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Efimov effect in the four-body case

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Introducing a suitable cutoff parameter r_0 for the effective potential in the three-body subsystem, we show that the number of four-body bound states is roughly $k \ln[\epsilon a(r_0)]$, where ϵ is a momentum cutoff and $2/3\pi \leq k \leq 1/3$, if the scattering length $a(r_0)$ for a two-body bound state and a third particle tends to infinity. In a numerical calculation we found that the value $r_0 \approx 1.82$ fm, which provides a sufficiently resonant effective interaction in the three-body subsystem, is independent of the applied two-body potentials within the numerical accuracy.

[NUCLEAR REACTIONS Efimov effect.]

I. INTRODUCTION

Some time ago, Efimov[1-3] discovered that if three nonrelativistic identical bosons interact via short-range two-body potentials $gv(r)$, then as the coupling constant g increases to that value g_0, which can support a single two-body bound state at zero energy, the number of bound states of the three-particle system increases without limit. All the levels are of the 0^+ kind. In other words, the pair interaction is required to become sufficiently resonant to produce a bound state near the energy $E=0$. Subsequently this remarkable and unexpected property of three-body systems was proved in a rigorous way by Amado and Noble,[4,5] who studied the eigenvalue spectrum of the Faddeev kernel in a particular singular limit. These results have also been confirmed by numerical calculations.[6,7] More recently, some attention has been dedicated to Efimov's effect in the special limiting case with two heavy and one light particle,[8,9] which had already been predicted in Ref. 5.

All these considerations had been restricted to three-body systems with interactions leading to a zero-energy two-body bound state. Because of the long-range character of the effective interaction in a three-body subsystem,[10,11] it seems impossible to observe a similar effect in the four particle case. In this paper however, we show that by introducing a suitable position space cutoff parameter r_0, which enables us to vary the range of the effective potential in the three-particle subsystem,[12] a denumerable set of 0^+ four-body bound levels appears, when r_0 decreases to a particular value r_E for which the scattering length of the two-body system and a third particle tends to infinity. The cutoff parameter procedure allows us to simulate the repulsion in an appropriate subsystem of the four particles. It turned out to be a helpful tool in studying the Tjon line[13] as well as cross sections and phase shifts of four interacting nucleons.[14] Furthermore, we discuss the relevant differences between our n-body equations and those studied by Amado and Greenwood.[19] In addition we sketch the proof that the necessary conditions for the Efimov effect for $n \geq 4$ is fulfilled in our approach.

III. ALT-GRASSBERGER-SANDHAS FORMALISM (AGS)

Let us briefly recall the general concept of our approach. Starting from a separable approximation

$$T_\gamma \sim |g_\gamma\rangle \hat{t}_\gamma \langle g_\gamma| \qquad (1)$$

of the two-body transition amplitude, the three-body equations can be reduced to effective two-body equations of the matrix LS form

$$T^\tau = V^\tau + V^\tau G_0 T^\tau , \qquad (2)$$

while we end up in the four-body case with the Faddeev type matrix relations[15]

$$U^{\sigma\rho} = \bar{\delta}_{\sigma\rho} G_0^{-1} + \sum_\tau \bar{\delta}_{\sigma\tau} T^\tau G_0 U^{\tau\rho}. \qquad (3)$$

Here σ, ρ, τ denote the partitions (ijk, l) or (ij, kl) of the four particles under consideration. Like the genuine three-body case, it is the essential aspect of Eq. (3) that its kernel is determined by the subsystem transition operators T^τ given as solutions of Eq. (2). This suggests that we can also approximate them by separable expressions of the form (1),

$$T^\tau \sim |F^\tau\rangle \tilde{t}_\tau \langle F^\tau| . \qquad (4)$$

Then, after partial wave decomposition, we end up with one-dimensional integral equations for four identical bosons with mass m:

$$T_{L;s,r}(q',q;z)=V_{L;s,r}(q',q;z)+\sum_t\int_0^\infty V_{L;s,t}(q',q'';z)\tilde{t}_t(z-a^tq''^2)T_{L;t,r}(q'',q;z)q''^2dq''\ . \tag{5}$$

