Nuclear vibrations with a zero-range interaction and the multipole condition

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Isoscalar monopole, quadrupole, and octopole states are calculated in closed shell nuclei. A delta interaction is used, its strength determined by the multipole condition, namely that the mean single-particle–single-hole potential energy difference is equal to the corresponding kinetic energy difference. This interaction is used to obtain particle-hole matrix elements appropriate for a random-phase approximation calculation. The strengths given by the different multipole conditions are different but (except for $L=0$) they appear to approach each other as the mass number A becomes large. The monopole mode has already collapsed somewhat before the strength implied by the quadrupole condition is reached. Some of the calculations were repeated using a zero-range Skyrme interaction. We observe a very high degeneracy in our calculation which we are able to explain in terms of L-S coupling and the fact that the particle-hole matrix element of a delta interaction is of the form $\int f_{P_A H_A}(r) f_{P_B H_B}(r) r^2 dr$.

[NUCLEAR STRUCTURE Closed shell nuclei, $0^+, 2^+, 3^-$ $T=0$ states, delta interaction with multipole condition, RPA calculations.]

I. INTRODUCTION

This work may be regarded as a continuation of a previous work[1] in which the delta interaction with the quadrupole condition was introduced. In that work we wanted to study, in as uncluttered a way as possible, the behavior of effective charge with mass number A. We therefore considered a model in which the spin orbit splittings were neglected so that the closed shells become $A=4$, 16, 40, 80, 140, 224, 336, 480.... We added a particle to the closed shell, allowed the particle to polarize the core, and calculated the isoscalar quadrupole effective charge $e^0=\delta e^n+\delta e^p$, i.e., the sum of the effective charge corrections when a valence neutron and valence proton are added to the core. It was noted that in random phase approximation (RPA) calculation e^0 increased from about 0.5 to 1 as A varied from 4 to infinity (we reached infinity by an extrapolation method).

The RPA calculation may be regarded as a coupling of the valence nucleon to the vibrations of the core. In this work we wish to study the vibrations by themselves using the same closed shells as in the previous work. We will consider here not only the isoscalar quadrupole state, but also the monopole, dipole, and octopole states as well. In the next section we will generalize our interaction from the quadrupole to the multipole condition.

Before proceeding, some comments are in order. First, we are not trying to duplicate experimental results. Rather, we define an interaction in a fixed way (multipole condition), rigorously stick to this interaction, and let the consequences be what they may. Although the authors have done, and hope to continue to do, work that is of experimental relevance, we feel that it is sometimes necessary to proceed as we are doing here. By following an experiment too closely it is never clear if one has really explained what is happening or merely parametrized what is happening.

What we do here is define in a sharp way the *zero order picture*, which we feel, strangely enough, has not up to now been done. This will better outline the path that must be taken via correlation rearrangements, etc., in order to reach agreement with experiment.

II. THE MULTIPOLE CONDITION

The delta interaction can be written as $-G(1+xP^\sigma)\delta(\vec{r}_1-\vec{r}_2)$, where P^σ is the spin exchange operator. Here we will be considering isoscalar vibrations in closed shell nuclei—quadrupole, monopole, dipole, and octopole. The mean energies of these states are independent of x. We therefore set $x=0$ and have only one parameter G to determine.

When harmonic oscillator wave functions are used, and we deal with closed shell nuclei, then the vibrations are linear combinations of one-particle–one-hole states. The unperturbed energy in the oscillator model for both the quadrupole and monopole is $2\hbar\omega$, for the dipole $1\hbar\omega$, and for the octopole a mixture of $1\hbar\omega$ and $3\hbar\omega$. In the case of the octopole mode for $A=4$ only $3\hbar\omega$ enters; for $A=16$ it is 70% $3\hbar\omega$ and 30% $1\hbar\omega$, and asymptotically one gets a 50% mixture of $1\hbar\omega$ and $3\hbar\omega$.

The parameter G is chosen for any given L pole so that the mean single-particle–single-hole po-

tential energy splitting is equal to the mean kinetic energy splitting. To obtain the mean we construct the L pole state

$$\psi = N\sum\left\langle 0\left|\sum r^L y^L (\mathrm{ph}^{-1})^L\right\rangle\left|(\mathrm{ph}^{-1})^L\right\rangle,$$

and assign a weighting factor to a given particle hole proportional to $|\langle 0|\sum r^L y^L(\mathrm{ph}^{-1})^L\rangle|^2$. Thus for the quadrupole mode the interaction $-G\delta(\vec{r}_i - \vec{r}_j)$ is chosen so that the mean potential energy difference for the $L=2$ mode is $1\hbar\omega$. The corresponding values for the dipole and monopole modes are $\hbar\omega/2$ and $\hbar\omega$.

The result for the octopole mode is a bit more complicated because the operator $\sum r^3 y^3$ raises $\Delta N = 1$ and $\Delta N = 3$ modes. The relative strengths of the modes are given in Bohr-Mottelson, Vol. II[2] as follows.

