Analytical theory of pion single and double charge exchange in resonance region. I. Geometrical limit

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We derive analytical formulas for angular distributions $\sigma(\theta)$ of pion single and double charge exchange to analog states. The result is based on the eikonal approximation and assumes that the interactions are invariant under isospin rotations. The theory reproduces semiquantitatively the results of exact coupled channel, lowest order optical model calculations and explicates the dependence of $\sigma(\theta)$ on the structure of the nuclear target. The $\sigma(\theta)$ for single charge exchange is proportional to the square of the ratio of the valence neutron density $\Delta\rho(\bar{R})$ to the total density $\rho(\bar{R})$. The theory predicts $\bar{R}$ as a function of energy and mass number A; for pion kinetic energy $T_\pi \simeq 180$ MeV, $\rho(\bar{R}) \simeq 0.1$–0.2 of central density. The $\sigma(\theta)$ for double charge exchange is proportional to $[\Delta\rho(\bar{R})/\rho(\bar{R})]^4$. The strong dependence of $\sigma(\theta)$ on $\Delta\rho/\rho$ suggests that the single and double charge exchange reactions may develop into a sensitive probe of the neutron halo. The dependence of $\sigma(\theta)$ on the diffuseness of ρ and $\Delta\rho$ is also evaluated. Assuming that the neutron and proton densities are proportional, the *relative* A dependence of $\sigma(\theta)$ for single and double charge exchange is calculated. The result is in qualitative agreement with recent measurements for single charge exchange and predicts that a similar set of measurements for double charge exchange may be feasible. Realistic densities are used to calculate the magnitude of $\sigma(\theta)$. Large enhancements are found in this case, but $\sigma(\theta)$ falls below the preliminary data by as much as a factor of 4. Difficulties in reproducing the observed angular distribution of the $^{18}O(\pi^+,\pi^-)^{18}Ne$ reaction are similar to those of other theories. The extent of discrepancy suggests the occurrence of strong modifications of the pion-nucleon interaction in the nuclear medium.

[NUCLEAR REACTIONS Pion single and double charge exchange; analytical formulas for angular distributions.]

I. INTRODUCTION

Angular distributions for pion single[1] (SCX) and double[1a,2] (DCX) charge exchange to the analogs of the target ground state are just becoming available. These data contain important information about the isospin dependence of the pion-nucleus interaction and also presumably about the higher order terms in the optical potential, which if identified, may reveal important information about the dynamics of the pion and Δ propagation in the nuclear medium. To extract this information, very thorough theoretical investigations are required. However, it is appropriate to be less ambitious at this early stage and examine the physics at a more elementary level. In this work we derive simple analytic expressions for the single and double charge exchange on the basis of a semiclassical theory which, as we shall argue, accurately reflects the sensitivity of more exact computer calculations to the geometrical aspects of the underlying theory. A comparison to the experiments will then give a clearer physical feeling for the process than is possible with large computer calculations and will also indicate the extent to which we will have to add physics beyond basic information about the nuclear size.

There have been many previous theoretical calculations[3] of both single and double charge exchange in semiclassical theories,[4] in coupled channel frameworks,[5] and in other approaches.[6] It has been emphasized[4-6] that charge exchange cross sections are sensitive to the nuclear density distributions. In this work we explicate this relationship and show that large mistakes can be made unless realistic densities are used in calculations, especially in the case of double charge exchange. Our work is to be distinguished from previous semiclassical results in the simple quantitative relationship which we find between the angular distribution and the underlying neutron and proton density distributions. Because of the semiclassical nature of the approach, we will restrict our investigation to the energy region near the (3-3) resonance. This is an interesting energy to study, because large discrepancies between theory and experimental total SCX cross sections to analog states have been noted here.[3]

II. BASIC THEORY

The most general form for the isotopic spin dependence of the pion optical potential is given by

$$U = u_0 + u_1 \vec{\phi}\cdot\vec{T} + u_2(\vec{\phi}\cdot\vec{T})^2, \tag{1}$$

where $\vec{\phi}$ is the pion isospin and $\vec{T}$ is the nuclear

isospin operator. The quantities u_0, u_1, and u_2 are complicated functions which depend on the nuclear spin and which are in general nonlocal. The optical potential is assumed to describe only the elastic scattering from, and charge exchange among, an isospin triplet of states. In this work we restrict ourselves to cases in which the nucleus has zero spin; we will then be able to study such interesting cases as ^{18}O, ^{48}Ca, ^{90}Zr, ^{120}Sn, and ^{208}Pb. Also, we ignore the Coulomb interaction. Koren[7] and Miller[5a] have shown that at least in the case of cross sections for single charge exchange that the results are very insensitive to the Coulomb effect.

Expressions for u_0 and u_1 may be found readily in standard lowest order optical potential models. In this work we shall for the most part ignore all partial waves except for P waves. The reasons for this are that (1) the P waves are the most important partial waves for the energy region in which we are interested, (2) the S waves are relatively easy to include but complicate the expressions which follow without adding to the qualitative understanding which we are after. The S waves should of course be included in any calculation where quantitative accuracy is needed.

As an example, in the lowest order Kisslinger theory,[8] the P wave pieces of u_0 and u_1 are given (see Appendix A)

$$u_0 = -\lambda_0^{(1)} \vec{\nabla} \cdot \rho(r) \vec{\nabla} \tag{2a}$$

and

$$u_1 = \frac{\lambda_1^{(1)}}{2T} \vec{\nabla} \cdot \Delta\rho(r) \vec{\nabla} . \tag{2b}$$

In these expressions T is the isospin of the target, and $\lambda_0^{(1)}$ and $\lambda_1^{(1)}$ are strengths for the isoscalar and isovector optical potentials, respectively. Normally λ_0 and λ_1 have different values, but in the example of a scattering entirely due to the (3-3) channel we have

$$\lambda_0^{(1)} = \beta = \lambda_1^{(1)}, \tag{2c}$$

where the constant β is related to the free pion-nucleon amplitude. In Eqs. (2a) and (2b) and throughout the rest of the paper, $\rho(r)$ is the total density,

$$\rho(r) = \rho_n(r) + \rho_p(r) \tag{3a}$$

and $\Delta\rho(r)$ is the valence neutron density,

$$\Delta\rho(r) = \rho_n(r) - \rho_p(r) . \tag{3b}$$

In these expressions $\rho_n(r)$ and $\rho_p(r)$ are normalized to the neutron number N and proton number Z, respectively.

The isotensor term in Eq. (1) is not well understood. The effect requires a collision between the pion and at least two nucleons, and thus the leading order term is a "higher order correction" to the optical potential. The u_0 and u_1 pieces also contain higher order corrections, but not as the leading order. There are many sources of these corrections, which will be treated carefully in a separate paper.

In order to solve the Kisslinger theory in the eikonal framework, we have to simplify the Klein-Gordon equation. The difficulties are (1) the coupled channel nature of the equation and (2) the nonlocal character of the optical potential. The first difficulty may be overcome by observing that the interaction is isotopic spin invariant. Thus, the interaction is diagonalized in the representation in which the pion and nuclear isospin are coupled to a definite total isotopic spin $\vec{\mathcal{T}} = \vec{T} + \vec{\phi}$. The coupled equations then reduce to three uncoupled equations for the wave functions $\psi_{\mathcal{T}}$,

$$(-\nabla^2 + U_{T+1})\psi_{T+1} = k^2 \psi_{T+1} , \tag{4a}$$

$$(-\nabla^2 + U_T)\psi_T = k^2 \psi_T , \tag{4b}$$

$$(-\nabla^2 + U_{T-1})\psi_{T-1} = k^2 \psi_{T-1} , \tag{4c}$$

where

$$U_{T-1} = u_0 - (T+1)u_1 + (T+1)^2 u_2 , \tag{5a}$$

$$U_T = u_0 - u_1 + u_2 , \tag{5b}$$

$$U_{T+1} = u_0 + T u_1 + T^2 u_2 . \tag{5c}$$

Assuming that $u_2 = 0$ and that u_0 and u_1 are given in Eq. (2), we have from Eq. (5)

$$U_{\mathcal{T}} = -\vec{\nabla} \cdot \xi_{\mathcal{T}}(r) \vec{\nabla} , \tag{6a}$$

where

$$\xi_{\mathcal{T}}(r) = \lambda_0^{(1)} \rho(r) + \gamma^{(1)}(\mathcal{T}) \lambda_1^{(1)} \Delta\rho(r) \tag{6b}$$

with $\gamma^{(1)}(\mathcal{T})$ given in Table I.

The second difficulty may be overcome by making a transformation to a local interaction. This transformation is well known[9] and some of the details are given in Appendix B. The resulting local interactions are, to lowest order in $\xi_{\mathcal{T}}$,

$$U_{\mathcal{T}} = k^2 \xi_{\mathcal{T}}(r) + \tfrac{1}{2} \nabla^2 \xi_{\mathcal{T}}(r) . \tag{7}$$

Because U has the form given in Eq. (1), the scattering amplitude $F(\theta)$ may also be written as

$$F(\theta) = f_0(\theta) + f_1(\theta) \vec{\phi} \cdot \vec{\mathcal{T}} + f_2(\theta) (\vec{\phi} \cdot \vec{\mathcal{T}})^2 . \tag{8}$$

When f_0, f_1, and f_2 are known, then the formula

TABLE I. Values for $\gamma^{(1)}(\mathcal{T})$.