The index L denotes the relative partial wave of the clusters, a^t is a kinematical factor, while s, r, t, collectively denote the fragmentations $(3+1)$ or $(2+2)$ and the further quantum numbers of the clusters. The effective potentials in the LS-type equation (5) are

$$V_{L;s,r}(q',q;z)=\int_{-1}^{+1}dxP_L(x)F_s(q_1)\hat{t}\left(z-\frac{P^2}{m}\right)F_r(q_2) \tag{6}$$

with

$$q_1{}^2=q'^2+\frac{1}{9}q^2+\tfrac{2}{3}qq'x,\quad q_2{}^2=q^2+\frac{1}{9}q'^2+\tfrac{2}{3}qq'x,\quad P^2=\tfrac{3}{4}(q'^2+q^2)-\tfrac{1}{2}qq'x \tag{7}$$

in the $(3+1)\to(3+1)$ channel. For $(2+2)\to(3+1)$ we have

$$q_1{}^2=q'^2+\frac{4}{g}q^2-\tfrac{4}{3}qq'x,\quad q_2{}^2=q^2+\tfrac{1}{4}q'^2-qq'x,\quad P^2=q^2+\tfrac{3}{4}q'^2-qq'x\ , \tag{8}$$

and for $(3+1)\to(2+2)$

$$q_1{}^2=q'^2+\tfrac{1}{4}q^2-qq'x,\quad q_2{}^2=q^2+\frac{4}{9}q'^2-\tfrac{4}{3}qq'x,\quad P^2=q^2+\tfrac{3}{4}q'^2-qq'x\ . \tag{9}$$

The three-body unitary pole approximation (UPA) form factors[12] are defined as

$$F_i(q)=\eta_i{}^{-1}\int_0^\infty v(q,q';z_{\text{fix}})\hat{t}\,(z_{\text{fix}}-Q^2)F_i(q')q'^2dq'\ , \tag{10}$$

with

$$Q^2=\begin{cases}\dfrac{3}{4m}q'^2 \text{ for } (3+1)\\[2mm] \dfrac{1}{m}q'^2 \text{ for } (2+2)\ .\end{cases} \tag{11}$$

The potentials v have the forms

$$v(q,q';z_{\text{fix}})=\int_{-1}^{+1}dx\,\frac{g[(\frac{1}{4}q^2+q'^2+qq'x)^{1/2}]g[(\frac{1}{4}q'^2+q^2+qq'x)^{1/2}]}{z_{\text{fix}}-\dfrac{1}{m}(q^2+q'^2+qq'x)}\ , \tag{12}$$

$$v(q,q';z_{\text{fix}})=\frac{g(q)g(q')}{z_{\text{fix}}-\dfrac{1}{m}(q^2+q'^2)} \tag{13}$$

in the $(3+1)$ and $(2+2)$ cases, respectively. The meaning of z_{fix} is explained elsewhere.[12,16] Now, after having developed the formalism we are able to investigate the appearance of Efimov's effect in the four-particle problem.

III. CUTOFF PARAMETER CONCEPT

Before we start our investigation in the four-body case we recall the reasons for the Efimov phenomenon in the simple case of the three spinless particles. If the pair forces are sufficiently resonant, then a whole family of bound states must appear for the three particles. That means that for a particular interaction, the radius of the forces is much smaller than the scattering length. In other words, the binding energy of two particles approaches its breakup threshold. In the three-body case the critical energy is $E=0$, while in the four-body problem this effect should occur at the deuteron binding energy E_d if we restrict our investigation to the $(3+1)$ channel. That means, if it is possible to shift the three-particle ground state energy E_t to E_d by variation of the two-body potentials coupling constant, then the number of four-body bound levels should become larger and reach infinity. On the other hand, it is known that the long-range nature of the effective interaction between the two-body subsystem and the third particle (known as triangle effect) is nearly independent of the underlying two-body interaction. So it seems impossible to induce the Efimov effect in the four-body case with the help of the mentioned manipulations. To escape these difficulties we introduced a cutoff parameter r_0, which allows us to simulate the repulsions in the three-body system by changing the *range* of the *effective* potentials. For that we replace in Eq. (10) v by $\tilde{v}$ which reads (for more details see Refs. 12, 13)