The transition strengths for a single completed shell with total quantum number N is

$$\sum_{\nu_N,\nu_{N+1}} |\langle \nu_{N+1}|r^3 y_{3,0}|\nu_N\rangle|^2 = \frac{21}{64\pi}\left(\frac{\hbar}{m\omega_0}\right)^3 N(N+1)(N+2)(N+3)(N+4),$$

$$\sum_{\nu_N,\nu_{N+3}} |\langle \nu_{N+3}|r^3 y_{3,0}|\nu_N\rangle|^2 = \frac{7}{64\pi}\left(\frac{\hbar}{m\omega_0}\right)^3 (N+1)(N+2)(N+3)(N+4)(N+5),$$

where ν_N represents the quantum numbers needed to specify the single particle state in the shell N. For the $\Delta N = 3$ case one has to add 3 terms corresponding to the transitions $\nu_N \to \nu_{N-3}$, $\nu_{N-1} \to \nu_{N+2}$, and $\nu_{N-2} \to \nu_{N+1}$.

Once the parameter G is chosen for a given L and A, we use this value for calculating the particle-hole matrix elements that enter into an RPA calculation of the collective vibrational states. This procedure then *defines* the multipole condition.

Our previous experience suggests that this interaction is not adequate for the monopole mode. A delta interaction does not lead to nuclear saturation. Since the monopole state is explained in terms of the vibration about a stable minimum, achieving saturation is crucial for describing this state. Indeed, we will get the wrong answer for this mode.

One motive for choosing such an interaction, and this was discussed to some extent in our previous work, is that it will hopefully shed some light on the Bohr-Mottelson dynamic self-consistency conditions[2], which the authors use extensively, as does Suzuki,[3] for calculating the energies of vibrational states and polarization charges. We shall see, however, that although the delta interaction leads to results which are in some cases the same as those of Bohr and Mottelson, there are some significant and interesting differences.

III. THE SINGLE PARTICLE SPLITTING—TWO CHOICES

Although we have chosen the *mean* value of the single-particle–single-hole splitting to be a simple multiple of $\hbar\omega$, e.g., $1\hbar\omega$ for quadrupole and monopole $\hbar\omega/2$ for the dipole case, this does not mean that the single particle spectrum obtained with our interaction reproduces the harmonic oscillator spectrum. In particular, with an oscillator all energy levels within a major shell are degenerate. With our delta interaction there is a very large spread in the energies. One example will suffice.

For ^{40}Ca with $\nu = 0.289$, $\hbar\omega = 11.988$, and $G = 336.61$ (octopole case) we get for potential energy in units of MeV:

$(0s, -43.654)$,
$(0p, -34.241)$,
$(1s, -27.353)$, $(0d, -27.838)$,
$(1p, -21.241)$, $(0f, -18.522)$,
$(2s, -20.667)$, $(1d, -17.861)$, $(0g, -12.685)$,
$(2p, -17.448)$, $(1f, -15.195)$, $(0h, -8.366)$.

We have performed the calculations with two different choices of the single particle energies, SPI and SPII. They are defined as follows:

SPI. We use the delta interaction with the multipole condition to calculate the single particle potential energies. The single particle kinetic energies are given by the harmonic oscillator values. Indeed, the single particle energies just presented above correspond to SPI.

SPII. Because the fluctuations in the single particle energies within a major shell are very large using SPI, and this may be unreasonable, here we take the single particle energies to be those of a harmonic oscillator, e.g., for the quadrupole and monopole state, every single-particle–single-hole splitting is taken to be $2\hbar\omega$.

The particle-hole interaction is the same for both of the above cases, SPI and SPII. Note that we have here chosen $\hbar\omega = 41/A^{1/3}$; hence $\nu = 0.9887/A^{1/3}$ and $b = 1/\sqrt{\nu}$.

However, because of the multipole condition we see that every matrix element of the RPA Hamiltonian is proportional to $\hbar\omega$. Hence if the resulting eigenvalues are expressed in units of $\hbar\omega$, the answers are independent of $\hbar\omega$. The coefficients X_k^α and Y_k^α which describe the RPA wave function of the αth state are also independent of $\hbar\omega$. Thus the specific prescription which is used to obtain $\hbar\omega$ is not very important.

The energy weighted sum rule for the $E1$ transition is of course proportional to b^{2L} (b^4 for $L=0$) and hence to $(1/\hbar\omega)^L$. The convention we use for RPA is given in Appendix B. Appendix C contains the sum rules.

IV. EXPLANATION OF HIGH DEGENERACY IN L-S COUPLING

Although a j-j coupling basis was used for these calculations, it should be noted that the interaction we use is central, and we are not introducing a one body spin orbit interaction. This means that the quantum numbers L and S are good. This manifests itself in the fact that most of the states that we obtain in an RPA diagonalization have vanishing $B(EL)$'s to the ground state. These correspond to states with $S\neq 0$.

Furthermore, if we choose the single particle energies to be those of a harmonic oscillator (this was described in more detail earlier as SPII), we find that even fewer states than are permitted by L-S coupling have finite $B(EL)$ transition to ground states.

