},\hat{\gamma}}$ is the antisymmetrized cluster-cluster effective interaction

$$\tilde{v}_{\hat{\lambda},\hat{\gamma}}(\nu_{\bar{\lambda}}\vec{k}_{\bar{\lambda}}|\nu_{\bar{\gamma}}\vec{k}_{\bar{\gamma}}) = \langle \phi_{\bar{\lambda}}(\nu_{\bar{\lambda}})\vec{k}_{\bar{\lambda}}|\tilde{u}_{\hat{\lambda},\hat{\gamma}}(A)|\phi_{\bar{\gamma}}(\nu_{\bar{\gamma}})\vec{k}_{\bar{\gamma}}\rangle . \quad (6.10)$$

From the definition of $\tilde{u}_{\hat{\lambda},\hat{\gamma}}(A)$, Eq. (5.1) we find

$$\tilde{v}_{\hat{\lambda},\hat{\gamma}}(\nu_{\bar{\lambda}}\vec{k}_{\bar{\lambda}}|\nu_{\bar{\gamma}}\vec{k}_{\bar{\gamma}}) = \bar{N}_{\hat{\lambda},\hat{\gamma}}\sum_{\lambda\in\hat{\lambda}} v_{\lambda,\bar{\gamma}}(\nu_{\bar{\gamma}}\vec{k}_{\bar{\gamma}}|\nu_{\bar{\gamma}}\vec{k}_{\bar{\gamma}}) \times \delta(P_{\lambda,\bar{\lambda}}) , \quad (6.11)$$

where $\delta(P_{\lambda,\bar{\lambda}})$ is defined in Appendix C.

Equations (6.1)–(6.11) together with the dynamical equations for $u_{\lambda,\gamma}(A)$ found in Sec. IV constitute an exact and well-defined formulation of the nuclear coupled reaction channel problem. It is of considerable interest, however, to see how the standard CRC formalisms[2,3,27] can be recovered from (6.2) and (6.6). The conventional CRC approximation is derived directly from the Schrödinger equation

$$(E - H)|\psi_\gamma(\nu_\gamma)\vec{k}_\gamma\rangle = 0 \quad (6.12)$$

in a manner which does not suggest how one can formulate systematic improvements. This shortcoming is removed by our demonstration of how the CRC is embedded in our general results.

If one makes the approximation

$$u_{\lambda,\gamma}(A) \simeq V^\gamma - \bar{\delta}_{\lambda,\gamma}G_\gamma^{-1} . \quad (6.13)$$

Eq. (6.2) reduces to the standard CRC form

$$\sum_{\gamma\in\mathfrak{B}} \mathcal{P}_\lambda(A)(E - H)\mathcal{P}_\gamma(A)|\psi_\beta^{\mathrm{CRC}}(\nu_\beta)\vec{k}_\beta\rangle = 0 , \quad (6.14)$$

where we have used the superscript CRC to signify the approximate nature of the wave function. The standard CRC equations[2,3] of the form (6.5) are obtained by taking the scalar product of (6.14) with $\langle\phi_\lambda(\nu_\lambda)\vec{k}_\lambda|$. The antisymmetrized counterparts of (6.14) are the resonanting group equations.[27] One finds, using (6.13) and (5.1), that (6.6) becomes

$$\sum_{\hat{\gamma}} N_{\hat{\gamma}}\mathcal{P}_{\bar{\gamma}}(A)R(E - H)\mathcal{P}_{\bar{\gamma}}(A)|\tilde{\psi}_\beta^{\mathrm{CRC}}(\nu_{\bar{\beta}})\vec{k}_{\bar{\beta}}\rangle = 0 . \quad (6.15)$$

The relationship of (6.13) to other systematic approximation procedures is discussed in the next section.

VII. APPROXIMATIONS

The exact determination of the effective interactions requires the solution of an N-body scattering problem. This is impractical for large N. It is necessary, therefore, to seek approximations appropriate to different physical situations which can result in practical calculations.

One class of such approximations are the low-density multiple scattering expansions.[10,22,26] These approximations have been used extensively in studies of nucleon-nucleus and pion-nucleus scattering at intermediate energies. We now consider similar approximations within the framework of the present formalism. The simplest approximations of this type in our equations is to ignore all but the N, $(N-1)$-connected terms in Eqs. (4.4) and (4.7). This corresponds to retaining only the two-particle scattering terms in the partition sums. One then finds that (4.6) reduces to the approximate form (cf. Appendix D):

$$u_{\alpha,\beta}(A) \cong \sum_{i'}\bar{\Delta}_{\alpha,i'}t_{i'}\bar{\Delta}_{\beta,i'} + V_\beta^\alpha + \bar{\delta}_{\alpha,\beta}G_\beta^{-1} - \sum_{\gamma\in\mathfrak{B}}\left(\bar{\delta}_{\alpha,\gamma} + \sum_{i'}\bar{\Delta}_{\alpha,i'}t_{i'}G_0\right) \times \mathcal{P}_\gamma(A)u_{\gamma,\beta}(A) . \quad (7.1)$$

Equation (7.1) represents the basis for our discussion of the various low-density approximations obtained in the present formalism. We note that the result (7.1) follows directly from either (4.4) and (4.7) or (B4) and (B5).