$$\tilde{v}(q,q';z_{\text{fix}})=\int_0^\infty f_0(q,q'';r_0)v(q'',q';z_{\text{fix}})dq'' , \tag{14}$$

with

$$f_0(q,q'';r_0)=\frac{r_0 q''}{\pi q}\{j_0[(q-q'')r_0]-j_0[(q+q'')r_0]\} , \tag{15}$$

where $j_0(kr_0)$ is the spherical Bessel function. Identifying $F_i(q)$ with the half on shell three-body T amplitude

$$\tilde{T}_i(q;z_{\text{fix}})=\eta_i^{-1}\int_0^\infty \tilde{v}(q,q';z_{\text{fix}})\hat{t}\,(z_{\text{fix}}-Q^2)\times\tilde{T}_i(q';z_{\text{fix}})q'^2dq' ; \tag{16}$$

setting $\eta_1^{-1}=1$ and $z_{\text{fix}}=E_t$, obviously the triton binding energy is a function of the cutoff parameter r_0. Moreover, we assume that it is possible to find an r_E for which

$$\lim_{r_0\to r_E} E_t(r_0)=E_d , \tag{17}$$

and consequently the scattering length

$$a(r_0)=\lim_{k\to 0}\,(2\pi m/3)\tilde{T}(k,r_0;E_d) \tag{18}$$

gets larger and larger, if r_0 reaches r_E. We have verified this assumption by a numerical calculation using Yamaguchi and Gaussian form factors as two-body input. As an interesting result of our calculation, shown in Fig. 1, it appears that the value $r_E=1.82$ fm, for which the scattering length $a(r_E)$ tends to infinity, is independent of the chosen two-body interaction within the numerical accuracy. The solution of Eq. (16) was accomplished by means of the Padé method. Already the [5, 5] approximant ensures sufficient accuracy in almost all cases (i.e., for different values of r_0). Numerical integration has been performed with an integration technique especially developed for that type of oscillating kernel (see Ref. 16). Now let us return to the problem of investigating Efimov's effect in the four-body case. Qualitatively we have demonstrated that in the three-body subsystem the effective interactions become sufficiently resonant ($a\gg r_0$) for the existence of a very large number of bound state levels in the four-body case. To prove this, we still have to calculate the number of four-body states when r_0 is larger than, but close to, r_E.

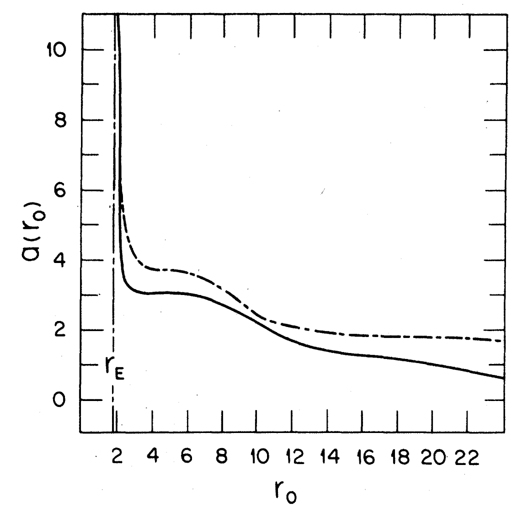

FIG. 1. Scattering length $a(r_0)$ of a two-body bound state and a third particle in dependence of the cutoff parameter r_0, $a(r_E)=\infty$. (—— Yamaguchi, -·-·-· Gaussian two-body input.)

As we know from the three-body case, the singularity leading to the infinite number of bound levels comes from the confluence of two different thresholds. Moreover, this divergence connected with the Efimov effect arises from the small-momentum behavior of the kernel.[4,5] Therefore it seems inappropriate to complicate the problem merely to deal with its irrelevant high-momentum behavior.