Furthermore, we observe that these states which have vanishing $B(EL)$'s have eigenvalues which are some multiples of $\hbar\omega$. For the quadrupole states there is a high degeneracy of $2\hbar\omega$; in the octopole case at $1\hbar\omega$ and $3\hbar\omega$. Take for example the $L=2^+$ states in ^{16}O. We find that there are 4 states which are degenerate at exactly $2\hbar\omega$. These states have zero $B(E2)$ strength to the ground state. This degeneracy can be partly explained by looking at the expression for the particle-hole interaction in L-S coupling which is given in Appendix A.

The particle-hole states forming the 2^+ states in ^{16}O can be divided into 3 classes.

Class (1) $L\neq 2$ $S=1$ $(0p^{-1}0f)L=3$ $(0p^{-1}1p)L=1$

Class (2) $L=2$ $S=0$ $(0s^{-1}0d)$ $(0p^{-1}0f)$ $(0p^{-1}1p)$

Class (3) $L=2$ $S=1$ $(0s^{-1}0d)$ $(0p^{-1}0f)$ $(0p^{-1}1p)$

The $3j$ symbol

$$\begin{pmatrix} l_h & l_p & L \\ 0 & 0 & 0 \end{pmatrix}$$

vanishes unless l_h+l_p+L is even. Hence all the matrix elements in class 1 will vanish. This explains two of the four states at $2\hbar\omega$.

We next note that for every matrix element in class 2 that vanishes there must be a corresponding one in class 3 that vanishes. This is because the entire spin dependence is contained in the factor $(1-4\delta_{S,0}\delta_{T,0}+2x\delta_{S,0}-2x\delta_{T,0})$. Since we are setting $x=0$, this factor will be 1 for $S=1$ and -3 for $S=0$. The particle-hole matrix elements in class 2 are -3 times those in class 3.

Thus two of the four degenerate states are from class 1, one is from class 2, and one from class 3. The ones from class 2 and class 3 are expected to have the same radial structure, differing only in the spins. Only class 2 states can have finite $B(E2)$'s. This is because the $E2$ operator $\sum r^2 y_2$ has no spin dependence and therefore cannot connect spin one to spin zero. Thus, in this example we are left with one *nontrivial* degeneracy (class 2) to explain, as well as the vanishing $B(E2)$.

To explain this, we note that the L-S coupling expression for the particle-hole matrix element (given in Appendix A) has the structure

$$\langle [\mathrm{p'h'^{-1}}]^{LST} V [\mathrm{ph^{-1}}]^{LST}\rangle = \int f_{k'}(r) f_k(r) r^2 dr\,,$$

where from here on we use the symbol k to designate ph.

Let us denote the class 2 state at $2\hbar\omega$ by ψ and expand it in terms of particle-hole components (we limit ourselves to TDA states in this discussion).

$$\psi = \sum z_k |k\rangle\,.$$

We expect

$$\langle \psi V_{\mathrm{ph}} \psi\rangle = 0\,.$$

This can be achieved by demanding

$$\sum_k f_k(r) z_k = 0\,.$$

Let us first consider the schematic approximation in which the integral $\int R_{n_1 l_1}(r) R_{n_2 l_2}(r) \times R_{n_3 l_3}(r) R_{n_4 l_4}(r) r^2 dr$ is replaced by a constant.[4] In this case the particle hole matrix element has the structure $\langle k' \mathrm{V}_{\mathrm{ph}} k\rangle = g_{k'} g_k$, where g does not depend on r. We can regard the g_k as elements of a vector of dimension D, where D is the number of particle-hole states for a given LST configuration, e.g., in the above example, ^{16}O $L=2$, $S=0$, $T=0$ we have $D=3$. The quantities z_k are also elements of a vector of dimension D, and the condition $\sum_k z_k g_k = 0$ simply means that the vector $\{z\}$ is orthogonal to the vector $\{g\}$.

There are clearly $(D-1)$ vectors $\{z\}$ which are orthogonal to $\{g\}$. Each of these $(D-1)$ vectors will have a vanishing particle-hole matrix element. Thus the class 2 degeneracy at $2\hbar\omega$ will be $(D-1)$. This will also be the class 3 degeneracy. Thus for the $L=2$, $S=0$, and $T=0$ states in ^{16}O we expect a twofold degeneracy from class 2. This was confirmed by a calculation. We obtain the same degeneracy in class 3.

We now consider the case where $f_k(r)$ is not

approximated by a constant. We note that $f_k(r)$ is proportional to the product of two harmonic oscillator radial wave functions $R_p(r)$, $R_h(r)$. Using the variable $x = r/b$ we note that the product can be written in the form of an exponential times a polynomial

$$f_{ph}(r) = e^{-x^2} X^{\sigma} \sum_{N_{MIN}(ph)}^{N_{MAX}(ph)} a_n(ph) X^{2n},$$

where $\sigma = 0$ or 1. Let N_{MIN} = minimum of all $N_{MIN}(ph)$ and let N_{MAX} = maximum of all $N_{MAX}(ph)$. To ensure that $\sum f_k(r) z_k = 0$ we demand that each coefficient of $X^{2n+\sigma}$ vanishes. That is,

$$\sum a_n(k) z_k = 0 \quad (k \equiv ph).$$

This leads to Δ conditions where $\Delta = N_{MAX} - N_{MIN} + 1$. We now have Δ vectors, the nth one of which is $[a_n(1), a_n(2), \ldots, a_n(D)]$. The vector $\{z(k)\}$ has to be orthogonal to all of these. There are clearly $(D - \Delta)$ such vectors $\{z(k)\}$. Hence the class 2 degeneracy is $D - \Delta$.