Let us consider now the integral equations for the effective potentials $v_{\alpha,\beta}$ defined by (6.4) which follow from (7.1). Let us define ($\nu_\alpha, \nu_\beta \in A$):

$$\mathcal{T}_{\alpha,\beta}(\nu_\alpha \vec{k}_\alpha | \nu_\beta \vec{k}_\beta) = \sum_{i'} \bar{\Delta}_{\alpha,i'} \langle \phi_\alpha(\nu_\alpha)\vec{k}_\alpha | t_{i'} \bar{\Delta}_{\beta,i'} + V_{i'} \Delta_{\beta,i'} | \phi_\beta(\nu_\beta)\vec{k}_\beta \rangle , \tag{7.2}$$

$$\mathcal{K}_{\alpha,\beta}(\nu_\alpha \vec{k}_\alpha | \nu_\beta \vec{k}_\beta) \equiv \left\langle \phi_\alpha(\nu_\alpha)\vec{k}_\alpha \middle| \sum_{i'} \bar{\Delta}_{\alpha,i'} t_{i'} G_0 \middle| \phi_\beta(\nu_\beta)\vec{k}_\beta \right\rangle, \tag{7.3}$$

and

$$\mathfrak{N}_{\alpha,\beta}(\nu_\alpha \vec{k}_\alpha | \nu_\beta \vec{k}_\beta) \equiv \langle \phi_\alpha(\nu_\alpha)\vec{k}_\alpha | \phi_\beta(\nu_\beta)\vec{k}_\beta \rangle \bar{\delta}_{\alpha,\beta} . \tag{7.4}$$

Then Eq. (7.1) can be written as ($\nu_\alpha, \nu_\beta \in A$)

$$\mathcal{U}_{\alpha,\beta}(\nu_\alpha \vec{k}_\alpha | \nu_\beta \vec{k}_\beta) = \mathcal{T}_{\alpha,\beta}(\nu_\alpha \vec{k}_\alpha | \nu_\beta \vec{k}_\beta) + [E - E(\nu_\beta, \vec{k}_\beta)]\mathfrak{N}_{\alpha,\beta}(\nu_\alpha \vec{k}_\alpha | \nu_\beta \vec{k}_\beta)$$
$$- \sum_{\gamma \in \mathcal{B}} \sum_{\nu_\gamma \in A} \int d\vec{k}'_\gamma [\mathfrak{N}_{\alpha,\gamma}(\nu_\alpha \vec{k}_\alpha | \nu_\gamma \vec{k}'_\gamma) + \mathcal{K}_{\alpha,\gamma}(\nu_\alpha \vec{k}_\alpha | \nu_\gamma \vec{k}'_\gamma)] \mathcal{U}_{\gamma,\beta}(\nu_\gamma \vec{k}'_\gamma | \nu_\beta \vec{k}_\beta) , \tag{7.5}$$

or in obvious matrix-operator notation,

$$\mathcal{U} = \mathcal{T} + \mathfrak{N}(E - \mathcal{E}) - (\mathfrak{N} + \mathcal{K})\mathcal{U} . \tag{7.6}$$

Here $\mathcal{E}$ has the elements $\delta_{\gamma,\beta}\delta_{\nu_\gamma,\nu_\beta}\delta(\vec{k}'_\gamma - \vec{k}_\beta) E(\nu_\beta, \vec{k}_\beta)$. After a partial wave analysis Eqs. (7.6) reduce to a set of coupled one-dimensional integral equations. Given the input functions $\mathcal{T}$, $\mathfrak{N}$, and $\mathcal{K}$ these equations can be solved numerically provided the number of coupled reaction channels is not too large.