$\mathcal{T}$	$T+1$	T	$T-1$
$\gamma^{(1)}(\mathcal{T})$	$-\left(\frac{T+1}{2T}\right)$	$-\frac{1}{2T}$	$\frac{1}{2}$

can be used to find elastic, single, and double charge exchange amplitudes. However, Eq. (4) is solved for the amplitudes $F_{\mathcal{T}}$ which are the projections of $F(\theta)$ in Eq. (8) onto states of good $\vec{\mathcal{T}}$. We find from Eq. (8),

$$F_{\mathcal{T}} \equiv \langle \mathcal{T}M; T, 1 | F(\theta) | \mathcal{T}M; T, 1\rangle$$
$$= f_0 + \left(\frac{\mathcal{T}(\mathcal{T}+1) - T(T+1) - 2}{2}\right) f_1$$
$$+ \left(\frac{\mathcal{T}(\mathcal{T}+1) - T(T+1) - 2}{2}\right)^2 f_2 . \quad (9)$$

Inverting this relationship we have

$$f_0 = F_T + \frac{1}{2T+1}(F_{T+1} - F_{T-1}), \quad (10a)$$

$$f_1 = \frac{(T+2)}{(2T+1)(T+1)} F_{T+1} - \frac{(T-1)}{T(2T+1)} F_{T-1} - \frac{F_T}{T(T+1)}, \quad (10b)$$

$$f_2 = \frac{-F_T}{T(T+1)} + \frac{F_{T-1}}{T(2T+1)} + \frac{F_{T+1}}{(T+1)(2T+1)} . \quad (10c)$$

On the other hand, the elastic amplitudes are easily found from Eq. (8):

$$F^{+} \equiv \langle \pi^{+}T | F(\theta) | \pi^{+}T\rangle = f_0 - Tf_1 + T(T+1)f_2 , \quad (11a)$$

$$F^{-} \equiv \langle \pi^{-}T | F(\theta) | \pi^{-}T\rangle = f_0 + Tf_1 + T^2 f_2 , \quad (11b)$$

$$F^{0} \equiv \langle \pi^{0}T | F(\theta) | \pi^{0}T\rangle = f_0 + Tf_2 , \quad (11c)$$

which in terms of F_T are

$$F^{+} = \frac{F_T}{T+1} + \frac{F_{T+1}}{(T+1)(2T+1)} + \frac{2T-1}{2T+1} F_{T-1} , \quad (12a)$$

$$F^{-} = F_{T+1} , \quad (12b)$$

$$F^{0} = \frac{T}{T+1} F_T + \frac{F_{T+1}}{T+1} . \quad (12c)$$

Similarly, the single charge exchange amplitude $F^{(0+)}$ and double charge exchange amplitude $F^{(-+)}$ are

$$F^{(0+)} \equiv \langle \pi^0 : T, -T+1 | F(\theta) | \pi^{+} : T, -T\rangle$$
$$= \sqrt{T}(f_1 - Tf_2), \quad (13a)$$

$$F^{(-+)} \equiv \langle \pi^{-} : T, -T+2 | F(\theta) | \pi^{+} : T, -T\rangle$$
$$= [T(2T-1)]^{1/2} f_2 . \quad (13b)$$

These may in turn be written in terms of $F_{\mathcal{T}}$:

$$F^{(0+)} = \frac{1}{\sqrt{T}} \frac{1}{2T+1} \frac{1}{T+1} [(2T^2 - 1)(F_T - F_{T-1}) + T(2F_{T+1} - F_T - F_{T-1})] , \quad (14a)$$

$$F^{(-+)} = -\left(\frac{2T-1}{T}\right)^{1/2} \frac{1}{T+1} \frac{1}{2T+1} \times [F_T - F_{T-1} - T(F_{T+1} + F_{T-1} - 2F_T)] . \quad (14b)$$

Equations (12) and (14) are the desired results, which show how to calculate elastic and charge exchange amplitudes once F_T, F_{T-1}, and F_{T+1} are known.

III. EIKONAL THEORY FOR ELASTIC SCATTERING AND CHARGE EXCHANGE

The theory of Ref. 10a gives simple, analytic expressions for the elastic scattering amplitude corresponding to a Klein-Gordon equation with an optical potential U of the form

$$U = \beta(k^2\rho + \tfrac{1}{2}\nabla^2\rho) . \quad (15)$$

The optical potential $U_{\mathcal{T}}$ has a similar form [see Eq. (7)], so we may apply the results of Ref. 10a, suitably generalized. (There is a minor change made in Eq. (17) from that appearing in Ref. 10a. See Ref. 10b.) In channel $\mathcal{T}$ the amplitude is

$$F_{\mathcal{T}}(\theta) \equiv F(\theta, R_{\mathcal{T}}) = ikR_{\mathcal{T}}^2 \frac{J_1(qR_{\mathcal{T}})}{qR_{\mathcal{T}}} + AR_{\mathcal{T}} J_0(qR_{\mathcal{T}}) . \quad (16)$$

Here k is the pion-nucleus center-of-mass momentum and q is the momentum transfer. The quantity $R_{\mathcal{T}}$ is defined by

$$\frac{1}{2k}(2\pi a_{\mathcal{T}} R_{\mathcal{T}})^{1/2} \mathrm{Im} U_{\mathcal{T}}(R_{\mathcal{T}}) = \ln 2 , \quad (17)$$

where $a_{\mathcal{T}}$ is the diffuseness in channel $\mathcal{T}$ evaluated at some appropriate $\bar{R}$,

$$-1/a_{\mathcal{T}} \equiv \mathrm{Im} U'_{\mathcal{T}}(\bar{R}) / \mathrm{Im} U_{\mathcal{T}}(\bar{R}) . \quad (18)$$

The quantity A in Eq. (16) is defined as

$$A = ika[C + \ln\ln 2 + \tfrac{1}{2}\ln(1 + Y^2)] + ka \tan^{-1} Y , \quad (19)$$

where C is Euler's constant

$$C \simeq 0.5772 \quad (20)$$

and

$$Y_{\mathcal{T}} \equiv \mathrm{Re} U_{\mathcal{T}}(\bar{R}) / \mathrm{Im} U_{\mathcal{T}}(\bar{R}) . \quad (21)$$

The cross section for elastic scattering from the target nucleus may be evaluated from Eqs. (16)–(20) and Eqs. (12a)–(12c). We shall not be further concerned with elastic scattering in this paper, but we shall use the expressions in Eqs. (16)–(21) to evaluate single and double charge exchange according to Eqs. (14a) and (14b). The formulas of Ref. 10a have been extended by Germond and Johnson[10b] to provide a better description of elastic scattering for angles between the first and second diffraction minima. In this work we shall

base our results on those of Ref. 10a because they are somewhat simpler. However, the results of Ref. 10b could be used to obtain analytical formulas giving more accurate results at larger angles.

The differences between the channel optical potentials U_T are rather small, and therefore the differences in Eq. (14) can be evaluated in terms of derivatives of the F in Eq. (16). Thus, the difference $F_T - F_{T-1}$ in Eq. (14) is found by making a Taylor series expression about $\bar{R}$. (A also depends on the channel T [see Eq. (19)] and the dependence of this term on T should also be expanded out. As long as we are close to the (3-3) resonance this term is small, and we shall neglect it.) For example,

$$F(\theta,R_T)\simeq F(\theta,\bar{R})+(R_T-\bar{R})\frac{dF}{d\bar{R}}(\theta,\bar{R}) +\tfrac{1}{2}(R_T-\bar{R})^2\frac{d^2F}{d\bar{R}^2}(\theta,\bar{R}). \tag{22}$$

We then easily find

$$F(\theta,R_T)-F(\theta,R_{T-1})\simeq (R_T-R_{T-1})\frac{dF}{d\bar{R}}(\theta,\bar{R}) +\tfrac{1}{2}(R_T-R_{T-1}) \times(R_T+R_{T-1}-2\bar{R})\frac{d^2F}{d\bar{R}^2}(\theta,\bar{R}). \tag{23}$$

We will choose $\bar{R}$ later to simplify the theory. We also need the double differences in Eq. (14). We find

$$F_{T+1}+F_{T-1}-2F_T=(F_{T+1}-F_T)-(F_T-F_{T-1}) =(R_{T+1}+R_{T-1}-2R_T)\frac{dF}{d\bar{R}} +\frac{1}{2}\frac{d^2F}{d\bar{R}^2}[(R_{T+1}-R_T)(R_{T+1}+R_T-2\bar{R})-(R_T-R_{T-1})(R_T+R_{T-1}-2\bar{R})]. \tag{24}$$

By choosing $\bar{R}$ appropriately the expression simplifies,

$$F_{T+1}+F_{T-1}-2F_T=(R_{T+1}+R_{T-1}-2R_T)\frac{dF}{d\bar{R}} +(R_{T+1}-R_T)(R_T-R_{T-1})\frac{d^2F}{d\bar{R}^2}, \tag{25a}$$

where

$$\bar{R}=\tfrac{1}{2}(R_{T+1}+R_{T-1}). \tag{25b}$$

The result in Eq. (24) may also be used to evaluate the difference $2F_{T+1}-F_T-F_{T-1}$. We find

$$2F_{T+1}-F_T-F_{T-1}=(2R_{T+1}-R_T-R_{T-1})\frac{dF}{d\bar{R}} -(R_T-R_{T+1})(R_{T+1}-R_{T-1})\frac{d^2F}{d\bar{R}^2}, \tag{26a}$$

where

$$\bar{R}=\tfrac{1}{2}(R_T+R_{T-1}). \tag{26b}$$

Combining Eqs. (14a), (23), and (26a) we find

$$F^{(0+)}(\theta)=\frac{1}{\sqrt{T}}\frac{1}{2T+1}\frac{1}{T+1}\left[S^{(0+)}\frac{dF}{d\bar{R}}(\bar{R})+P^{(0+)}\frac{d^2F}{d\bar{R}^2}(\bar{R})\right], \tag{27a}$$

where

$$S^{(0+)}=(2T^2-1)(R_T-R_{T-1})+T(R_{T+1}-R_T) +T(R_{T+1}-R_{T-1}), \tag{27b}$$

$$P^{(0+)}=T(R_{T+1}-R_{T-1})(R_{T+1}-R_T), \tag{27c}$$

and

$$\bar{R}=\tfrac{1}{2}(R_T+R_{T-1}). \tag{27d}$$

Similarly, combining Eqs. (14b), (23), and (25) we find

$$F^{(-+)}(\theta)=-\left(\frac{2T-1}{T}\right)^{1/2}\frac{1}{T+1}\frac{1}{2T+1} \times\left[S^{(-+)}\frac{dF}{d\bar{R}}(\bar{R})+P^{(-+)}\frac{d^2F}{d\bar{R}^2}(\bar{R})\right], \tag{28a}$$

where

$$S^{(-+)}=(T+1)(R_T-R_{T-1}) -T(R_{T+1}-R_T), \tag{28b}$$

$$P^{(-+)}=-\tfrac{1}{2}(2T+1)(R_{T+1}-R_T) \times(R_T-R_{T-1}), \tag{28c}$$

and

$$\bar{R}=\tfrac{1}{2}(R_{T+1}+R_{T-1}). \tag{28d}$$

There are several observations to be made at this point. The first is that Eqs. (27) and (28)

have been derived consistently from the expression in Eq. (14) in which only the first two terms in the Taylor series were retained. As we shall see, this is essentially an expansion in $\Delta\rho(\bar{R})/\rho(\bar{R})$, which is a small quantity in the nuclear surface. In what follows, the terms will therefore be kept consistently through order $(\Delta\rho/\rho)^2$. The second point is that the derivatives of F are easily evaluated from Eq. (16):

$$\frac{dF}{dR} = ik\bar{R}J_0(q\bar{R}) + A[J_0(q\bar{R}) - q\bar{R}J_1(q\bar{R})], \tag{29}$$

$$\frac{d^2F}{dR^2} = ik[J_0(q\bar{R}) - q\bar{R}J_1(q\bar{R})]$$

$$-Aq[J_1(q\bar{R}) + q\bar{R}J_0(q\bar{R})]. \tag{30}$$

It should be clear that the higher order derivatives vanish as progressively higher powers of q as $q \to 0$. Thus, truncating the expansion in Eq. (22) is also reasonable for evaluating the cross section for small angles. The quantity A in Eqs. (29) and (30) is given by Eq. (19).