Near $E=E_d$ the UPA three-body subsystem propagator $\tilde{t}$ in Eq. (5) is approximated by

$$\tilde{t}^{ER}(E;r_0)=-\frac{3}{2\pi m}\left\{\frac{1}{a(r_0)}+[-m(E-E_d)]^{1/2}\right\}^{-1} , \tag{19}$$

so the homogeneous form of Eq. (5) reads

$$\chi_{L;s,r,i}(q)=\lambda_i^{-1}(E,r_0)\left\{\int_0^\infty \mathrm{V}_{L;s,(3+1)}(q,q';E)\tilde{t}_{(3+1)}{}^{ER}(E-a^{(3+1)}q'^2)\chi_{L;(3+1),r,i}(q')q'^2dq' + \int_0^\infty \mathrm{V}_{L;s,(2+2)}(q,q';E)\tilde{t}_{(2+2)}(E-a^{(2+2)}q'^2)\chi_{L;(2+2),r,i}(q')q'^2dq'\right\} \tag{20}$$

Because of the formal equivalence (one-variable equation) of the three- and four-body cases, here we may apply exactly the same method which leads to the well known result in the three-particle problem.

IV. NUMBER OF FOUR-BODY BOUND LEVELS

In this section we show that the number of four-body bound states increases like $\ln(\epsilon |a|)$ with increasing three-body scattering length a in the case of four identical bosons interacting via a separable two-body potential. ϵ is a suitable chosen momentum cutoff. As described in Sec. II, one gets a one-dimensional integral equation by a twofold separable approximation of the original four-body equation. So we may adapt the mathematical tools used by Amado and Noble[4,5] to prove the existence of Efimov's effect in a rigorous way. Therefore we restrict ourselves to a short description of the idea of the proof but show in more detail the significant differences to the three-body case.

In order to study the kernel and the eigenvalues as a function of the energy and the cutoff parameter r_0, we rewrite the four-body bound state problem [see Eq. (20)]

$$K(E,r_0)\chi_i = \lambda_i(E,r_0)\chi_i \,. \tag{21}$$

For $E<E_d$ and $r_0>r_E$ the eigenvalues $\lambda_i(E,r_0)$ are real and discretely distributed and we have $\lim_{i\to\infty}\lambda_i(E,r_0)=0$. χ_i is a solution of the Schrödinger equation if the corresponding eigenvalue is equal to one. Thus for fixed r_0 the number of bound states is given by the number of eigenvalues equal to one, if the energy E varies from $-\infty$ to the two-body bound state E_d. Since $\lim_{E\to-\infty}\mathrm{tr}[K^2(E,r_0)]=0$, obviously we have for each cutoff parameter r_0 a most tightly bound state $E_1(r_0)$ with $-\infty<E_1(r_0)<E_d$. That means all energy eigenvalues lie in the interval $[E_1(r_0),E_d]$. These facts are the starting point of the idea of Amado and Noble to prove the Efimov effect: Since each eigenvalue $\lambda_i(E,r_0)$ is a real and continuous function of E and r_0 for $E<E_d$ and $r_0>r_E$, it must pass through $\lambda_i=1$ in order to become larger than unity. If it is possible to show that an infinite number of eigenvalues λ_i become larger than 1 in the limit $E=E_d$ and $r_0=r_E$, we would prove the existence of an infinite number of four-body bound states. Our task therefore is to demonstrate that the sequence $S_i=\lambda_i(E_d,r_E)$ has a denumerable set of accumulation points $H_L\neq 0$, $L=0,1,2,\ldots$ with $H_L>1$ for particular L.

In the first step we write the kernel of Eq. (21) as a sum of a singular and a regular part

$$K(E_d,r_E)=K^S(E_d,r_E)+K^R(E_d,r_E)\,. \tag{22}$$

So K^S is defined after partial wave decomposition as

$$K_L^S(q,q')=(-1)^L\frac{9}{\pi q q'}\theta(\epsilon-q)\theta(\epsilon-q')Q_L\left(\frac{\frac{3}{2}(q^2+q'^2)-2m(E-E_d)}{qq'}\right)\left\{\frac{1}{a}+[\tfrac{2}{3}q'^2-m(E-E_d)]^{1/2}\right\}^{-1}. \tag{23}$$