In our example ($L = 2$, $S = 0$, and $T = 0$ in ^{16}O) the polynomial has terms in x^2 and x^4. Thus $N_{MIN} = 1$, $N_{MAX} = 2$, and hence $\Delta = 2$. Thus the degeneracy $(D - \Delta)$ is *one*.

In a major shell n, l, N ($N = 2n + l$) the lowest power that appears in a polynomial is x^l; the highest power is x^N. Thus if the particle is the shell n_p, l_p, N_p, and the hole is the shell n_h, l_h, N_h, then we have

$$N_{MAX}(ph) = \tfrac{1}{2}(N_p + N_h) \text{ if } N_p + N_h \text{ is even}$$
$$= \tfrac{1}{2}(N_p + N_h - 1) \text{ if } N_p + N_h \text{ is odd},$$
$$N_{MIN}(ph) = \tfrac{1}{2}(l_p + l_h) \text{ if } l_p + l_h \text{ is even}$$
$$= \tfrac{1}{2}(l_p + l_h - 1) \text{ if } l_p + l_h \text{ is odd}.$$

Just to give another example, consider the $L = 2$ and $T = 0$ states in ^{40}Ca. There are seventeen states in all, five in class 1, six in class 2, and six in class 3.

The entire degeneracy at $2\hbar\omega$ for the schematic model is 15. Five of the states are from class 1. Since D is 6 for class 2 the degeneracy here is $D - 1 = 5$. It is also 5 for class 3.

For the delta interaction the entire degeneracy at $2\hbar\omega$ is eleven. Five still come from class 1, leaving 3 from class 2 and 3 from class 3. We find that the quantity Δ is equal to three. The class 2 degeneracy is therefore $D - \Delta = 3$, as expected.

It is very easy to show that the $B(EL)$ to ground is zero for these degenerate states.

The $B(EL)$ is proportional to

$$\int r^L \sum_k f_k(r) z_k r^2 dr.$$

Since the integrand is zero, the integral will also be zero. The above is not true when we approximate the radial integral by a constant.

It should be emphasized that while part of the above argument involved spin isospin symmetry (as discussed many years ago by de Shalit and Walecka[5]), the crucial part involving class 2 degeneracies did not.

It is also worthwhile noting that for L-S coupling the expression for the particle-hole matrix element the spin-isospin factor is very simple $(1 - 4\delta_{S,0}\delta_{T,0} + 2x\delta_{S,0} - 2x\delta_{T,0})$. The values of this factor for the four different modes are -3 for $S = 0, T = 0$; $(1 - 2x)$ for $S = 1, T = 0$; $(1 + 2x)$ for $S = 0, T = 1$; and 1 for $S = 1, T = 1$.

Note that even if there is no spin dependence, i.e., $x = 0$, the $S = 0, T = 0$ mode comes down low and is well separated from the other modes. This affords a nice concrete example of the ideas of de Shalit and Walecka.[5]

V. THE STRENGTH G

We have obtained the strength G by applying the multipole condition. The values of G for the various closed shells and various multipoles are listed in Table I.

It should be noted that for large A, say $A = 336$, it appears that the parameters for the quadrupole and octopole conditions are nearly the same. This may also be true for the dipole mode. Although we cannot be completely certain, it may be that the strengths for these different multipoles approach each other as A goes to infinity.

TABLE I. Strength for various multipoles (in units of MeVfm^3).

L \ A	4	16	40	80	140	224	336	480
0	738.60	502.57	447.50	425.30	413.22	403.69	400.61	397.58
1	274.12	293.47	298.81	304.50	305.57	306.58	307.72	309.81
2	366.24	330.68	317.73	312.79	309.57	306.04	305.98	305.35
3	471.14	370.15	336.61	323.33	314.60	308.87	306.20	

VI. THE QUADRUPOLE MODE

We recall that the main motivation for using the interaction in this work came from the quadrupole mode. As mentioned in our previous work, a deformed oscillator variational calculation with a delta interaction yields the result that the potential energy is independent of deformation. This further yields the Mottelson condition[6]

$$\frac{b_z^2}{b_x^2}=\frac{2\sum_z}{\sum_x+\sum_y},$$

where $\sum_z$ is the sum $(Nz+\frac{1}{2})$ with Nz the number of quanta in the z direction for a given state.

When the Mottelson conditions are linearized, one gets the simple result that when a nucleon is added to a closed shell the core deforms so that the mass quadrupole moment of the core is equal to the quadrupole moment of the valence nucleon. Thus the isoscalar polarization charge, defined as $Q_{\mathrm{CORE}}/Q_{\mathrm{VALENCE}}$ is equal to unity.