According to Eqs. (27) and (28) the single and double charge exchange cross sections depend on combinations of the following three differences of radii

$$R_T - R_{T-1},\ R_{T+1} - R_T, \text{ and } R_{T+1} - R_{T-1}. \tag{31}$$

The physics of single and double charge exchange thus reduces to an understanding of these three differences. In the eikonal theory the connection between the differences and the quantities of the underlying theory is given by Eq. (17).

To solve Eq. (17) and obtain the differences $\Delta R_{ij} \equiv R_i - R_j$ of Eq. (31), we write

$$R_i = R' + \Delta R_{ij}/2, \tag{32}$$

$$R_j = R' - \Delta R_{ij}/2 \tag{33}$$

and

$$a_i = a_{ij} + \delta/2, \tag{34}$$

$$a_j = a_{ij} - \delta/2, \tag{35}$$

and assume that

$$U_i(R_i) = U_i(R)\exp(R - R_i)/a_i(R). \tag{36}$$

We choose R to be the same as $\bar{R}$ in Eqs. (25b) and (26b). For notational simplicity it is to be understood in the remainder of the paper that U, ξ, and λ mean ImU, Imξ, and Imλ unless stated otherwise. Thus,

$$\exp\left(-\frac{\Delta R_{ij}}{2a_i} + \frac{\Delta R_{ij}}{2a_j}\right) = \frac{U_i(R)}{U_j(R)}\left(\frac{a_jR_i}{a_iR_j}\right)^{1/2} \tag{37}$$

or

$$\Delta R_{ij}\left(\frac{a_{ij}}{a_{ij}^2 - \delta^2/4}\right) = \ln\frac{U_i(R)}{U_j(R)} + \tfrac{1}{2}\ln\left(\frac{1+\delta/2a_{ij}}{1-\delta/2a_{ij}}\right) + \tfrac{1}{2}\ln\left(\frac{1+\Delta R_{ij}/2R'}{1-\Delta R_{ij}/2R'}\right). \tag{38a}$$

Using the definition of ξ in Eq. (6) this may be written

$$\Delta R_{ij}\left(\frac{a_{ij}}{a_{ij}^2 - \delta^2/4}\right) = \ln\frac{\xi_i(R)}{\xi_j(R)} + \ln\left[\frac{1 + \frac{1}{2k^2}\frac{\nabla^2\xi_i(R)}{\xi_i(R)}}{1 + \frac{1}{2k^2}\frac{\nabla^2\xi_j(R)}{\xi_j(R)}}\right] + \tfrac{1}{2}\ln\left(\frac{1+\delta/2a_{ij}}{1-\delta/2a_{ij}}\right) + \tfrac{1}{2}\ln\left(\frac{1+\Delta R_{ij}/2R'}{1-\Delta R_{ij}/2R'}\right). \tag{38b}$$

Rearranging terms in Eq. (38b), we get

$$\Delta R_{ij}\left(\frac{a_{ij}}{a_{ij}^2 - \delta^2/4}\right) = \ln\left(\frac{1+x}{1-x}\right) + \ln\left(\frac{1+y}{1-y}\right) + \tfrac{1}{2}\ln\left(\frac{1+\delta/2a_{ij}}{1-\delta/2a_{ij}}\right) + \tfrac{1}{2}\ln\left(\frac{1+\Delta R_{ij}/2R'}{1-\Delta R_{ij}/2R'}\right), \tag{39a}$$

where

$$x = [\xi_i(R) - \xi_j(R)]/[\xi_i(R) + \xi_j(R)], \tag{39b}$$

$$y = \frac{[U_i(R)/\xi_i(R) - U_j(R)/\xi_j(R)]}{[U_i(R)/\xi_i(R) + U_j(R)/\xi_j(R)]}. \tag{39c}$$

[Note that y is the difference of the second derivative at R; see Eq. (38b).] The quantities x and y are generally small numbers because for large R the nuclear densities and their rates of falloff are roughly the same. Expanding the logarithms in Eq. (39a), we get

$$\frac{\Delta R_{ij}}{a_{ij}} = \frac{\Delta R_{ij}}{2R'} + 2x + 2y + \frac{\delta}{2a_{ij}}, \tag{40}$$

where the omitted terms are *third* order in the small parameters. Solving Eq. (40) for ΔR_{ij} we have

$$\Delta R_{ij} = \Delta R_{ij}(1) + \Delta R_{ij}(2) + \Delta R_{ij}(3), \tag{41a}$$

$$\Delta R_{ij}(1) = \frac{4Ra_{ij}}{2R - a_{ij}}\left(\frac{\xi_i(R) - \xi_j(R)}{\xi_i(R) + \xi_j(R)}\right), \tag{41b}$$

$$\Delta R_{ij}(2) = \frac{4Ra_{ij}}{2R - a_{ij}}\left(\frac{U_i(R)/\xi_i(R) - U_j(R)/\xi_j(R)}{U_i(R)/\xi_i(R) + U_j(R)/\xi_j(R)}\right), \tag{41c}$$

$$\Delta R_{ij}(3) = \frac{2Ra_{ij}}{2R - a_{ij}}\left(\frac{a_i - a_j}{2a_{ij}}\right). \tag{41d}$$

In this equation we have let $R' = R$, which intro-

duces only small error at order a/R.

We have not made any special assumption about the radial dependence of $U_T(R)$ except that it is approximately exponential in some region about $\bar{R}$ in channel T [see Eq. (36)]. Thus, the results of this section could be used to calculate elastic scattering and charge exchange for rather general forms of $U_T(R)$. The procedure for charge exchange would be to evaluate the differences expressed in Eq. (41) and to use these results in Eqs. (27) and (28). If the assumptions about the form of $U_T(R)$ are sufficiently simple, then it is possible to evaluate ΔR_{ij} analytically in terms of a few parameters characterizing the theory. In the next section we shall make the evaluations in the framework of a lowest order optical potential.

IV. CHARGE EXCHANGE IN THE GEOMETRICAL LIMIT

We are assuming in this paper that the dynamics of charge exchange is governed by the lowest order optical potential. By definition, the strength of this interaction is determined by the scattering of a pion from a *free* nucleon. Therefore the details of the charge exchange cross sections, e.g., the variation of the magnitude with N and Z, are determined by geometrical properties of the nucleus. The geometry is defined by the effective nuclear radius $\bar{R}$, the diffuseness of the surface in the vicinity of $\bar{R}$, and the relative valence nucleon density $\Delta\rho/\rho$ at this point. In this section we shall derive simple expressions for the cross sections which explicate the dependence on the nuclear geometry. In Sec. IV A we assume equal diffuseness for neutrons and protons, but this assumption is relaxed in Sec. IV B.

A. Equal diffuseness for ρ and $\Delta\rho$

Suppose we consider a model in which *locally* the neutron and proton densities are varying exponentially with the *same* diffuseness, i.e.,

$$\rho_n(r) \sim e^{-r/a}, \quad r \approx \bar{R} \tag{42a}$$

$$\rho_p(r) \sim e^{-r/a}, \quad r \approx \bar{R}. \tag{42b}$$

Because we are dropping ρ^2 terms, Eq. (42) implies that $\Delta R_{ij}(2) = 0 = \Delta R_{ij}(3)$, and thus

$$R_i - R_j = \frac{4Ra}{2R-a}\left[\frac{\xi_i(R)-\xi_j(R)}{\xi_i(R)+\xi_j(R)}\right]. \tag{43a}$$

The sums and differences may be determined from Eq. (6), which gives

$$\xi_i(R) - \xi_j(R) = [\gamma^{(1)}(i) - \gamma^{(1)}(j)]\lambda_1^{(1)}\Delta\rho(R), \tag{43b}$$

$$\xi_i(R) + \xi_j(R) = 2\lambda_0^{(1)}\rho(R) + [\gamma^{(1)}(i) + \gamma^{(1)}(j)]\lambda_1^{(1)}\Delta\rho(R). \tag{43c}$$

Combining the results in Eqs. (42) and (43), we find

$$R_i - R_j = \frac{2Ra}{2R-a}[\gamma^{(1)}(i) - \gamma^{(1)}(j)]\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\frac{\Delta\rho(R)}{\rho(R)} \times \frac{1}{1+[\gamma^{(1)}(i)+\gamma^{(1)}(j)]\frac{\lambda_1^{(1)}}{2\lambda_0^{(1)}}\frac{\Delta\rho(R)}{\rho(R)}}. \tag{44}$$

Expanding this through second order in $\Delta\rho/\rho$, we find

$$R_i - R_j \simeq \frac{2Ra}{2R-a}\left[[\gamma^{(1)}(i)-\gamma^{(1)}(j)]\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\frac{\Delta\rho(R)}{\rho(R)} - \tfrac{1}{2}[\gamma^{(1)^2}(i) - \gamma^{(1)^2}(j)]\left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2 \times \left(\frac{\Delta\rho(R)}{\rho(R)}\right)^2\right]. \tag{45}$$

The sums and differences of the γ's in Eq. (45) may be determined from Table I and the results are given in Table II.

The single charge exchange amplitude, Eq. (27a), requires that we evaluate the quantities $S^{(0+)}$ and $P^{(0+)}$ [see Eqs. (27b) and (27c)]. Here we are interested in evaluating the expressions for single and double charge exchange to lowest order in the small parameters. It is clear from Eq. (45) that the term $P^{(0+)}$ does not contribute to lowest order in ρ. We easily find from Table II

$$(2T^2-1)[\gamma^{(1)}(T) - \gamma^{(1)}(T-1)] + T[\gamma^{(1)}(T+1) - \gamma^{(1)}(T)] + T[\gamma^{(1)}(T+1) - \gamma^{(1)}(T-1)] = \tfrac{1}{2}(2T+1)(T+1). \tag{46a}$$

Hence, to lowest order in $\Delta\rho/\rho$, we find from Eqs. (45), (27b), and (46a)

$$S^{(0+)} = \frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\frac{\Delta\rho(R)}{\rho(R)}(2T+1)(T+1)\frac{aR}{2R-a}. \tag{46b}$$

Hence, the expression for the single charge exchange amplitude, Eq. (27a), gives

$$F^{(0+)} = \frac{k\bar{R}}{\sqrt{T}}\left(\frac{\Delta\rho(\bar{R})}{\rho(\bar{R})}\right)\left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right) i\frac{a\bar{R}}{2\bar{R}-a}J_0(q\bar{R}), \tag{47}$$

where, according to Eq. (26b), $\bar{R} = \frac{1}{2}(R_T + R_{T-1})$.