Here we approximate the three-body subsystem propagator by its effective range formula [see Eq. (19)]. The three-body scattering length is denoted by a and the momentum cutoff ϵ^{-1} has to be small in correspondence to a in the case $E=E_d$. $Q_L(x)$ is the Legendre function of the second kind. Recalling that K^R is a compact operator with a finite trace for $E=E_d$ and $r_0=r_E$, a theorem by Weyl[17] proves that

$$\lim_{i\to\infty}S_i=\lim_{i\to\infty}S_i^S\,, \tag{24}$$

where $S_i^S=\lambda_i^S(E_d,r_E)$ is the eigenvalue spectrum of the operator $K^S(E_d,r_E)$. Since we are interested only in the *limit points* of the eigenvalues S_i, we are allowed to concentrate our attention on the sequence S_i^S. According to that we define

$$R_L(E,r_0)=\frac{\mathrm{tr}\{[K_L^S(E,r_0)]^2\}}{\mathrm{tr}[K_L^S(E,r_0)]} \tag{25}$$

and arrive at

$$(-1)^L R_L(E_d,r_0)=\frac{\sum_{i=1}^{\infty}[\lambda_i^S(L,E_d,r_0)]^2}{\sum_{i=1}^{\infty}(-1)^L\lambda_i^S(L,E_d,r_0)} \leq \frac{\sum_{i=1}^{\infty}\lambda_i^S(L,E_d,r_0)S_i^S(L)}{\sum_{i=1}^{\infty}(-1)^L\lambda_i^S(L,E_d,r_0)}, \tag{26}$$

because of the well known fact that a positive compact operator is equal to the sum of its eigenvalues, and of the monotonicity conditions

$$(-1)^L\lambda_i^S(L,E,r_0)\leq(-1)^L S_i^S(L) \text{ for } E\leq E_d,\quad r_0>r_E\,. \tag{27}$$

Following the arguments given by Amado and Noble we emphasize that

$$(-1)^L H_L' = \lim_{\substack{E\to E_d\\ r_0\to r_E}} R_L(E,r_0)(-1)^L \tag{28}$$

is a lower bound for

$$(-1)^L H_L=\lim_{i\to\infty}(-1)^L S_i^S(L)\,, \tag{29}$$

and that the excess of $(-1)^L H'_L$ is just a geometric factor. To get this lower bound H'_L we have to calculate the traces $\mathrm{tr}(K_L^S)$ and $\mathrm{tr}((K_L^S)^2)$:

$$\mathrm{tr}[(K_L^S)^2] \leq (3)^{2L+2}\frac{3^5}{4\pi}\ln[\epsilon a(\tfrac{2}{3})^{1/2}]\frac{\Gamma(L+1)}{\Gamma(L+\frac{3}{2})}Q_L{}^2(3), \tag{30}$$

$$\mathrm{tr}(K_L^S) = 3(\tfrac{3}{2})^{3/2}\pi^{-1}(-1)^L Q_L(3)\ln[1+\epsilon a(\tfrac{2}{3})^{1/2}], \tag{31}$$

where $\Gamma(x)$ is the well known gamma function. Finally we find that

$$(-1)^L H'_L \geq \frac{(\frac{3}{2})^{1/2}Q_L(3)3^{2L+2}\Gamma(L+1)}{\pi\Gamma(L+\frac{3}{2})}. \tag{32}$$

Applying a theorem of Tiktopoloulos[18] we also get an upper bound

$$\|K_L^S\| \leq 3^{L+2}\pi^{-1}(3)^{1/2}Q_L(3)\frac{\Gamma^2[(L+1)/2]}{\Gamma(L+1)} \tag{33}$$

for the accumulation points H_L. For $L=0$ we see that

$$1.214 \leq H'_0 \leq H_0 \leq 2.697, \tag{34}$$

but in the case of $L=2$ we find $H_2 \leq 0.3$. So we conclude that for $L>0$ there is no further accumulation point $H_L>1$, because the right side of (33) is a monotonic decreasing function of the partial wave quantum number L. For odd L H_L obviously is negative. To summarize, we have shown that in the four-body case an infinite number of bound states appear in the limits $E_t \to E_d$ and $r_0 \to r_E$ if $L=0$. Analogous to the three-body case we observe only in the 0^+ state Efimov's effect. The number of bound states goes with $N \approx \mathrm{tr}[K_0^S(E_d, r_E)]/H_0$. That means in the four-body case