In this work, where we examine the vibrations themselves, we wish to test another result--that a suitably weighted mean energy of the isoscalar quadrupole state should equal $\sqrt{2}\,\hbar\omega$.

From the results in Table II we have obtained parametric expressions for the mean energy E_1/E_0 and for the quantity $(E_3/E_1)^{1/2}$. It was shown by Bohr and Mottelson[2] and Suzuki,[3] using dynamic self-consistency (or what is equivalent, the use of a separable quadrupole-quadrupole interaction), and by Golin and Zamick,[7] using zero range Skyrme interactions and scaling arguments, that the energy of the isoscalar quadrupole state is $\sqrt{2}\,\hbar\omega$. It was then shown by Lane *et al.*[8] that the scaling expression corresponds to $(E_3/E_1)^{1/2}$.

We express the results in units of $\hbar\omega$:

$$E_1/E_0=\eta\hbar\omega,\quad (E_3/E_1)^{1/2}=\eta'\hbar\omega\,.$$

We obtain for SPI

$$\eta=1.496+0.028/j+0.27/j^2,$$

$$\eta'=1.499+0.051/j+0.23/j^2,$$

and for SPII

$$\eta=1.484-0.137/j+0.544/j^2,$$

$$\eta'=1.507+0.089/j+0.154/j^2.$$

We see that with both SPI and SPII the results for η' are close but somewhat higher than the

TABLE II. Eigenvalues with nonzero strengths of $J^\pi=2^+, T=0$ states.

	SPI[a]			SPII[a]		
A	Eigenvalues (MeV)	$B(E2)$ (fm^4)		Eigenvalues (MeV)	$B(E2)$ (fm^4)	
4	42.22	3.76	S[b]$=158.86$ E[c]$=42.22$	42.22	3.76	$S=158.86$ $E=42.22$
16	35.80 25.35 18.61	0.22 58.44 1.48	$S=1\,516.70$ $E=25.22$	28.42 20.74	31.14 30.47	$S=1\,516.72$ $E=24.62$
40	25.80 22.39 19.86 18.42 17.24 13.31	0.94 0.40 0.02 361.8 1.25 10.10	$S=6\,853.20$ $E=18.30$	21.85 19.37 14.94	77.57 132.76 173.23	$S=6853.19$ $E=17.87$
80	20.36 19.22 18.24 17.11 16.48 14.54 14.29 12.77 10.66	2.27 0.72 0.05 1.03 2.12 1409.20 37.33 2.72 46.19	$S=21\,658.00$ $E=14.42$	17.72 16.29 15.47 11.81	178.09 9.29 662.07 686.61	$S=21\,658.07$ $E=14.10$

[a] $\nu=0.9887/A^{1/3}\ \mathrm{fm}^{-2}$, $\hbar\omega=41/A^{1/3}$ MeV.
[b] S is energy weighted sum in units of MeV fm^4.
[c] E is mean energy in units of MeV.

collective result $\sqrt{2}\hbar\omega$.

Concern for this deviation may seem like too fine a point to the reader. However, people are using the mean energy of the quadrupole state to analyze the effective mass. When effects of finite range are taken into account the collective expression for the value $(E_3/E_1)^{1/2}$ becomes $\sqrt{2}\hbar\omega(m^*/m)^{1/2}$ (see also the generalization by Kohno and Ando).[9] The delta interaction used here should correspond to an effective mass of one.

It is not clear why we do not get the collective result exactly. We will see later that results of Golin,[10] using a density dependent zero range interaction, are much closer to $\sqrt{2}\hbar\omega$.

A striking feature with SPI is that well below the main strength there is an $L=2$ state with a small $B(E2)$ strength. For example, in ^{16}O the strongest state is at 25.35 MeV with a $B(E2)$ value of 58.444 fm^4. At 18.61 MeV there is a state with a $B(E2)$ strength at 1.47 fm^4. At the other extreme $A=480$ the main strength is at 7.94 MeV with $B(E2)=51\,511$ fm^4, but there is a state at 5.99 MeV with $B(E2)=1998$ fm^4.

We call these low lying states quadrupole "pigmy" resonances, in analogy with similar states which were obtained in calculations (whether they are seen or not, experimentally, is another matter) for isovector dipole modes.

When we change from SPI to SPII, the lowest state acquires even more strength, so it is no longer appropriate to call it a pigmy resonance. In fact we now get several states of comparable strength. For example, in ^{16}O we have a state at 28.42 MeV with $B(E2)$ of 31.14 fm^4 and a state at 20.74 MeV with $B(E2)=30.47$ fm^4.

In ^{40}Ca the states at 21.85, 19.37, and 14.94 MeV have $B(E2)$ values of 77.57, 132.76, and 173.23 fm^4, respectively.

These results do not seem to correspond to experiment. The *mean* energy of the quadrupole resonance is quite reasonable, but the strong fragmentation of strength is evidently not there.