For double charge exchange, Eq. (28a), we

TABLE II. Differences of $\gamma^{(1)}$ and $\gamma^{(1)^2}$.

i	j	$\gamma^{(1)}(i)-\gamma^{(1)}(j)$	$\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)$
T	$T-1$	$\frac{1}{2}$	$-\frac{1}{4}\left(\frac{T+2}{T}\right)$
$T+1$	T	$\frac{T+1}{2T}$	$\frac{T^2-1}{4T^2}$
$T+1$	$T-1$	$\frac{T+1}{2T}$	$-\left(\frac{2T+1}{4T^2}\right)$

need the combinations $S^{(-+)}$ and $P^{(-+)}$ in Eqs. (28b) and (28c). We easily find from Table II

$$(T+1)[\gamma^{(1)}(T)-\gamma^{(1)}(T-1)] - T[\gamma^{(1)}(T+1)-\gamma^{(1)}(T)]=0 \quad (48)$$

$$(T+1)[\gamma^{(1)^2}(T)-\gamma^{(1)^2}(T-1)] - T[\gamma^{(1)^2}(T+1)-\gamma^{(1)^2}(T)]=-\frac{(2T+1)(T+1)}{4T} \quad (49)$$

Hence, to lowest order in $\Delta\rho/\rho$, we find from Eqs. (45), (28b), (28c), (48), and (49)

$$S^{(-+)}=\frac{a}{2}\frac{(2T+1)(T+1)}{4T}\left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2 \times\left(\frac{\Delta\rho(R)}{\rho(R)}\right)^2\frac{2R}{2R-a}, \quad (50a)$$

$$P^{(-+)}=\tfrac{1}{2}(2T+1)\left(\frac{T+1}{2T}\right)^{\frac{1}{2}}a^2\left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2 \times\left(\frac{\Delta\rho(R)}{\rho(R)}\right)^2\frac{4R^2}{(2R-a)^2}, \quad (50b)$$

and thus from Eq. (28a)

$$F^{(-+)}=-\left(\frac{2T-1}{T}\right)^{1/2}\left(\frac{\Delta\rho(R)}{\rho(R)}\right)^2\frac{a}{8T}\left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2 \times\left(\frac{dF}{dR}-a\frac{d^2F}{dR^2}\right)\frac{2R}{2R-a}. \quad (50c)$$

Using Eqs. (29) and (30), setting $A=0$, gives

$$F^{(-+)}=-\left(\frac{2T-1}{T}\right)^{1/2}\frac{ik\bar{R}}{8T}\left(\frac{\Delta\rho}{\rho}\right)^2\left(\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2\frac{2\bar{R}a}{2\bar{R}-a} \times\{J_0(q\bar{R})-(a/\bar{R})[J_0(q\bar{R})-q\bar{R}J_1(q\bar{R})]\}, \quad (50d)$$

where $\bar{R}=\frac{1}{2}(R_{T+1}+R_{T-1})$.

In order to get some feeling for the validity of these formulas, we have solved the Klein-Gordon equation for elastic scattering using the program PIRK.[11] We included only the (3-3) pion-nucleon scattering amplitude to construct the optical potential. The target nucleus was $N=48$, $Z=40$ with ρ_n proportional to ρ_p. For the density we chose a Woods-Saxon form with half radius $=3.74$ fm and diffuseness $=0.564$ fm. The Coulomb interaction was turned off. The elastic scattering amplitudes F_{T-1}, F_{T+1}, and F_T were used to obtain the solid curve in Fig. 1 using Eq. (14a). The dashed curve is the semiclassical formula result corresponding to the present work. We included all terms in the amplitude required by Eq. (16). The quantity plotted is $\sigma(\theta)$ at a pion kinetic energy of 180 MeV:

$$\sigma(\theta)=|F^{(0+)}(\theta)|^2, \quad (51)$$

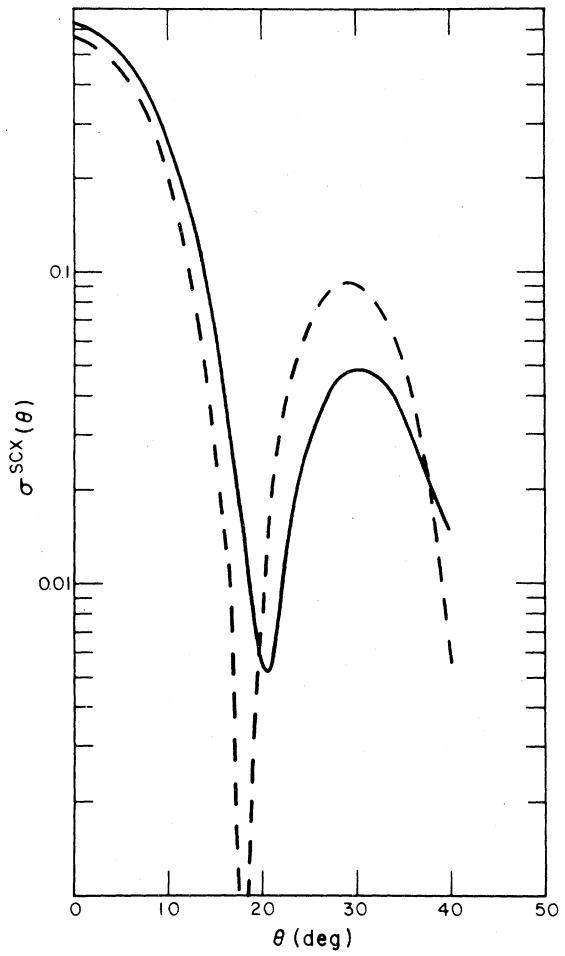

FIG. 1. Angular distribution for single charge exchange in mb/sr. The solid curve is an exact coupled channel calculation and the dashed curve is the semiclassical approximation. The pion energy is 180 MeV; the nucleus is $N=28$, $Z=20$.

where

$$F^{(0+)}(\theta)=\frac{ika}{2}\frac{\Delta\rho(\bar{R})}{\rho(\bar{R})}\left(\frac{\bar{R}^2}{2\bar{R}-a}\right) \times\{J_0(q\bar{R})+(a/\bar{R})[J_0(q\bar{R})-q\bar{R}J_1(q\bar{R})](A-iB)\}, \quad (52a)$$

$$A=C+\ln\ln 2+\tfrac{1}{2}\ln(1+Y^2)=0.2165, \quad (52b)$$

$$B=\tan^{-1}Y=0.1079, \quad \Delta\rho/\rho=\tfrac{8}{48}, \quad (52c)$$

$$Y=0.1083, \quad \bar{R}=5.13, \quad a=0.690. \quad (52d)$$

The values of $\bar{R}$ and a were obtained by solving Eqs. (17) and (18). This calculation demonstrates the semiquantitative validity of our approximations. The calculations agree to within 10% at 0° and becomes progressively worse in magnitude at larger angles. At the position of the second maximum the approximation overestimates the exact result

by about a factor of 2. However, the locations of the first minimum agrees to within 2° and the position of the second maximum seems to be correctly reproduced.

Figure 2 shows the corresponding calculation for double charge exchange. The double charge exchange amplitude is

$$F^{(-+)}(\theta)=\frac{ika}{16}\left(\tfrac{1}{4}\right)^{1/2}\left[\frac{\Delta\rho(\overline{R})}{\rho(\overline{R})}\right]^2\left(\frac{\overline{R}^2}{2\overline{R}-a}\right)\times\{J_0(q\overline{R})-(a/\overline{R})[J_0(q\overline{R})-q\overline{R}J_1(q\overline{R})]\times(1-A+iB)\},\tag{52e}$$

where we used the values given in Eqs. (52b)–(52d), except now $\overline{R}=5.10$ and $a=0.697$. At $\theta=0°$, the approximation for $\sigma^{DCX}(\theta)$ is 44% below the exact result. The locations of the minima now agree to 4°, and the maxima are shifted by several degrees.

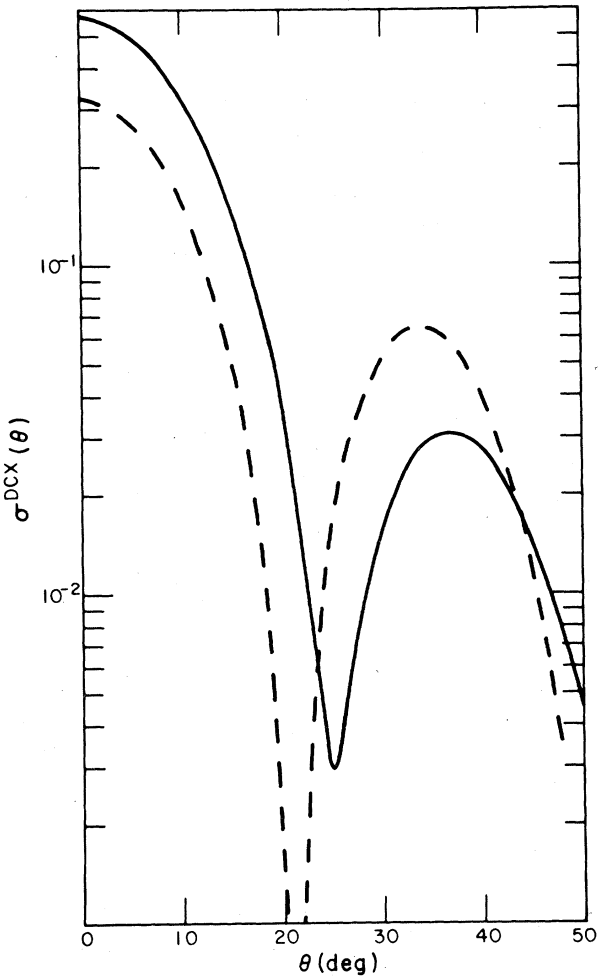

FIG. 2. Angular distribution for double charge exchange in μb/sr. The solid curve is an exact coupled channel calculation and the dashed curve is the semiclassical approximation. The pion energy is 180 MeV; the nucleus is $N=28$, $Z=20$.