It is interesting to consider the generalization of the impulse approximation which follows from (7.6). To obtain this we introduce the operator[28]

$$\bar{\mathcal{U}} = \mathcal{U} - (I + \mathfrak{N})^{-1}\mathfrak{N}(E - \mathcal{E}) . \tag{7.7}$$

From (7.6), $\bar{\mathcal{U}}$ can be shown to satisfy the integral equation

$$\bar{\mathcal{U}} = (I + \mathfrak{N})^{-1}\mathcal{T} - \bar{\mathcal{K}}\mathfrak{N}(E - \mathcal{E}) - \bar{\mathcal{K}}(I + \mathfrak{N})\bar{\mathcal{U}} . \tag{7.8}$$

In (7.8) we have defined

$$\bar{\mathcal{K}} = (I + \mathfrak{N})^{-1}\mathcal{K}(I + \mathfrak{N})^{-1} . \tag{7.9}$$

The advantage of (7.8) over (7.6) lies in the fact that all the single-scattering terms (i.e., those linear in $t_{i'}$) arise only from the inhomogeneity. A form of the impulse approximation can then be obtained from (7.8) by neglecting the kernel term, namely,

$$\bar{\mathcal{U}} \simeq (I + \mathfrak{N})^{-1}\mathcal{T} - \bar{\mathcal{K}}\mathfrak{N}(E - \mathcal{E}) . \tag{7.10}$$

Combining (7.7) and (7.10) we finally obtain

$$\mathcal{U} \simeq (I + \mathfrak{N})^{-1}[\mathcal{T} + \mathfrak{N}(E - \mathcal{E})] - \bar{\mathcal{K}}\mathfrak{N}(E - \mathcal{E}) . \tag{7.11}$$

Equation (7.11) contains channel nonorthogonality effects arising from the overlap kernel $\mathfrak{N}$ in a nonperturbative fashion, while the target-projectile dynamics (contained in the two-body potentials $V_{i'}$ and t matrices $t_{i'}$) appear only in the lowest order. The nonorthogonality effects may be small.[2,3,13] If this is the case then it is consistent to retain only terms linear in $\mathfrak{N}$, $\mathcal{T}$, and $\mathcal{K}$ in (7.11). We then obtain the matrix version of the inhomogeneous terms of (7.1),

$$\mathcal{U} = \mathcal{T} + \mathfrak{N}(E - \mathcal{E}) . \tag{7.12}$$

Equation (7.12) represents a multichannel generalization of the impulse approximation. Calculations based upon (7.12) are well within the present day computational capabilities. It should be noted that the two-particle transition operator $t_{i'}$ in (7.1)–(7.12) is defined solely in terms of two-nucleon information. Also, the kinematics relating the total c.m. system to the two-nucleon c.m. is uniquely defined. In other formulations of the impulse approximation[10,26] the recovery of similar low-order approximations is often complicated by ambiguities in the definition of the appropriate kinematics.

Equation (7.12) yields effective potentials very similar to those employed, implicitly, in standard CRC calculations [cf. Eq. (6.13)] if we make the further approximation $t_{i'} \approx V_{i'}$. We then obtain from (6.2)

$$\sum_\gamma \mathcal{P}_\lambda(A)(E - H)\mathcal{P}_\gamma(A)|\psi^{IA}_\beta(\nu_\beta)\vec{k}_\beta\rangle$$
$$= \sum_{\gamma \neq \lambda} \mathcal{P}_\lambda(A)(G_\gamma^{-1} + G_\lambda^{-1})\mathcal{P}_\gamma(A)|\psi^{IA}_\beta(\nu_\beta)\vec{k}_\beta\rangle , \tag{7.13}$$

where we have used the superscript IA to signify the approximate nature of the wave function. Equation (7.13) differs from (6.14) in that its

right-hand side is not zero. If we rewrite (7.13) in momentum space [cf. (6.5)], its right-hand side becomes

$$\sum_{\gamma}\int d\vec{k}'_{\gamma}\,[2E - E(\nu_{\gamma},\vec{k}'_{\gamma}) - E(\nu_{\lambda},\vec{k}_{\lambda})] \times \mathfrak{R}_{\lambda,\gamma}(\nu_{\lambda}\vec{k}_{\lambda}|\nu_{\gamma}\vec{k}'_{\gamma})\mathfrak{X}^{IA}_{\gamma,\beta}(\nu_{\gamma}\vec{k}'_{\gamma}|\nu_{\beta}\vec{k}_{\beta})\,. \tag{7.14}$$

Although (7.14) may be negligible if $\mathfrak{R}_{\lambda,\gamma}$ is sharply peaked about the on-shell values of $\vec{k}_{\lambda}$ and $\vec{k}_{\gamma}$, it is not obvious for what types of fragment wave functions this will be true. Calculations comparing (6.14) and (7.13) are needed to settle the question of the importance of the term (7.14) which provides the measure of their difference. This comparison should present no more difficulty than standard CRC numerical calculations [based on (6.14)] because (7.14) is nearly of the same form as the nonorthogonality corrections which appear in these works.[2,3,27]

It must be stressed that (6.14) and (7.13) are the different end results of two distinct series of approximations. Thus, the significance of a comparison between the predictions of (6.14) and (7.13) lies in the possible superiority of one approximation over the other under a given set of physical circumstances. The usual CRC formalism, embodied in Eq. (6.14), is a consequence of variational reasoning applied directly to the Schrödinger equation (6.12). It is not easy to see how (6.13) and (6.14) result from a plausible series of approximations within our particular N-particle formulation[29] since the physical significance of the term (7.14) is, at present, not understood. Nonetheless, we have shown how both (6.14) and (7.13) are embedded in a full dynamical description wherein higher-order corrections can be executed in a consistent and systematic fashion.