We here note that the use of a density dependent interaction can eliminate this strong fragmentation, as is evident from Golin's thesis[10] at Rutgers for ^{16}O using a zero range Skyrme interaction[11]

$$V=-t_0\,\delta(\vec{r}_i-\vec{r}_j)+t_3\delta(\vec{r}_i-\vec{r}_j)\delta(\vec{r}_i-\vec{r}_k).$$

The parameters t_0 and t_3 were chosen to give the correct binding energy and radius of ^{16}O. Golin obtained three states with nonzero strengths; they were at 31.20, 24.93, and 18.00 with strengths being 0.09, 2.14, and 106.78, respectively. She found a mean energy of 18.1458, and $(E_3/E_1)^{1/2}$ as 18.2454 MeV, so $\eta'=1.4197$ (mean energy $=\eta\hbar\omega$, $\eta=1.411\,97$).

Almost all the strength lies in one state at 18 MeV. We note that Golin used $\nu=0.31$ fm^{-2} whereas we are using $\nu=0.39$ fm^{-2}. This leads to some confusion, which can be avoided by stating the results in units of $\hbar\omega$. Golin's strongest state is at 1.4 $\hbar\omega$. For SPI our strongest state is at 1.55 $\hbar\omega$ and the pigmy state is at 1.14 $\hbar\omega$. For SPI we have 2 states of nearly equal strength at 1.75 $\hbar\omega$ and 1.27 $\hbar\omega$.

Despite the fact that density dependence wipes out the pigmy resonance, it might not be imprudent for the experimentalist to look for low lying fragments anyway. The density dependent interactions currently in use are really still very phenomenological; it would be useful to put them to severe experimental tests.

VII. THE MONOPOLE MODE

For the eight nuclei $A=4,\ldots 480$, the lowest monopole state collapses with SPI and SPII for the monopole condition strengths. Even the weaker quadrupole condition strengths lead to a collapse in all but the $A=4$ nuclei. In fact we found that the monopole mode comes to zero at strengths 4.1, 5.3, and 6.3% weaker than the respective quadrupole strengths for $A=16$, 40, and 80 nuclei, respectively.

The use of a density dependent interaction will raise the monopole energy to a high energy. For example, with all Skyrme interactions[11] the energy comes out to be greater than $2\hbar\omega$.[12] We refer the reader to interesting work by Kirson[13] (and similar, unpublished comments by Yoshida[14]).

VIII. THE OCTOPOLE MODE

In Table III we give the results of the eigenvalues and $B(E3)_{0-3}$ values for the 3^- states. We do this for SPI for 5 nuclei and for SPII for 3 nuclei. We list only the states which have finite $B(E3)$ values, i.e., the states with $L=3$ and $S=0$. For SPI the results are complex. For $A=4$ the first 3^- state is at a very high energy 71.76 MeV. This is due to the fact that there is no $\Delta N=1$ component. In startling contrast, for $A=16$ the lowest state has collapsed. This nucleus offers a nice example of Towner's remark[15] that the energy weighted sum rule (EWSR) is violated when there is a zero eigenvalue (see Appendix C). The Lane expression of EWSR yields a value 27 058.59 MeV fm^6, but our calculation gives 21 414.347 MeV fm^6. The lowest state comes at 1.9 MeV in ^{40}Ca, is at 2.054 MeV for $A=80$, and is at 1.95 MeV for $A=140$. We should also mention that for collapsed state and zero energy states we obtained rather strong transition strengths.

The results for SPII were mentioned previously.

TABLE III. Eigenvalues with nonzero strength for $J^{\pi}=3^{-}$, $T=0$ states (using octopole condition).

	SPI			SPII		
A	Eigenvalues (MeV)	$B(E3)$ (fm^6)		Eigenvalues (MeV)	$B(E3)$ (fm^6)	
4	71.76	18.66	S[a] = 1 338.75 E[a] = 71.76	71.76	18.66	S = 1 338.75 E = 71.76
16	54.47 44.27 33.24 collapse	20.75 418.51 52.81	S = 21 414.35 E = 11.83	46.30 41.12 collapse	430.62 41.55	S = 21 646.36 E = 16.85
40	42.48 38.41 32.86 31.91 29.69 25.89 24.16 14.24 8.25 1.9	20.92 201.74 3 342.77 422.61 358.43 8.53 358.90 138.64 171.67 29 886.80	S = 211 671.66 E = 6.06	34.44 32.72 28.25 9.38 collapse	3195.12 1080.50 255.66 547.42	S = 157 744.77 E = 7.56

[a] S is energy weighted sum in units of MeV fm^6.
[b] E is mean energy in units of MeV.

The lowest state for all closed shell nuclei with $A>4$ collapses to zero energy. This means that with our multipole interaction the nuclei are unstable with respect to octopole deformations.

For SPII the octopole condition consistently leads to a collapse. One would, of course, be interested in finding which strengths take the octopole mode to zero value. Interestingly enough, we found that the lowest octopole mode comes to zero energy at a strength which is within 0.5% of the quadrupole condition value. The strengths as determined from the multipole condition differ for different multipoles. Excluding the $L=0$ case they seem to converge towards each other for large A. This suggests that the lowest octopole state comes to zero exactly as A goes to infinity.