The approximation for the double charge exchange amplitude is somewhat inferior to that of the single charge exchange. Both calculations underestimate the exact results at forward angles and have their minima at slightly smaller angles. The approximations overestimate the height of the second maximum, but this discrepancy would be diminished by using the analytical formulas of Ref. 10b rather than 10a. The extent of disagreement in Figs. 1 and 2 at forward angles would presumably tend to be removed by retaining higher powers of $\Delta\rho/\rho$ in the expansions derived in this paper, at the expense of making the formulas more complicated. The simpler formulas are a sufficient reproduction of the exact result for the purposes of our present investigation. Wallace[12] has made a careful study of the eikonal theory and written explicitly a set of corrections which are needed to improve the extent of agreement between the eikonal theory and an exact solution of the wave equation. These corrections are second order and higher in the density and one might therefore expect the double charge exchange cross section to be more strongly affected by their absence than the single charge exchange.

It is interesting to note that the first minimum of the SCX and DCX cross sections occur at a smaller angle than the minima in the elastic scattering angular distributions. To a first approximation the *second maximum* of the SCX and DCX cross sections occur at the same angle as the *first minimum* of the elastic scattering angular distributions [compare Eqs. (16), (47), and (50d)]; it is this angle which is usually determined by the effective nuclear radius $\overline{R}$.

B. Different diffuseness for ρ and $\Delta\rho$

In Sec. IV A we made the approximation of assuming that $\rho(r)$ and $\Delta\rho(r)$ had the same rate of falloff in the nuclear surface. In this subsection we shall investigate the corrections to this approximation. In order to do this we must carefully study the terms $\Delta R_{t_j}(2)$ and $\Delta R_{t_j}(3)$ in Eq. (41).

Let us begin by considering $\Delta R_{i_j}(3)$:

$$\Delta R_{i_j}(3)=\tfrac{1}{2}(a_i-a_j),\tag{53}$$

where

$$a_i=-\frac{k^2\xi_i(R)+\nabla^2\xi_i(R)/2}{k^2\xi_i'(R)+\nabla^2\xi_i'(R)/2}.\tag{54}$$

Now, we define the diffuseness a_ρ and a_Δ so that

$$a_\rho\equiv-\frac{k^2\rho(R)+\nabla^2\rho(R)/2}{k^2\rho'(R)+\nabla^2\rho'(R)/2}\tag{55a}$$

and

$$a_\Delta \equiv -\frac{k^2\Delta\rho(R)+\nabla^2\Delta\rho(R)/2}{k^2\Delta\rho'(R)+\nabla^2\Delta\rho'(R)/2}\,. \tag{55b}$$

Taking ξ_i from Eq. (6a), Eq. (54) becomes [using Eq. (55)]

$$a_i = a_\rho \frac{\lambda_0^{(1)}(k^2\rho+\frac{1}{2}\nabla^2\rho)+\gamma^{(1)}(i)\lambda_1^{(1)}(k^2\Delta\rho+\frac{1}{2}\nabla^2\Delta\rho)}{\lambda_0^{(1)}(k^2\rho+\frac{1}{2}\nabla^2\rho)+\frac{a_\rho}{a_\Delta}\gamma^{(1)}(i)\lambda_1^{(1)}(k^2\Delta\rho+\frac{1}{2}\nabla^2\Delta\rho)}\,. \tag{56}$$

Dividing numerator and denominator by $\lambda_0^{(1)}(k^2\rho+\frac{1}{2}\nabla^2\rho)$, we find

$$a_i = a_\rho \frac{1+\gamma^{(1)}(i)\alpha\kappa_1}{1+(a_\rho/a_\Delta)\gamma^{(1)}(i)\alpha\kappa_1}\,, \tag{57a}$$

where we define

$$\alpha \equiv \frac{\Delta\rho}{\rho}\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}} \tag{57b}$$

and

$$\kappa_1 \equiv \frac{1+\dfrac{1}{2k^2}\dfrac{\nabla^2\Delta\rho}{\Delta\rho}}{1+\dfrac{1}{2k^2}\dfrac{\nabla^2\rho}{\rho}}\,. \tag{57c}$$

Expanding Eq. (57a) to second order in the small parameter α, we find

$$a_i = a_\rho + a_\rho\left(1-\frac{a_\rho}{a_\Delta}\right)\left[\alpha\kappa_1\gamma^{(1)}(i) - \frac{a_\rho}{a_\Delta}\alpha^2\kappa_1{}^2\gamma^{(1)^2}(i)\right]. \tag{58}$$

Using Eq. (58), we find from Eq. (53)

$$\frac{\Delta R_{ij}(3)}{1-a_\rho/a_\Delta} = \frac{a_\rho}{2}\alpha\kappa_1[\gamma^{(1)}(i)-\gamma^{(1)}(j)] - \frac{a_\rho}{2}\left(\frac{a_\rho}{a_\Delta}\right)\alpha^2\kappa_1{}^2 \times[\gamma^{(1)}(i)^2-\gamma^{(1)}(j)^2]\,. \tag{59}$$

Next consider $\Delta R_{ij}(2)$ [Eq. (41)], which may be rearranged to read

$$\Delta R_{ij}(2) = \frac{a_{ij}}{k^2}\,\frac{\nabla^2\xi_i/\xi_i-\nabla^2\xi_j/\xi_j}{2+(\nabla^2\xi_i/\xi_i+\nabla^2\xi_j/\xi_j)/(2k^2)}\,. \tag{60}$$

We shall first find an expression for $\nabla^2\xi_i/\xi_i$. Using Eq. (6a),

$$\nabla^2\xi_i/\xi_i = \frac{\nabla^2\rho/\rho+\gamma^{(1)}(i)\alpha\nabla^2\Delta\rho/\Delta\rho}{1+\gamma^{(1)}(i)\alpha}\,, \tag{61}$$

where we have used the definition of α, Eq. (57b). Expanding Eq. (61) to second order in α, we get

$$\frac{\nabla^2\xi}{\xi} = \nabla^2\rho/\rho - (\nabla^2\rho/\rho-\nabla^2\Delta\rho/\Delta\rho) \times[\alpha\gamma^{(1)}(i)-\alpha^2\gamma^{(1)^2}(i)]\,. \tag{62}$$

Substituting Eq. (62) into Eq. (60) we find, dropping the second order term in the denominator,

$$\Delta R_{ij}(2) = -\frac{a_{ij}\bar{\kappa}_1\{\alpha[\gamma^{(1)}(i)-\gamma^{(1)}(j)]-\alpha^2[\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)]\}}{1-(\alpha\bar{\kappa}_1/2)[\gamma^{(1)}(i)+\gamma^{(1)}(j)]}\,, \tag{63}$$

where

$$\bar{\kappa}_1 = (\nabla^2\rho/\rho-\nabla^2\Delta\rho/\Delta\rho)(2k^2+\nabla^2\rho/\rho)^{-1}\,. \tag{64}$$

Expanding Eq. (63) to second order in α, we find

$$\Delta R_{ij}(2) = -a_{ij}\bar{\kappa}_1\alpha[\gamma^{(1)}(i)-\gamma^{(1)}(j)] + a_{ij}\bar{\kappa}_1\alpha^2[1-(\bar{\kappa}_1/2)][\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)]\,. \tag{65}$$

Making use of the following relationship between κ_1 and $\bar{\kappa}_1$

$$1-\kappa_1 = \bar{\kappa}_1\,, \tag{66}$$

Eq. (65) becomes

$$\Delta R_{ij}(2) = -a_{ij}\alpha(1-\kappa_1)[\gamma^{(1)}(i)-\gamma^{(1)}(j)] + \frac{a_{ij}\alpha^2}{2}(1-\kappa_1{}^2)[\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)]\,. \tag{67}$$

Now, we need a_{ij} in order to evaluate $\Delta R_{ij}(1)$ and $\Delta R_{ij}(2)$. This quantity may be evaluated from Eq. (58). To first order in α,

$$a_{ij} \equiv a_i + a_j = a_\rho + \frac{a_\rho}{2}\left(1-\frac{a_\rho}{a_\Delta}\right)\alpha\kappa_1[\gamma^{(1)}(i)+\gamma^{(1)}(j)]\,. \tag{68}$$

Using this in Eq. (45),

$$\Delta R_{ij}(1) \simeq a_\rho\left\{\alpha[\gamma^{(1)}(i)-\gamma^{(1)}(j)] - \tfrac{1}{2}\alpha^2\left[1-\kappa_1\left(1-\frac{a_\rho}{a_\Delta}\right)\right][\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)]\right\}, \tag{69}$$

and in Eq. (67),

$$\Delta R_{ij}(2) = -a_\rho(1-\kappa_1)\alpha[\gamma^{(1)}(i)-\gamma^{(1)}(j)] + \frac{a_\rho(1-\kappa_1)\alpha^2}{2}[1+\kappa_1 a_\rho/a_\Delta][\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)]\,. \tag{70}$$

Finally, putting together the results of Eqs. (59), (69), and (70) we find

$$\Delta R_{ij} = \alpha a_\rho\left[\frac{\kappa_1}{2}(3-a_\rho/a_\Delta)\right][\gamma^{(1)}(i)-\gamma^{(1)}(j)] - \frac{\alpha^2 a_\rho}{2}[(a_\rho/a_\Delta)\kappa_1^2(2-a_\rho/a_\Delta)][\gamma^{(1)^2}(i)-\gamma^{(1)^2}(j)]\,. \tag{71}$$

In the limit that $a_\rho = a_\Delta$, $\kappa_1 = 1$, this result is the same as Eq. (45), as it should be.

We may now evaluate the modified SCX and DCX amplitudes. Repeating the arguments following Eq. (45), we have

$$S^{(0+)}(\text{new}) = S^{(0+)}(\text{old})\left[\frac{\kappa_1}{2}(3-a_\rho/a_\Delta)\right] \tag{72a}$$

and

$$S^{(-+)}(\text{new}) = S^{(-+)}(\text{old}) \times[(a_\rho/a_\Delta)\kappa_1^2(2-a_\rho/a_\Delta)]\,. \tag{72b}$$

The $P^{(0+)}$ function is unchanged to the order we are considering. However, $P^{(-+)}$ becomes

$$P^{(-+)}(\text{new}) = P^{(-+)}(\text{old})\left[\frac{\kappa_1}{2}(3-a_\rho/a_\Delta)\right]^2\,. \tag{72c}$$

The amplitudes may be evaluated from Eqs. (27a) and (28a); explicit expressions for cross sections are given in Eqs. (73) and (74) below.

In the next section we shall try to understand the importance of the various terms of the theory by making a comparison to the experimental data which exists.

V. COMPARISON OF GEOMETRICAL MODEL TO EXPERIMENT

The theory of Sec. IV was derived under the following set of approximations:

(1) The underlying interactions are isospin invariant.

(2) The optical potential is first order in ρ and $\Delta\rho$.