It is also interesting to note that Eqs. (7.13) contain explicit nonorthogonality terms $\mathfrak{R}_{\lambda,\gamma}$. This may be relevant to the recent controversy[19] concerning the presence of such terms in many-body formulations of the CRC-type equations. Such terms appear in our formalism because of our constraints on the discontinuity structure of $\mathfrak{U}_{\lambda,\gamma}$. Specifically, the nonorthogonality terms are generated by the $\bar{\delta}_{\lambda,\gamma}G_{\gamma}^{-1}$ piece of the AGS off-shell extension (2.11). Evidently the appearance and structure of the nonorthogonality terms depend crucially upon the off-shell extension of $T_{\lambda,\gamma}$ used in (3.3) for the definition of $\mathfrak{U}_{\lambda,\gamma}$.

A class of higher-order approximations to $\mathfrak{U}_{\alpha,\beta}(A)$ can be obtained if we include the $(N-2)(N-3)\ldots$ connected terms in the inhomogenity and kernel of the integral equations for $\mathfrak{U}_{\alpha,\beta}$. Such a procedure is limited primarily by its computational feasibility, and actual numerical calculations are expected to be restricted to the equations not much more complicated than those of the type given in this section. It should be noted that the practical utilization of the approximation discussed here is based on the assumption that the nucleon-nucleon interaction and the bound-state wave functions are known. In practice, this is often not the case and an approximate cluster description is used, e.g., an inert core plus two external nucleons interacting via effective potentials whose parameters are adjusted to fit the experimental separation energies.[3,13] Finally, we note that the results of this section can be easily extended to include the effects of particle identiy using the results of the previous two sections.

It should be emphasized that all the approximations discussed in this section are expected to be valid at relatively high energies at which the effects of target binding and multiparticle correlations are small, i.e., where the terms of higher connectivity in $I_{\lambda,\gamma}$ and $K_{\lambda,\gamma}$ can be neglected. Approximations to $\mathfrak{U}_{\alpha,\beta}(A)$ corresponding to other physical situations cannot be discussed within the present, multiple scattering approach and other formulations of the problem must be developed. However, the present study suggests that any such approach should use the AGS off-shell extension (2.11) and (3.3) to define the effective interactions. This ensures that the $\mathfrak{U}_{\alpha,\beta}(A)$ are free of all A-type discontinuities and remain so even when approximated. This is not the case with many approximations[1-3,22,26] often used in analyses of nuclear reactions.

VIII. SUMMARY

In this paper we have presented an N-body theory of rearrangement collisions. This theory is expressed as a coupled reaction channel formalism and includes all effects of the Pauli principle in a practical fashion. As in Ref. 16, it is seen that the use of the AGS transition operator in the definition of the effective interactions is essential in order that the effective interactions be free of all discontinuities corresponding to the channels included in the reaction model. The effective cluster-cluster interactions are shown to be matrix elements of operators which satisfy connected-kernel integral equations with an explicit multiple-scattering structure. This feature of the theory suggests various systematic approximations. For example, in the lowest order of a density-type expansion we obtain equations, similar to the standard CRC equations, which represent a generalization of the impulse approximation to

rearrangement collisions. The effects of particle identity are handled consistently and straightforwardly in all orders of approximation.

We have shown exactly how the standard CRC formalism is embedded in a full and consistent N-body scattering theory. However, we were not able to obtain this approximation from our equations in an entirely plausible fashion. We have suggested some specific numerical calculations which should help to decide whether the standard CRC or our lowest-order approximation constitutes a better starting point for formulating higher-order corrections. Whatever the conclusion of such an investigation, the present formulation of the many-nucleon scattering problem or a formalism equivalent to it provides a vehicle for carrying out such calculations in a well-defined and practical manner.

This work was supported in part by the National Science Foundation under the Grant No. PHY78-26595.