IX. THE OCTOPOLE MODE WITH A ZERO RANGE SKYRME INTERACTION

Since we obtain a collapse or near collapse with the delta interaction plus octopole condition, we now repeat the calculations using a zero range Skyrme interaction with parameters chosen to give the correct binding energy and radius. The Skyrme interactions lead to saturation. We can therefore test if saturation is the key element for removing the collapse, as it was in the case of the monopole mode.

The calculations were performed for $A=16$, 40, and 80. We list the parameters, the value of $\hbar\omega$, and the energy of the lowest 3^- state in TDA and RPA, in Table IV.

Note that even with this saturating interaction we get a collapse in ^{40}Ca and ^{80}Zr. The results here are a bit tricky. Had we just looked at ^{16}O we might have concluded that the zero range Skyrme interaction gave reasonable results. Only by systematically exploring heavier nuclei are we able to see that this interaction also leads to difficulties.

By examining the calculations of Blaizot and Gogny,[16] who calculated the octopole states for more general Skyrme interactions, we are led to the conclusion that by introducing repulsive finite range terms one raises the energy of the lowest 3^- state. We can characterize the various Skyrme interactions by the effective mass. A zero range Skyrme corresponds to $m^*/m=1$. The above authors considered SKIII with $m^*/m=0.76$ and SKIV with $m^*/m=0.41$.

In ^{16}O they obtain the lowest 3^- state at 6.77 MeV for SKIII and 8.90 MeV for SKIV. In ^{40}Ca corresponding values are 2.76 and 3.48 MeV.

Their results, together with ours, show that as m^*/m increases the lowest 3^- state comes down. This may be connected in part with the fact that as m^*/m increases the single-particle–single-hole splitting decreases. However, this is not the whole story. We recall that a collective for-

TABLE IV. Lowest octopole mode state with zero range Skyrme interaction.

A	t_0 (MeV fm^3)	t_3 (MeV fm^6)	$\hbar\omega$ (MeV)	E_3 (TDA) (MeV) SPI	E_3 (TDA) (MeV) SPII	E_3 (RPA) (MeV) SPI	E_3 (RPA) (MeV) SPII
16	1085.132	20196.307	13.353	9.854	9.686	8.313	8.144
40	1107.294	19563.724	10.608	0.934	4.922	collapse	collapse
80	1121.384	19070.514	8.916	−1.784	2.884	collapse	collapse

mula for the isoscalar quadrupole state yields a result $E_{2^+} = \sqrt{2}\hbar\omega/\sqrt{m^*/m}$. This energy also decreases as m^*/m increases, but not in the same way as the single particle splittings decrease.

It would be nice to be able to develop an analytical formula for the dependence of the lowest 3^- state on effective mass, but we have up to now not been able to do so.

X. SUMMARY AND CONCLUSIONS

In this and previous work we have made a careful study of the application of a delta interaction to study the vibrational states of closed shell nuclei and the $E2$ effective charges of closed shell plus one nuclei. Rather than adopting an empirical approach we chose the strength of the delta interaction by the multipole condition. Part of our motivation was to see if we could realize some of the results of Bohr and Mottelson[2] with an explicit two body interaction. We are also interested in comparing our results with the Skyrme interaction. Which of the many results obtained with such an interaction depend upon the fact that one has achieved saturation? One cannot obtain saturation with the delta interaction that is used here.

In the previous work we found that the isoscalar $E2$ polarization charge varied from about 0.5 for mass to about 1 as the mass number A went to infinity. Thus only asymptotically did it agree with the Bohr and Mottelson result.

Since a delta interaction has no velocity dependence it should bear some similarity to the zero range Skyrme interaction (i.e., where $t_1 = t_2 = 0$). There are some similarities and some differences.

The fact that the mean energy of the quadrupole state [or more precisely $(E_3/E_1)^{1/2}$] comes close to $\sqrt{2}\hbar\omega$ with a delta interaction whose strength is chosen by the quadrupole condition shows the similarity with Skyrme as well as with the results of Bohr and Mottelson. However, we found a fragmentation of the $B(E2)$ strength not present with a Skyrme interaction.

Also, for reasons which are not yet understood, the zero range Skyrme interaction seems to yield a very large $E2$ isoscalar polarization charge (~ 1.7 in ^{40}Ca). This is much larger than the value of unity obtained with the Mottelson conditions, or the value of 0.9 obtained with the delta interaction.

As expected, with a delta interaction the monopole mode undergoes a collapse. This occurs with a strength even less than the quadrupole condition. With a saturating Skyrme interaction the monopole mode comes at a high energy ($>2\hbar\omega$).

However, for other modes we have shown in some sense that saturation is not crucial. When we apply the multipole condition we find that every matrix element in the RPA Hamiltonian is proportional to $\hbar\omega$ so that the eigenvectors are independent of $\hbar\omega$ and the eigenenergies are proportional to $\hbar\omega$. The isoscalar effective charge is independent of $\hbar\omega$ in this model.

We found that somewhere between the quadrupole condition and the octopole condition there was a collapse of the octopole mode to zero energy. Apparently these two conditions approach each other for large A, so we expect that the collapse will be more sharply defined at large A. That this should occur asymptotically was noted by Bohr and Mottelson, and this stimulated our investigation.