$$N \approx k\ln(\epsilon a), \tag{35}$$

with

$$2/3\pi \leq k \leq \tfrac{1}{3}.$$

V. n-BODY CASE

Amado and Greenwood[19] showed that there is no Efimov effect for four or more particles. In the light of their work our result seems to be very strange, but let us analyze what the decisive differences are between the approaches under consideration. In their paper a four-body kernel is investigated which contains the full (not approximated) subsystem operator. As a consequence their trace for small E, p, q, is proportional to

$$\int \frac{p^2q^2dp\,dq}{(E-p^2-q^2)(E-\frac{2}{3}q^2)^y}, \tag{36}$$

and thus does not become divergent as $E \to 0$, because $y=1$ or $\frac{1}{2}$ in the four-body case. The divergence of tr$K$ however is a necessary condition for the appearance of the Efimov effect. Hence, taking into account the full subsystem amplitude, an $(n-1)$-body zero-energy bound state will not produce an infinite number of bound states in the n-body case for $n \geq 4$. In contrast to such an attempt our final four-body equation is provided by a *twofold separable* approximation of the original four-body operator identity [Eq. (3)]. This means that we have already integrated over the internal momenta, and hence only the relative momentum between the fragmentations $(3+1)$ and $(2+2)$ respectively, appears explicitly in the resulting one-dimensional integral equation. The particles, which build up the clusters, have no longer any degree of freedom, but they are "frozen" to a bound state or a "quasiparticle." Thus the volume of our phase space is reduced to the three components of the relative variable $\vec{q}$ between the clusters. Therefore the trace of the relevant part of our kernel is proportional to

$$\int_0^\epsilon \frac{F^2_{(3)}(\frac{4}{3}q)dq}{[a(r_0)]^{-1}+(\frac{3}{2})^{1/2}q} \tag{37}$$

after partial wave decomposition, if $E \to E_d$. Obviously this integral is logarithmically divergent as $a(r_0) \underset{r_0 \to r_E}{\to} \infty$. So here the necessary condition for the Efimov effect is fulfilled. Moreover, as shown in the preceding section, this divergence is sufficient for the existence of the effect in the four-body case.

We are now ready to extend our arguments to the n-body case $(n>4)$. Starting from general n-body operator identities,[15] an $(n-1)$-fold successive separable approximation of the subamplitudes leads to a system of one-dimensional integral equations, which variable describes the relative movement of the clusters. By introducing a position space cutoff parameter $r_0^{(N-1)}$ in the $(n-1)$ subsystem and adjusting it to a value, which provides the coincidence of the $(n-1)$ and $(n-2)$ bound states, we arrive at a similar expression for the trace of the kernel as in the four-body case, after splitting up its irrelevant parts (ω^N, κ^N are kinematical factors):

$$\int_0^\epsilon \frac{F^2_{(N-1)}(\omega^N q)dq}{[a(r_0^{(N-1)})]^{-1}-\kappa^N q}. \tag{38}$$

$F_{(N-1)}$ is the form factor of the $(n-1)$ bound state obtained as a solution of an eigenvalue equation analogous to Eq. (10). Thus, in contrast to the more general case discussed by Amado and Greenwood, here the necessary condition for the existence of an infinite number of n-body bound levels $(n>4)$ is not violated. The problem of a general proof of the existence of Efimov's effect in the (separable approximated) n-particle case is under investigation.

The essential point in Amado and Greenwood's paper is the fact that there are $(n-2)$ free momenta and one propagator in the part of the equation containing the $(n-1)$ connected kernel. The dimension of the propagator is always the same, but each additional particle gives three more powers of momentum in the numerator. Since the $(n-1)$ amplitudes do not diverge more and more strongly at an $(n-1)$-body zero-energy bound state, the integrals are finite for small momentum cutoff parameters and small momenta for $n>3$. As emphasized above, this situation is avoided by applying an $(n-1)$-fold separable approximation, which does not change the dimension of the relevant propagators by incorporating the dominant subsystem structures (bound states, resonances, etc.), but reduces the n-body to an effective two-body phase space.