APPENDIX A: CONNECTIVITY STRUCTURE OF $W^{a,b}(c)$

We consider the connectivity structure of the $W^{a,b}(c)$ operators. Let $[\mathcal{O}]_c$ be the c-connected part of $\mathcal{O}$. By definition $W^{a,b}(c)$ is given by[10,11,16]

$$W^{a,b}(c)=[V^{a,b}+V^aGV^b]_c=[V_c^{a,b}+V_c^aG_cV_c^b]_c . \tag{A1}$$

The only discontinuous part of $W^{a,b}(c)$ is the term $[V_c^aG_cV_c^b]_c$. Since A contains only two-cluster channels, this term cannot have any A-type discontinuities unless $c=\gamma\in\mathfrak{B}$. The part of G discontinuous across the A-type cut is the sum of the pole terms $\mathcal{P}_\gamma(A)G_\gamma=g_\gamma(A)$. We therefore conclude that

$$\overline{W}^{a,b}(\gamma)\equiv W^{a,b}(\gamma)-V_\gamma^a g_\gamma(A)V_\gamma^b \tag{A2}$$

is continuous across all A-type unitarity cuts. Consider now $b=0$. Since $V_\gamma^0=V_\gamma$ and

$$V_\gamma=G_0^{-1}-G_\gamma^{-1} , \tag{A3}$$

we conclude that

$$\overline{W}^{a,0}(\gamma)=[W^{a,0}(\gamma)G_0-V_\gamma^a g_\gamma(A)]G_0^{-1}+V_\gamma^a\mathcal{P}_\gamma(A) \tag{A4}$$

is continuous across all A-type unitarity cuts. Since $V_\gamma^a\mathcal{P}_\gamma(A)$ is a nonsingular operator it follows that the kernel of (4.1) is free of any A-type singularities. Evidently $W^{a,0}(\gamma)G_0G_\gamma^{-1}$ is also free of these singularities.

APPENDIX B: KERNEL AND INHOMOGENEITY OF (4.6)

If we combine (2.21), (4.3), and (4.5), the kernel of (4.6) is found to be

$$K_{\alpha,\lambda}(A)=\sum_\gamma\Lambda_{\alpha,\gamma}(A)\left[V\!\!\!/_\lambda-\overline{\delta}_{\gamma,\lambda}G_\lambda^{-1}-W^{\gamma,0}(\lambda)G_0G_\lambda^{-1}-\sum_b{}'W^{\gamma,0}(b)G_0G_\lambda^{-1}\overline{\delta}_{n_b,2}\right]g_\lambda(A) . \tag{B1}$$

Then using (4.2) in combination with (B1) we obtain

$$K_{\alpha,\lambda}(A)=\left\{\delta_{\alpha,\lambda}-\sum_\gamma\Lambda_{\alpha,\gamma}(A)\left[1+\sum_b{}'W^{\gamma,0}(b)G_0\overline{\delta}_{n_b,2}\right]\right\}\mathcal{P}_\lambda(A) , \tag{B2}$$

which is manifestly free of all A-class singularities. Equation (B2) can be rewritten in an alternate form using the identity

$$\sum_{\lambda,\gamma}\Lambda_{\alpha,\gamma}(A)[\delta_{\gamma,\lambda}-W^{\gamma,0}(\lambda)G_0]\overline{\delta}_{\lambda,\beta}\delta(\lambda\in\mathfrak{B})=\overline{\delta}_{\alpha,\beta}\delta(\alpha\in\mathfrak{B}) \tag{B3}$$

and we find

$$K_{\alpha,\lambda}(A)=\left[\delta_{\alpha,\lambda}-\sum_{\gamma\in\mathfrak{B}}\Lambda_{\alpha,\lambda}(A)-\sum_\gamma\Lambda_{\alpha,\gamma}(A)\sum_{b\notin\mathfrak{B}}{}'W^{\gamma,0}(b)G_0\right]\mathcal{P}_\lambda(A) . \tag{B4}$$

Here $\delta(\gamma\in\mathfrak{B})=1$ if $\gamma\in\mathfrak{B}$, and zero otherwise. Also, $\delta(\gamma\notin\mathfrak{B})=1-\delta(\gamma\in\mathfrak{B})$. Similarly, the inhomogeneous term (4.4) can be written in an alternative form using (B3). We find

$$I_{\alpha,\beta}(A)=\sum_\gamma\Lambda_{\alpha,\gamma}(A)\left\{\left[\sum_{b\notin\mathfrak{B}}{}'W^{\gamma,0}(b)+W^{\gamma,0}(\beta)\delta(\beta\in\mathfrak{B})\right]G_0+\delta(\gamma\in\mathfrak{B})\overline{\delta}_{\gamma,\beta}\right\}G_\beta^{-1}+\overline{\delta}_{\alpha,\beta}\delta(\alpha\notin\mathfrak{B})G_\beta^{-1} . \tag{B5}$$

Many other transformations of $I_{\alpha,\beta}(A)$ are possible, one of which is (D1) given in Appendix D.