(3) Only the leading terms in $\Delta\rho/\rho$ are retained in the expressions for the single and double charge exchange amplitudes.

We also assumed that there is no S-wave pion-nucleon scattering, which is approximately true in the region of the (3-3) resonance.

Let us begin by recording the complete expressions for the angular distributions in the semiclassical theory. We have, from Eqs. (27), (28), (46), (50), and (72),

$$\sigma^{(0+)}(\theta) = \left(\kappa_1\frac{\Delta\rho(\bar R)}{\rho(\bar R)}\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^2\frac{(ka_\rho)^2}{4T} \times\frac{1}{4}\frac{(3-a_\rho/a_\Delta)^2}{(\bar R-a_\rho/2)^2}\bar R^4 J_0^2(q\bar R) \tag{73a}$$

with

$$\bar R = \tfrac{1}{2}(R_T + R_{T-1})\,, \tag{73b}$$

and

$$\sigma^{(-+)}(\theta) = \left(\kappa_1\frac{\Delta\rho(\bar R)}{\rho(\bar R)}\frac{\lambda_1^{(1)}}{\lambda_0^{(1)}}\right)^4\frac{(ka_\rho)^2}{8}\frac{2T-1}{(2T)^3}\frac{\bar R^4}{(\bar R-a_\rho/2)^2} \times\{(a_\rho/a_\Delta)(2-a_\rho/a_\Delta)J_0(q\bar R) - \tfrac{1}{4}(3-a_\rho/a_\Delta)^2(a_\rho/\bar R)[J_0(q\bar R)-q\bar R J_1(q\bar R)]\}^2 \tag{74a}$$

with

$$\bar R = \tfrac{1}{2}(R_{T+1} + R_{T-1})\,, \tag{74b}$$

where κ_1, a_ρ, and a_Δ are defined in Eqs. (57c), (55a), and (55b), respectively. These simple expressions express the dependence of the single and double charge exchange cross sections on the nuclear geometry. We have omitted small terms involving the quantity A [see Eqs. (29) and (30)]. These become slightly more important away from resonance and should be included for more accurate studies of the angular distribution.

If the assumptions (1)–(3) are valid, then a comparison of the theory to experiment provides an experimental means of verifying the nuclear model from which $\bar R$, ρ, $\Delta\rho$, a_ρ, and a_Δ are obtained. Because the cross sections are exceedingly sensitive to these quantities in the tail of the nucleus, single and double charge exchange could become a valuable tool for studies of nuclear structure, especially the neutron halo.

However, it is by no means clear that the assumptions expressed here are sufficient for a complete theory. Difficulties of previous theories to reproduce experiment[1-3] suggest that they are not. Perhaps the most interesting information is contained in the *discrepancies* between the experiment and the geometrical model with realistic nuclear densities, as this is a measure of the extent of modification of the pion-nucleon interaction in the medium. Because the data analysis for the single charge exchange measurements is still in progress, a detailed comparison between the theory and experiment will have to be made at a later time. Although there are preliminary data published in Ref. 1, we will not quote these results because there are some sizeable corrections to some of these points. However, there

is double charge exchange data available and a comparison between the theory and experiment will be made in this case. A comparison between this theory and the data is an important step in understanding the nature of the charge exchange reactions. It complements the more comprehensive computer solutions by providing direct physical insight into the causes of success or failure of the calculation. We are not able to calculate the total cross sections to analog states in this theory, because the eikonal theory on which our results are based is valid only for forward scattering.

First, consider the A dependence of the single charge exchange cross sections. We shall display the *relative* A dependence in which case the following three assumptions are plausible:

(a) a is constant throughout the periodic table.

(b) $\overline{R} \propto A^{1/3}$.

(c) The neutron and proton densities are related to the total density $\rho(r)$ by

$$\rho_n(r) = \frac{N}{A}\rho(r), \tag{75a}$$

$$\rho_p(r) = \frac{Z}{A}\rho(r), \tag{75b}$$

in which case

$$\frac{\Delta\rho(r)}{\rho(r)} = \frac{N-Z}{A}. \tag{75c}$$

It then follows that for a given energy

$$\sigma^{(0+)}(\theta) \propto (N-Z)A^{-4/3}J_0^2(q\overline{R}). \tag{76}$$

We show a plot of Eq. (76) at $\theta = 0°$ for selected nuclei throughout the periodic table, in Fig. 3. The structure seen in this figure is due to the diffractive character of the scattering and would presumably be seen in single charge exchange for any strongly absorbed probe. The theory reproduces the general trend of the data,[1] but tends to underestimate the scattering from the lighter elements.

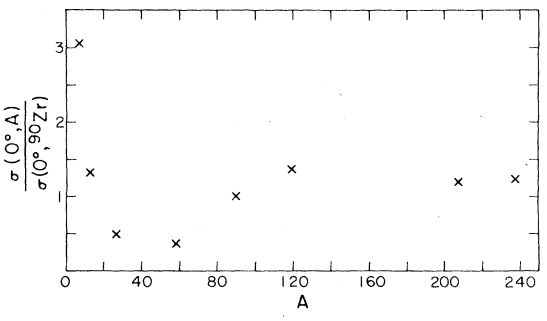

FIG. 3. Relative A dependence according to Eq. (76). The magnitude is normalized to unity for ^{90}Zr.

Next, let us consider the absolute magnitude of the angular distribution. In order to calculate this we shall need $\overline{R}$, a, and $\Delta\rho/\rho$. According to the theory of Ref. 10 the radius combination in Eqs. (73b) and (74b) should be evaluated from Eq. (17). However, it is important to verify that the theory used to evaluate charge exchange is consistent with elastic scattering data, and to accomplish this we have determined $\overline{R}$ from empirical total cross sections rather than from the theory. This will ensure that our cavalier treatment of such refinements as Fermi averaging and relativistic transformations between the pion-nucleon and pion-nucleus center mass frames would not lead to an incorrect value for $\overline{R}$. To obtain $\overline{R}$ we use the relationship

$$\sigma_{\text{tot}} = 2\pi\overline{R}^2 + 4\pi a\overline{R}[C + \ln\ln 2 + \tfrac{1}{2}\ln(1+Y^2)]. \tag{77}$$

This expression follows from applying the optical theorem to the elastic scattering amplitude [see Eq. (23) of Ref. 10a]. We took σ_{tot} as the average of the π^+ and π^- total cross sections.[13] The data is not sufficiently accurate to distinguish between R_T, R_{T+1}, and R_{T-1}. The quantities a and Y are defined in Eqs. (18) and (21); we used realistic Hartree-Fock densities[14a] evaluating $a(\overline{R})$ and $\Delta\rho(\overline{R})/\rho(\overline{R})$ for heavy nuclei and densities of Ref. 14b for light nuclei. The results of this calculation are shown in Table III for $T_\pi = 180$ MeV. For energies as low as 100 MeV this procedure for obtaining $\overline{R}$ does not yield sensible results, except for the heavier nuclei, due to the increased nuclear transparency and scattering from the real part of the optical potential.

From the results of Table III we calculate the single charge exchange cross sections from Eq. (73a) and show the results in Table IV under the column headed σ^{th} (realistic ρ). The enhancement factor between the simple theory and the realistic theory is also shown in Table IV. We have assumed that the diffuseness in the simple theory is the same as the diffuseness of ρ in the realistic theory and that $\overline{R}$ is the same in both theories. We conclude that in order to evaluate realistically the single charge exchange cross sections, it is necessary to use the best available densities. The enhancement is largest in the far nuclear tail; for energies different from 180 MeV, which corresponds to a smaller $\overline{R}$, the enhancement will be smaller because $\Delta\rho/\rho$ is more similar to $(N-Z)/A$. No comparison to the data will be made here for the reason stated above, but there is a systematic tendency for the theory to underestimate the preliminary data in Ref. 1a, by as much as a factor of 4 in some cases. In the case of the lighter nuclei, particularly ^{7}Li, the method of calculating the diffuseness a in terms of the deriva-

TABLE III. Parameters needed for evaluating $\sigma(\theta)$ in the geometrical model at $T_\pi = 180$ MeV. R_0 is defined as $R_0 = (\sigma_{tot}/2\pi)^{1/2}$, with values of σ_{tot} taken from Ref. 12. R is defined in Eq. (77). We use $\rho_0 = 0.16$ fm^{-3}.

Target	R_0 (fm)	$\bar{R}$ (fm)	Simple theory $\Delta\rho/\rho$	Realistic theory $\rho/\rho_0(\bar{R})$	$\Delta\rho/\rho(\bar{R})$	$a_\rho(\bar{R})$	$a_\rho/a_\Delta(\bar{R})$	κ_1
^{7}Li [a]	2.84	2.61	0.143	0.185	0.172	0.854	0.393	0.697
^{13}C [a]	3.39	3.25	0.0769	0.128	0.114	0.685	0.618	0.742
^{58}Ni [b]	5.64	5.50	0.0345	0.0781	0.0526	0.670	0.642	0.743
^{90}Zr [b]	6.51	6.37	0.111	0.0639	0.160	0.642	0.937	0.931
^{120}Sn [b]	7.02	6.87	0.167	0.0765	0.366	0.696	0.667	0.786
^{208}Pb [b]	8.27	8.12	0.212	0.0710	0.432	0.681	0.751	0.826

[a] Densities from Ref. 14b. [b] Densities from Ref. 14a.

tive of the density at $\bar{R}$ becomes questionable. Because densities $\rho < \rho(b)$ contribute to the amplitude for scattering at a given impact parameter b, a better prescription for a in the light nuclei would be to take for it the distance Δ such that $\rho(\bar{R}+\Delta) = e^{-1}\rho(\bar{R})$. This has been done in Table III for the case of ^{7}Li.

Finally, in this section, let us consider double charge exchange. To estimate the relative A dependence, we make the assumptions (a)–(c) above Eq. (75) and apply these to Eq. (74). We then find

$$\sigma^{(--)}(0) \sim A^{-10/3}(N-Z)(N-Z-1)\,. \tag{78}$$

In Table V we show the cross sections at $\theta = 0°$ relative to those of ^{18}O. According to this table, the double charge exchange cross section to the double analog of ^{48}Ca should be the same as for ^{18}O. The experiments[2] have resolved a clear signal for ^{18}O, and a measurement for ^{48}Ca would appear to be feasible at present. According to this table it may be possible to measure the double analog in heavy nuclei with modest improvements in the statistics.

According to Table IV there are large corrections to the SCX cross section arising from the use of realistic nuclear densities. The same is true for DCX, but the effects are larger due to the fact that the $\Delta\rho/\rho$ term in Eq. (74a) is raised to the fourth power. In Table V we show the corresponding DCX enhancement factor for several heavy nuclei evaluated at $T_\pi = 180$ MeV. The effect of using realistic densities is particularly pronounced in the case of ^{120}Sn.