What does that mean physically? The approximated n-body amplitudes are not able to describe breakup processes of the clusters into smaller fragments or elementary particles. This is a shortcoming of the equations in the scattering region, but does not affect seriously the bound state problem. In actual calculations this problem usually is cured by introduction of breakup amplitudes, which are built up by subsystem scattering states and transition operators. Moreover, numerical calculations[12,13,20] show that the four-body bound state is mainly determined by the position of the subsystem poles (e.g. triton energy) that means by the dominant structure of the subsystem amplitudes. So in the n-body bound state problem it seems physically justified to reduce the complexity of the full integral equation system by applying the described clustering concept.

VI. CONCLUSION

We have investigated the Efimov effect in the four boson case by varying the range of the effective potential in the three-particle subsystem via an appropriate chosen cutoff parameter. Having taken over a proof of Amado and Noble[4,5] applied in the three boson case, we have shown that a denumerable set of 0^+ bound state levels occurs for four bosons, if a particular scattering length is forced to tend to infinity by adjusting the cutoff r_0 at the value $r_E = 1.82$ fm. There are some indications for the existence of such a pathology in the $(n>4)$-body case, if the complexity of the n-body equations is reduced by successive separable approximations of the subsystem amplitudes.

The study of Efimov's effect in the three-body system has not been restricted to the proof of its existence. Efimov[1] himself emphasized that, as an application of the effect, the existence of the level in ^{12}C with excitation energy 7.65 MeV could be understood at least qualitatively in terms of a three-α-particle system with resonant pair interaction of the α particles in the s state (the resonance is the ground state of ^8Be). In recent attempts[8] the possibility has been investigated to describe the vibration rotation structure of molecular spectroscopy from the Efimov viewpoint. So we are optimistic that also in a four-body bound state model the cutoff parameter induced appearance of a family of 0^+ levels will turn out to be an appropriate tool in the analysis of empirical data. Investigations are under way.

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[1]V. Efimov, Yad. Fiz. 12, 1080 (1970) [Sov. J. Nucl. Phys. 12, 569 (1971)].
[2]V. Efimov, Phys. Lett. 33B, 563 (1970).
[3]V. Efimov, Nucl. Phys. A210, 157 (1973).
[4]R. D. Amado and J. V. Noble, Phys. Lett. 35B, 25 (1971).
[5]R. D. Amado and J. V. Noble, Phys. Rev. D 5, 1992 (1972).
[6]A. T. Stelbovics and L. R. Dodd, Phys. Lett. 39B, 450 (1972).
[7]A. G. Antunes, V. L. Baltar, and E. M. Ferreira, Nucl. Phys. A265, 365 (1976).
[8]A. G. Fonseca and P. E. Shanley, Ann. Phys. (N.Y.) 117, 268 (1979).
[9]A. G. Fonseca, E. F. Redish, and P. E. Shanley, Nucl. Phys. A320, 273 (1979).
[10]H. P. Noyes, Phys. Rev. D 5, 1547 (1972).
[11]H. P. Noyes, Phys. Rev. Lett. 23, 1201 (1969).
[12]R. Perne and W. Sandhas, Phys. Rev. Lett. 39, 788 (1977).
[13]R. Perne and H. Kröger, Phys. Rev. C 20, 340 (1979).
[14]R. Perne and W. Sandhas, in *Few Body Systems and Nuclear Forces*, Proceedings of the VIII International Conference on Few Body Systems and Nuclear Forces, Graz, Austria, 1978, edited by H. Zingl, M. Haftel, and H. Zankel (Springer, Berlin, 1978).
[15]P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967).
[16]R. Perne, Ph.D. thesis, University of Bonn, 1978 (unpublished).
[17]F. Riesz and B. S. Nagy, *Functional Analysis* (Ungar, New York, 1955).

[18]G. Tiktopoulos, J. Math. Phys. 6, 573 (1965).
[19]R. D. Amado and F. C. Greenwood, Phys. Rev. D 7, 2517 (1973).
[20]J. A. Tjon, Phys. Lett. 56B, 217 (1975); H. Kröger and W. Sandhas, Phys. Rev. Lett. 40, 834 (1978); S. Sofianos, H. Fiedeldey, and N. J. McGurk, Phys. Lett. 68B, 117 (1977); B. F. Gibson and D. R. Lehman, Phys. Rev. C 14, 685 (1976).