We now demonstrate that (4.6) has a connected kernel after one iteration. Iterating (4.6) we get

$$\mathfrak{U}_{\alpha,\beta}(A)=I_{\alpha,\beta}(A)+\sum_{\gamma\in\mathfrak{G}} I_{\alpha,\gamma}(A)K_{\gamma,\beta}(A) + \sum_{\gamma,\lambda\in\mathfrak{G}} K_{\alpha,\gamma}(A)K_{\gamma,\lambda}(A)\mathfrak{U}_{\lambda,\beta}(A) . \tag{B6}$$

Since $K_{\alpha,\gamma}(A)=K_{\alpha,\gamma}(A)\mathcal{P}_\gamma(A)$ it is sufficient to show that $\mathcal{P}_\alpha(A)K_{\alpha,\gamma}(A)$ is connected. Using (4.2) we can rewrite (B2) as

$$K_{\alpha,\lambda}(A)=\Big\{\overline{\delta}_{\alpha,\lambda}-\sum_b{}' W^{\alpha,0}(b)G_0\overline{\delta}_{n_b,2} - \sum_{\sigma,\gamma}[W^{\alpha,0}(\sigma)G_0-V^\alpha_\sigma g_\sigma(A)]\Lambda_{\sigma,\gamma}(A) \times\Big[1+\sum_b{}' W^{\gamma,0}(b)G_0\overline{\delta}_{n_b,2}\Big]\Big\}\mathcal{P}_\lambda(A) . \tag{B7}$$

Since $W^{\alpha,0}(b)$, and V^α_σ are constructed from interactions external to partition α, these terms will become connected upon multiplication by $\mathcal{P}_\alpha(A)$, while $\mathcal{P}_\alpha(A)\overline{\delta}_{\alpha,\lambda}\mathcal{P}_\lambda(A)$ is connected since A contains only two-cluster channels. Thus the kernel of (4.6) is connected after one iteration.

For use in Eqs. (3.3) one requires only the projected operators $\mathcal{P}_\alpha(A)\mathfrak{U}_{\alpha,\beta}(A)\mathcal{P}_\beta(A)$. We see that (4.6) yields a closed set of coupled integral equations which by the preceding argument have connected kernels even before iteration.

APPENDIX C: ANTISYMMETRIZATION

We outline briefly the formalism[16,18] required to obtain symmetrized scattering equations. For the sake of simplicity we assume that all particles are identical fermions. The generalization to include several species of identical particles is straightforward and is not considered here. Let S be the permutation symmetry group and let $P\in S$ represent an interchange of particles. Then, for any partition a of the N-particle system,

$$P(a)=a' \tag{C1}$$

is another partition which is physically equivalent to a. The set of all such permutation-related partitions defines an equivalence class

$$\hat{a}=\{P(a)|\ P\in S\} . \tag{C2}$$

The transition operators which occur in the scattering theory of identical particles are labeled by classes rather than by partitions.

The symmetry group S is realized on the N-particle Hilbert space by the unitary operators $U(P)$. A partition-labeled operator $\mathcal{O}$ is called *label transforming* if

$$U(P)\mathcal{O}^{a,b\cdots}U(P)^{-1}=\mathcal{O}^{P(a),P(b)\cdots} . \tag{C3}$$

Let $N_{\hat{a}}$ be the number of physically equivalent partitions in the class $\hat{a}$ and $\delta(P)$ be the parity of the permutation with respect to fermion interchange. Also, let R_a be the antisymmetrizer with respect to the interchange of the particles within the clusters of the partition a. (It it obvious that if $\mathcal{O}^a$ is label transforming, $[\mathcal{O}^a,R_a]=0$.) Then the antisymmetrizer with respect to S is given by

$$R=\Big(\frac{1}{N_{\hat{a}}}\Big)\sum_{a\in\hat{a}} R_{\bar{a}}U(P_{a,\bar{a}})^\dagger\delta(P_{a,\bar{a}})\equiv\Big(\frac{1}{N_{\hat{a}}}\Big)\sum_{a\in\hat{a}}\mathcal{A}_a(\bar{a}) , \tag{C4}$$

where $\bar{a}$ is a canonical partition of the class $\hat{a}$. In what follows we suppress the dependence of $\mathcal{A}_a(\bar{a})$ upon $\bar{a}$.