We have treated DCX from ^{18}O as a special case because of the existence of an angular distribution at $T_\pi = 164$ MeV. In Fig. 4 we show the experimental angular distribution.[2b,2e] The value of $\bar{R}$ is determined from Eq. (77), with the experimental total cross section taken from Ref. 15, and the a_ρ, a_ρ/a_Δ, and $\Delta\rho/\rho$ evaluated according to the densities of Ref. 14b. [The quantity A in Eqs. (29) and (30) increases the cross section by 20% and was included in the calculation.]

The theoretical curve corresponds to Eq. (74) assuming that the neutrons and protons have the *same* diffuseness in the surface (i.e., $a_\rho/a_\Delta = 1 = \kappa_1$). The mark at $\sigma(0) = 0.31$ μb is the forward cross section, assuming that a_ρ/a_Δ and κ_1 are calculated according to the model of Ref. 14b. Its shape is essentially unchanged from the curve shown. The parameters in the geometrical limit for the two cases are shown in Table VI. We found some difficulty in obtaining a_Δ due to the fact that the effective density $\Delta\rho + \nabla^2\Delta\rho/(2k^2)$ from which a_Δ is calculated according to Eq. (55b) peaks in the vicinity of $\bar{R}$. In this case we have taken a_Δ as the distance over which $\Delta\rho(r)$ falls by $1/e$ rather than evaluating it according to Eq. (55b). Because the actual density for ^{18}O does not fit

TABLE IV. Experimental and theoretical SCX angular distributions at 0°. The incident pion energy is 180 MeV.

Target	σ^{th}(realistic ρ)	Enhancement	σ(realistic ρ)/$\sigma(^{90}$Zr)
^{7}Li	1.78	1.20	6.6
^{13}C	0.56	1.72	0.48
^{58}Ni	0.128	1.78	0.43
^{90}Zr	0.271	1.91	1.0
^{120}Sn	0.810	4.05	3
^{208}Pb	0.530	3.22	2

TABLE V. A dependence for forward scattering to double analog of target.

Target	$\sigma^{DCX}/\sigma(^{18}O)$	$\sigma^{DCX}(0°,$ realistic $\rho)/\sigma^{DCX}(0°,$ simple $\rho)$
^{18}O	1.00	0.99
^{48}Ca	1.06	
^{58}Ni	0.01	1.05
^{90}Zr	0.21	3.22
^{120}Sn	0.34	6.02
^{208}Pb	0.27	6.67
^{238}U	0.26	

very naturally into the set of assumptions under which the theory was derived, we believe that the reduction factor due to different diffuseness of neutrons and protons is probably not very reliable. However, for heavier nuclei these difficulties are not encountered and theory should be more reliable, accordingly. The theoretical angular distributions for $^{18}O(\pi^+,\pi^-)^{18}Ne$ as shown in Refs. 2 and 16 are similar in shape to ours. The two limits which we have considered span the model dependence in these works. The minimum in our theoretical curve gets filled in by the real part of the amplitude, which consists of contributions from S and P waves. We have omitted these terms as discussed earlier.

The most puzzling feature of the $^{18}O(\pi^+,\pi^-)^{18}Ne$ double analog angular distribution is the minimum in the angular distribution, which occurs in the experiment at about 20°, and which occurs in theories at about 40°. From an inspection of Eq. (74) and Table VI, we see that the theoretical curve is dominated by the $J_0(q\bar{R})$ term in the angular distribution. The term $J_0(q\bar{R}) - q\bar{R}J_1(q\bar{R})$ has its zero at about 15°. Within the context of the present theory what is needed in order to reproduce the location of the first minimum is an effect which provides a coherent superposition of these two terms with $J_0(qR)$ having the smaller coefficient. The main effects which have been omitted from this investigation are the ρ^2 terms in the optical potential. Our calculations show that the greatest sensitivity of the DCX scattering to the ρ^2 terms occurs in the coefficient of J_0^{4a}, and that this coefficient is affected most strongly by the isotensor piece of the optical potential. The important implication of this is that the position of the minimum in the cross section of the $^{18}O(\pi^+,\pi^-)^{18}Ne$ analog transition may be an essentially direct measure of the second order optical potential.[17] The ρ^2 terms are also presumably responsible for a large part of the disagreement in the SCX comparisons. The details of this calculation will be presented in a forthcoming paper on the role of ρ^2 terms in pion charge exchange.

VI. SUMMARY AND DISCUSSION

We have derived analytical expressions for pion single and double charge exchange. To lowest order in a systematic expansion the results are given in Eqs. (73a) (single charge exchange) and (74a) (double charge exchange). These expressions are applicable in energy regions for which the nuclear scattering is diffractive. The expressions are intended to accurately reproduce the charge exchange corresponding to the Laplacian model for the lowest order optical potential. In order to arrive at expressions corresponding to the Kisslinger model[8] it is necessary to evaluate corrections arising from second order terms in the optical potential. A careful treatment of higher order terms in U will be given in a separate paper utilizing the framework established in Secs. I–IV of the present paper.

As the development in this paper assumes that the dynamics of the charge exchange is governed by the lowest order optical potential, the strength of which may be deduced from free pion-nucleon scattering, the variation of the cross section as a function of A is determined by the nuclear geometry, i.e., the diffuseness parameters, the radius $\bar{R}$, and the ratio $\Delta\rho/\rho$ of the density of the neutron excess to the total density evaluated at $\bar{R}$. Our results show explicitly the dependence of the angular distribution on this geometry. The

TABLE VI. Parameters for the geometrical model of ^{18}O at $T_\pi = 164$ MeV.

$\bar{R}$ (fm)	a_ρ (fm)	a_ρ/a_Δ	κ_1	$\rho/\rho_0(\bar{R})$	$\Delta\rho/\rho(\bar{R})$	$\frac{\kappa_1^2}{4}(3-a_\rho/a_\Delta)^2\frac{a_\rho}{\bar{R}}$	$(a_\rho/a_\Delta)\kappa_1^2(2-a_\rho/a_\Delta)$
3.50	0.969	1	1	0.184	0.208	0.276	1
3.50	0.969	0.781	0.580	0.184	0.208	0.115	0.320

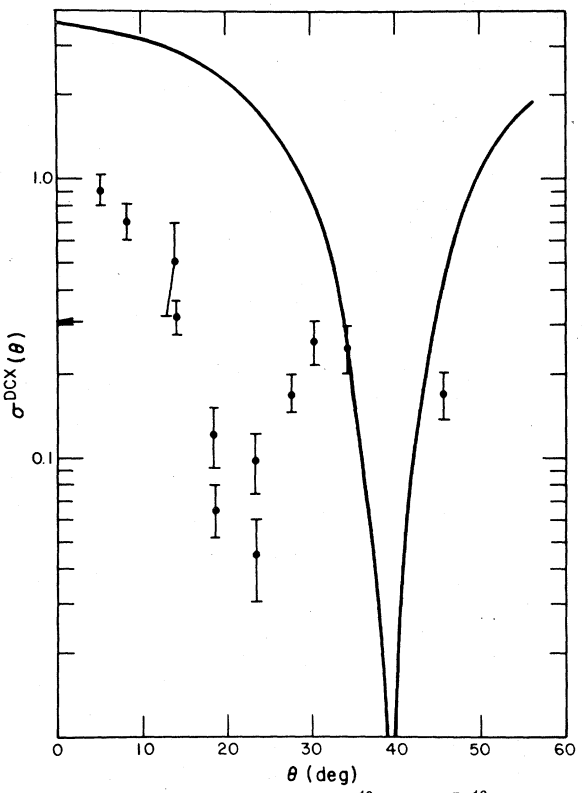

FIG. 4. Angular distribution for $^{18}O(\pi^+,\pi^-)^{18}Ne$ at $T_\pi = 164$ MeV in μb/sr. The data is from Refs. 1a and 2. The solid curve is the theory without corrections for different diffuseness for $\rho(r)$ and $\Delta\rho(r)$. The mark at 3.1 μb/sr is the value of $\sigma^{DCX}(0)$ when this correction is made.

strong dependence on $\Delta\rho/\rho$ suggests that the SCX and DCX experiments may develop into a sensitive probe of the neutron halo.

Even with the best available densities, the calculations of SCX in the geometrical model underestimate the data by a factor of 2–4. However, the general trend in the A dependence is qualitatively reproduced. There are puzzling discrepancies between the DCX cross section and experiment, particularly regarding the position of the first minimum in the angular distribution. As we have stressed, it is very interesting to understand the nature of the residual disagreement.

It is possible that the disagreement arises from a breakdown of one of the three assumptions stated at the beginning of Sec. V and/or that the nuclear model used is wrong. Isospin violating corrections could arise from Coulomb energy shifts in the pion-nucleus interactions, from the $\pi^\pm - \pi^0$ mass difference, from level shifts due to the energy difference of the nuclear ground state and the analog state, or from differences in the wave functions in the analog and ground states. There is no careful, systematic treatment of all these effects, and the traditional view is that isospin violating terms are small.[5a,7]

It is likewise unlikely that the nuclear model can be in error by enough to account for the discrepancies noted. The nuclear wave functions have been taken from theories in which careful consideration was given to consistency with electron scattering and removal energies of the bound nucleons. However, for some targets considered, the assumptions underlying the nuclear models are less well justified than for others. For example, ^{120}Sn has a closed proton shell but the neutrons do not form a closed shell. It is well known[18] that assuming that the ground state consists of a single Slater determinant of filled single-particle orbitals is not a good approximation for this nucleus. We have estimated this effect to be moderately small at $T_\pi = 180$ MeV. However, in the case of ^{208}Pb, the nucleus is doubly magic and the occupation probabilities of the levels outside the Fermi sea are much closer to zero. Sub-Coulomb pickup experiments have given strong support[19] to the neutron tail in ^{208}Pb for the model of Ref. 14a.

In the present paper, we have made the assumption that the pion-nucleon form factor is zero ranged. We have also made the "factorization approximation" which effectively ignores certain momentum dependent terms in the π-nucleon interaction. One consequence of these additional nonlocalities is to increase the diffuseness of the nuclear surface. This result can be easily seen following the arguments of Ref. 20, where the spreading effect of finite range form factors was shown to be taken into account approximately through a convolution of the nuclear density and the square of the pion-nucleon form factor. An increase in the diffuseness would raise the cross section as can be seen in Eqs. (73) and (74), assuming the effect is the same for ρ and $\Delta\rho$. However, the increase arising from the form factors should be a small effect, since the modern estimates for the range of this interaction suggest a very *short* range.[21] In any case we have determined $\overline{R}$ from experiment, which partly accounts for these terms.