Let us introduce the antisymmetrized transition operators

$$\tilde{T}_{\hat{a},\hat{b}}=\overline{N}_{\hat{a},\hat{b}}\sum_{a\in\hat{a}}\mathcal{A}_a T_{a,\bar{b}} , \tag{C5}$$

where $\overline{N}_{\hat{a},\hat{b}}=(N_{\hat{b}}/N_{\hat{a}})^{1/2}$ and $\bar{b}\in\hat{b}$ denotes a canonical partition. The matrix element $\langle\phi_a(\nu_a)\vec{k}_a|\tilde{T}_{\hat{a},\hat{b}}|\phi_b(\nu_b)\vec{k}_b\rangle$ is the physical scattering amplitude where, e.g., $\mathcal{A}_a|\phi_a(\nu_a)\vec{k}_a\rangle=|\phi_a(\nu_a)\vec{k}_a\rangle$. If the $T_{a,b}$ operators satisfy the generic integral equations

$$T_{a,b}=I_{a,b}+\sum_c K_{a,c}T_{c,b} , \tag{C6}$$

where $I_{a,b}$, $K_{a,b}$ are label transforming, then it is shown in Ref. 18 that the $T_{a,b}$ are necessarily label transforming and

$$\tilde{T}_{\hat{a},\hat{b}}=\tilde{I}_{\hat{a},\hat{b}}+\sum_{\hat{c}}\tilde{K}_{\hat{a},\hat{c}}\,\tilde{T}_{\hat{c},\hat{b}} , \tag{C7}$$

where $\tilde{I}_{\hat{a},\hat{b}}$ and $\tilde{K}_{\hat{a},\hat{b}}$ are defined as in (C5). The sum in (C7) runs over classes rather than partitions and one can show[18] that (C7) has a connected kernel provided (C6) is a connected-kernel equation. The reduction of (C6) to (C7) holds if we replace $T_{a,b}$ by any other partition-labeled operators, such as, e.g., $\mathfrak{U}_{\alpha,\beta}(A)$.[16]

APPENDIX D: APPROXIMATE FORMS OF THE DYNAMICAL EQUATIONS

For the purpose of obtaining Eq. (7.1) we note that Eq. (4.4) can be rewritten using the identity (2.19) as

$$I_{\alpha,\beta}(A)=\sum_{\lambda}\Lambda_{\alpha,\lambda}(A)\times\left[W_{\mathrm{MS}}^{\lambda,\beta}+\bar{\delta}_{\lambda,\beta}G_{\beta}^{-1}+V_{\beta}^{\lambda}-\sum_{\gamma}W^{\lambda,0}(\gamma)G_0\bar{\delta}_{\gamma,\beta}(G_{\beta}^{-1}+V_{\beta}^{\gamma})\right] . \quad \text{(D1)}$$

The kernel of the integral equation (4.1) for $\Lambda_{\alpha,\beta}(A)$ is two-cluster connected. Thus a low-connectivity approximation to $\Lambda_{\alpha,\beta}(A)$ is $\delta_{\alpha,\beta}$. If we also retain only (N)- and $(N-1)$-connected pieces in the square brackets in (D1) we find

$$I_{\alpha,\beta}(A)\simeq\sum_{i'}W^{\alpha,\beta}(i')+V_{\beta}^{\alpha}+\bar{\delta}_{\alpha,\beta}G_{\beta}^{-1} , \quad \text{(D2)}$$

which with (2.18) yields the inhomogeneous term of (7.1). With similar approximations the kernel (4.7) becomes

$$K_{\alpha,\beta}(A)=-\left[\bar{\delta}_{\alpha,\beta}+\sum_{i'}W^{\alpha,0}(i')G_0\right]\mathcal{P}_{\beta}(A) . \quad \text{(D3)}$$

It is instructive to start from the alternative form (4.4) rather than (D1). From (2.18) and (2.19) we have

$$\sum_{i'}[W^{\alpha,0}(i')G_0G_{\beta}^{-1}]_{i'}=\sum_{i'}\bar{\Delta}_{\alpha,i'}(t_{i'}-t_{i'}G_0V_{i'}\Delta_{\beta,i'})=\sum_{i'}\bar{\Delta}_{\alpha,i'}t_{i'}\bar{\Delta}_{\beta,i'}+V_{\beta}^{\alpha} . \quad \text{(D4)}$$

Using (D4) in (4.4) we again obtain (D2) if only the (N)- and $(N-1)$-connected terms are retained and both (4.4) and (D1) yield the same result in this approximation.

---

*Present address: Department of Physics, FM-15, University of Washington, Seattle, Wash. 98195.

[1]W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961); N. Austern, *Direct Nuclear Reaction Theory* (Wiley, N.Y., 1970).

[2]T. Ohmura, B. Imanishi, M. Ichimura, and M. Kawai, Prog. Theor. Phys. 41, 391 (1969); 43, 347 (1970); 44, 1242 (1970); T. Udagawa, H. H. Wolter, and W. R. Coker, Phys. Rev. Lett. 31, 1507 (1973).

[3]S. R. Cotanch and C. M. Vincent, Phys. Rev. C 14, 1739 (1976); Phys. Rev. Lett. 36, 21 (1976).

[4]K. L. Kowalski, in *Few Body Systems and Nuclear Forces*, edited by H. M. Haftel and H. Zankel, Lecture Notes in Physics 87 (Springer, Berlin, 1978), Vol. II, p. 393. This reference provides the most recent review of the N-body scattering theory.