We are thus led to consider the effects of higher order terms in the optical potential. These are very poorly understood, and estimates of their sizes have shown them to have a nonnegligible effect on the charge exchange cross section. The effects calculated include the Pauli exclusion principle,[22,4d] the energy shift of the π-nucleon scattering from the valence nucleon due to the core,[23,25] and the spreading interaction in the isobar-hole model.[26] It is likely that a sizable amount of the discrepancy with experiment comes from

these terms, and that the charge exchange reaction will provide important information about the dynamics of the pion-nucleon interaction in the nuclear medium.

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APPENDIX A: ISOVECTOR AND ISOSCALAR OPTICAL POTENTIAL TERMS LINEAR IN DENSITY

If the optical potential for elastic scattering is known, then the functions u_0, u_1, and u_2 of Eq. (1) may be inferred from the isospin invariance of the strong interaction. For simplicity we shall take the optical potential to be purely P wave and the target to be spinless. Then,

$$U=-\vec{\nabla}\cdot\xi(r)\vec{\nabla}, \tag{A1a}$$

where

$$\xi=\xi_0+\xi_1\vec{\phi}\cdot\vec{\mathrm{T}}+\xi_2(\vec{\phi}\cdot\vec{\mathrm{T}})^2. \tag{A1b}$$

In this appendix the subscript on ξ refers to the term in Eq. (A1b) and not to the total isospin quantum number $\vec{T}$. The relationship between ξ_0, ξ_1, and ξ_2 and the corresponding quantities $\xi^{(j)}$, defined as

$$\xi^{(j)}\equiv\langle\pi^j:T_N,\ T_z=-T_N\,|\,\xi\,|\,\pi^j:T_N,\ T_z=-T_N\rangle, \tag{A1c}$$

where $|\pi^j:T_N,\ T_z=-T_N\rangle$ is the isospin wave function for the target nucleus and incident pion, is

$$\xi_2=\frac{1}{T_N(2T_N-1)}(\xi^{(-)}+\xi^{(+)}-2\xi^{(0)}), \tag{A2a}$$

$$\xi_1=\frac{1}{T_N}(\xi^{(0)}-\xi^{(+)})+T_N\xi_2, \tag{A2b}$$

$$\xi_0=\xi^{(0)}-T_N\xi_2. \tag{A2c}$$

We now want to determine the dependence of ξ_i on the neutron and proton densities. With ρ_n and ρ_p normalized to the neutron and proton numbers we may write $\xi^{(i)}$ as

$$\xi^{(i)}=\alpha_n^{(i)}\rho_n+\alpha_p^{(i)}\rho_p, \tag{A3}$$

where $\alpha_n^{(i)}$ and $\alpha_p^{(i)}$ is a set of six complex numbers independent of N and Z and proportional to the pion-nucleon scattering amplitudes. Invariance under isospin rotations implies constraints among the ξ's. Because the pion and nucleon may couple only to total isospin $\frac{1}{2}$ and $\frac{3}{2}$ it is easy to show that

$$\alpha_n^{(+)}=\alpha_p^{(-)}\ \text{and}\ \alpha_p^{(+)}=\alpha_n^{(-)}. \tag{A4a}$$

Furthermore, because the scattering from an isotopic spin zero target cannot depend on the isospin of the projectile, we have

$$\alpha_n^{(+)}+\alpha_p^{(+)}=\alpha_n^{(-)}+\alpha_p^{(-)}=\alpha_n^{(0)}+\alpha_p^{(0)} \tag{A4b}$$

and

$$\alpha_n^{(0)}=\alpha_p^{(0)}\equiv\lambda_0^{(1)}. \tag{A4c}$$

It now follows from Eqs. (A4a) and (A4c) that

$$\alpha_n^{(-)}+\alpha_n^{(+)}-2\alpha_n^{(0)}=0=\alpha_p^{(-)}+\alpha_p^{(+)}-2\alpha_p^{(0)} \tag{A5}$$

and hence from Eqs. (A3) and (A2a) we find

$$\xi_2=0. \tag{A6}$$

Equation (A4b) implies

$$\alpha_n^{(0)}-\alpha_n^{(+)}=-(\alpha_p^{(0)}-\alpha_p^{(-)}) \tag{A7}$$

which gives according to Eqs. (A2b) and (A6)

$$\xi_1=\frac{1}{T_N}(\alpha_n^{(0)}-\alpha_n^{(+)})(\rho_n-\rho_p)$$

$$\equiv\frac{1}{2T_N}\lambda_1^{(1)}\Delta\rho, \tag{A8}$$

where $\Delta\rho\equiv\rho_n-\rho_p$. Finally, we easily show

$$\xi_0=\lambda_0^{(1)}(\rho_n+\rho_p)\equiv\lambda_0^{(1)}\rho. \tag{A9}$$

Thus, to lowest order in the density the pion-nucleus optical potential becomes in the Kisslinger theory

$$U=-\vec{\nabla}\cdot\left[\lambda_0^{(1)}\rho(r)+\frac{1}{2T_N}\lambda_1^{(1)}\Delta\rho(r)\vec{\phi}\cdot\vec{\mathrm{T}}\right]\vec{\nabla}. \tag{A10}$$

Similar arguments give the form for the S-wave optical potential $\bar{U}$ to be

$$\bar{U}=\bar{\lambda}_0^{(1)}\rho(r)+\frac{1}{2T_N}\bar{\lambda}_1^{(1)}\Delta\rho(r)\vec{\phi}\cdot\vec{\mathrm{T}}, \tag{A11}$$

where $\bar{\lambda}_0^{(1)}$ and $\bar{\lambda}_1^{(1)}$ are the isoscalar and isovector strengths for S waves. A relationship between the λ parameters and the pion-nucleus phase shifts may be found in standard references[5]; we have presented the above derivation of the form of the potential because it easily generalizes to terms quadratic and higher in the density. For the case that the scattering occurs entirely through the (3-3) resonance we have

$$\lambda_0^{(1)}=\lambda_1^{(1)}. \tag{A12}$$

This relationship holds for energies close to $T_\pi=180$ MeV.

The isovector potential [Eq. (A8)] depends on $\rho_n-\rho_p$. For all practical purposes this difference gives the *valence* neutron distribution, since the core neutrons and protons fill identical orbits when the Coulomb interaction V_C is turned off. Even when $V_C\neq 0$, the core orbits are nearly the same and $\Delta\rho$ will continue to be very nearly the valence

neutron distribution. However, for heavy nuclei the Coulomb interaction leads to unbound analog states, and therefore the wave function for the final state is probably not very accurately described in this model.

APPENDIX B: THE LOCAL REPRESENTATION

We are assuming that the optical potential in the channel $\mathcal{T}$ of good total isospin may be written

$$U_{\mathcal{T}} = -\vec{\nabla}\cdot\xi_{\mathcal{T}}(r)\vec{\nabla} + \bar{\xi}_{\mathcal{T}}(r)\,, \tag{B1}$$

where $\xi_{\mathcal{T}}$ is the matrix element of ξ [see Eq. (A1b)] between isospin states of good $\vec{\mathcal{T}} = \vec{T} + \vec{\phi}$, and $\bar{\xi}_{\mathcal{T}}$ is the equivalent matrix element for S waves. Following Ref. 9 we make the substitution [$\psi(r)$ is the wave function of the Klein-Gordon equation with U given in Eq. (B1)]:

$$\psi_{\mathcal{T}}(r) = \frac{1}{[1+\xi_{\mathcal{T}}(r)]^{1/2}}\chi_{\mathcal{T}}(r) \tag{B2}$$

and find that $\chi_{\mathcal{T}}(r)$ satisfies the equation

$$-\nabla^2\chi_{\mathcal{T}}(r) + \frac{1}{1+\xi_{\mathcal{T}}(r)}[k^2\xi_{\mathcal{T}}(r) + \bar{\xi}_{\mathcal{T}}(r) + \tfrac{1}{2}\nabla^2\xi_{\mathcal{T}}(r)]\chi_{\mathcal{T}}(r) - \frac{1}{4}\left[\frac{\xi'_{\mathcal{T}}(r)}{1+\xi_{\mathcal{T}}(r)}\right]^2\chi_{\mathcal{T}}(r) = k^2\chi_{\mathcal{T}}(r)\,. \tag{B3}$$

Expanding Eq. (B3) to second order in $\xi_{\mathcal{T}}(r)$, we find

$$-\nabla^2\chi_{\mathcal{T}}(r) + \{k^2\xi_{\mathcal{T}}(r) - k^2\xi_{\mathcal{T}}{}^2(r) + \bar{\xi}_{\mathcal{T}}(r) - \bar{\xi}_{\mathcal{T}}(r)\xi_{\mathcal{T}}(r) + \tfrac{1}{2}\nabla^2\xi_{\mathcal{T}}(r) - \tfrac{1}{2}\xi_{\mathcal{T}}(r)\nabla^2\xi_{\mathcal{T}}(r) - \tfrac{1}{4}[\xi'_{\mathcal{T}}(r)]^2\}\chi_{\mathcal{T}}(r) = k^2\chi_{\mathcal{T}}(r)\,. \tag{B4}$$

Now, using the identity,

$$\xi\nabla^2\xi = \tfrac{1}{2}\nabla^2\xi^2 - (\xi')^2\,, \tag{B5}$$

we find that the (local) optical potential in Eq. (B4) may be written

$$U_{\mathcal{T}}(r) \equiv k^2[\xi_{\mathcal{T}}(r) - \xi_{\mathcal{T}}{}^2(r)] + \tfrac{1}{2}\nabla^2[\xi_{\mathcal{T}}(r) - \xi_{\mathcal{T}}{}^2(r)] + \bar{\xi}_{\mathcal{T}}(r) - \bar{\xi}_{\mathcal{T}}(r)\xi_{\mathcal{T}}(r) + \tfrac{1}{4}[\nabla^2\xi_{\mathcal{T}}{}^2(r) + [\xi'(r)]^2\}\,. \tag{B6}$$

This expression for U is the local form of the Kisslinger potential to second order in the density. If we omit the ξ^2 terms, then we get what is commonly called the Laplacian model,[27]

$$U\ (\text{Laplacian}) = k^2\xi_{\mathcal{T}}(r) + \xi_{\mathcal{T}}(r) + \tfrac{1}{2}\nabla^2\xi_{\mathcal{T}}(r)\,. \tag{B7}$$

The theory in this paper can easily accommodate the Kisslinger and Laplacian theories, but to see the differences we must calculate through order ρ^2. These terms will be treated in a separate paper.

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