[5]E. F. Redish, in *Modern Three-Hadron Physics*, edited by A. W. Thomas (Springer, Berlin, 1977), p. 181; E. F. Redish, Phys. Rev. C 10, 67 (1974).

[6]W. N. Polyzou and E. F. Redish, Ann. Phys. (N.Y.) 119, 1 (1979).

[7]R. Goldflam and W. Tobocman, Phys. Rev. C 20, 904 (1979).

[8]R. M. Dixon and E. F. Redish, J. Math. Phys. 21, 372 (1980).

[9]R. Goldflam, K. L. Kowalski, and W. Tobocman, J. Math. Phys. 21, 1888 (1980).

[10]K. L. Kowalski, Ann. Phys. (N.Y.) 120, 328 (1979).

[11]R. Goldflam and K. L. Kowalski, Phys. Rev. C 21, 483 (1980).

[12]K. L. Kowalski, E. R. Siciliano, and R. M. Thaler, Phys. Rev. C 18, 2017 (1978).

[13]J. M. Greben and F. S. Levin, Nucl. Phys. A325, 145 (1979). These calculations employ a three-body description of the collisions discussed.

[14]M. Baer and D. J. Kouri, J. Math. Phys. 14, 1637 (1973); D. J. Kouri and F. S. Levin, Phys. Lett. 50B, 421 (1974); W. Tobocman, Phys. Rev. C 9, 2466 (1974).

[15]This approach has been used by the authors to obtain a well-defined integral equation for the optical potential for a system of identical particles. See Ref. 16

[16]R. Goldflam and K. L. Kowalski, Phys. Rev. Lett. 44, 1044 (1980); also Phys. Rev. C 22, 949 (1980).

[17]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. 2B, 167 (1967); P. Grassberger and W. Sandhas, *ibid.* 2B, 181 (1967).

[18]G. Bencze and E. F. Redish, Nucl. Phys. A238, 240 (1975); J. Math. Phys. 19, 1909 (1978).

[19]M. Kawai (private communication); N. Austern, Phys. Lett. 90B, 33 (1980); F. S. Levin and C. T. Li (private communication).

[20]K. L. Kowalski, Phys. Rev. C 16, 7 (1977).

[21]W. N. Polyzou, J. Math. Phys. 21, 506 (1980); K. I. Kowalski, W. N. Polyzou, and E. F. Redish, J. Math. Phys. (to be published).

[22]A. L. Fetter and K. M. Watson, in *Advances in Theoretical Physics*, edited by K. A. Brueckner (Academic, New York, 1965), p. 113.

[23]A. Picklesimer and K. L. Kowalski, Phys. Lett. (to be published).

[24]Here, we will use the term *A-type discontinuities* to mean the discontinuities across the unitarity cuts corresponding to the channels contained in the RM A.

[25]Lippmann-Schwinger type equations for $\mathfrak{U}_{\alpha,\beta}(A)$ have been given in Ref. 23. These can be used to develop low-order approximations to $\mathfrak{U}_{\alpha,\beta}(A)$.

[26]K. M. Watson, Phys. Rev. 89, 575 (1953); A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) 8, 551 (1959).

[27]M. LeMere and Y. C. Tang, Phys. Rev. C 19, 391 (1979); K. Wildermuth and Y. C. Tang, *A Unified Theory of Nucleus* (Vieweg, Braunschweig, 1977).

[28]In deriving (7.7)–(7.11) we have assumed implicitly that the inverse $(I+\mathfrak{N})^{-1}$ exists. This is not necessarily the case since $\mathfrak{N}$ can have, in principle, eigenvalues -1. This circumstance, however, appears to be asso-

ciated with very special forms of the cluster wave functions. See Ref. 3; H. Feshbach, Ann. Phys. (N.Y.) 19, 287 (1962); S. Saito, Prog. Theor. Phys. (Kyoto) 41, 705 (1969); S. Saito, S. Okai, R. Tamagaki, and M. Yasuno, *ibid.* 50, 1561 (1973); Y. I. Kukulin, V. G. Neudatchin, and V. N. Pomerantsev, J. Phys. G 4, 1409 (1978). We also remark that the validity of our *general* formalism does not appear to depend upon the existence of $(I+\mathfrak{N})^{-1}$.

[29] One can regard the standard CRC, viz. *Eqs.* (6.12)–(6.14), as the lowest-order approximation which is realized in an alternative formalism obtained using the off-shell extension $\overline{T}_{\alpha,\beta}=T_{\alpha,\beta}-\overline{\delta}_{\alpha,\beta}(G_\alpha{}^{-1}+G_\beta{}^{-1})$. In this instance, however, the effective interactions are not free of A-type discontinuities in higher-order approximations.