We find that the octopole state has collapsed or is near collapse (this depends somewhat on how one chooses the single particle energies, i.e., SPI or SPII, and which condition one chooses, quadrupole or octopole) for all nuclei with $A \geq 16$. We found the same result with a zero range Skyrme interaction for ^{40}Ca and ^{80}Zr (but, surprisingly, not for ^{16}O).

It would therefore appear that the use of an effective mass less than unity is crucial to obtaining the octopole mode, and possibly higher negative parity multipoles, at the right energy. This point is relevant not only for 3^- vibrations but also for states built out of these vibrations. For example, Feshbach and Iachello[17] described the first excited 0^+ state in ^{16}O at 6.05 MeV in terms of four 3^- vibrations. How does the energy of the 6.05 MeV state depend on effective mass?

This point then becomes important in the context of recent work of Brown, Dehesa, and Speth.[8]

They argue that although the empirical single particle energies near the Fermi surface are consistent with an effective mass of *unity*, in doing calculations of collective states one should use bare single particle energies corresponding to an effective mass less than one, i.e., about 0.7.

As a bonus, we discovered some very interesting properties of the delta interaction when used in conjunction with oscillator single particle states. Many of the eigenfunctions had vanishing particle-hole matrix elements; i.e., the corresponding eigenvalues coincided with the single-particle-single-hole splittings. These states had vanishing $B(EL)$'s to the ground state. We were able to explain this in terms of the structure of the particle-hole matrix elements of a delta interaction in L-S coupling.

In summary, we feel that we have shed some light on several problems. At the same time we have clearly indicated that there are some remaining problems which we, and we hope others, will attempt to resolve.

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APPENDIX A

Particle-particle and particle-hole matrix elements for the interaction

$$V=-G(1+xP^{\sigma})\delta(\vec{r}_1-\vec{r}_2),\quad P^{\sigma}=\left(\frac{1+\vec{\sigma}_1\cdot\vec{\sigma}_2}{2}\right)$$

in jj and LS coupling are as follows:

jj coupling.

$$\langle[j_1j_2]^{JT}V[j_3j_4]^{JT}\rangle=-\frac{G}{2}\mathcal{R}[1+x(-1)^T][(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)]^{1/2}$$
$$\times\left\{[1+(-1)^T]\begin{pmatrix}j_1 & j_2 & J\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}\begin{pmatrix}j_3 & j_4 & J\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}\right.$$
$$\left.+(-1)^{j_2+j_4}[(-1)^{l_2+l_3+J+T}-(-1)^{l_1+l_3}]\begin{pmatrix}j_1 & j_2 & J\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}j_3 & j_4 & J\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}\right\},$$

$$\langle[j_{p_1}j_{h_2}^{-1}]^{JT}V[j_{p_2}j_{h_2}^{-1}]^{JT}\rangle=\frac{G}{2}\mathcal{R}[(2j_{p_1}+1)(2j_{h_1}+1)(2j_{p_2}+1)(2j_{h_2}+1)]^{1/2}$$
$$\times\left\{(-1)^{j_{h_1}+j_{h_2}+1}\begin{pmatrix}j_{p_1} & j_{h_1} & J\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}j_{p_2} & j_{h_2} & J\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}\right.$$
$$\times[(1+x)(1-2\delta_{T,0})+x(-1)^{l_{p_1}+l_{h_1}+J}-2\delta_{T,0}(-1)^{l_{p_1}+l_{h_1}+J}]$$
$$\left.+(-1)^{l_{h_1}+l_{h_2}}\begin{pmatrix}j_{p_1} & j_{h_1} & J\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}\begin{pmatrix}j_{p_2} & j_{h_2} & J\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}(1-2x\delta_{T,0})\right\}.$$

LS coupling.

$$\langle[l_1l_2]^{LST}V[l_3l_4]^{LST}\rangle=-G\mathcal{R}[(2l_1+1)(2l_2+1)(2l_3+1)(2l_4+1)]^{1/2}[1-x(-1)^S][1-(-1)^{S+T}]\begin{pmatrix}l_1 & l_2 & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_3 & l_4 & L\\ 0 & 0 & 0\end{pmatrix},$$

$$\langle[l_{p_1}l_{h_1}^{-1}]^{LST}V[l_{p_2}l_{h_2}^{-1}]^{LST}\rangle=G\mathcal{R}[(2l_{p_1}+1)(2l_{h_1}+1)(2l_{p_2}+1)(2l_{h_2}+1)]^{1/2}$$
$$\times(-1)^{l_{h_1}+l_{h_2}}\begin{pmatrix}l_{h_1} & l_{p_1} & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_{h_2} & l_{p_2} & L\\ 0 & 0 & 0\end{pmatrix}(1-4\delta_{S,0}\delta_{T,0}+2x\delta_{S,0}-2x\delta_{T,0}),$$

where

$$\mathcal{R}=\frac{1}{4\pi}\int \mathcal{R}_{n_1 l_1}(r)R_{n_2 l_2}(r)R_{n_3 l_3}(r)R_{n_4 l_4}(r)r^2dr\,.$$

APPENDIX B

For the RPA equations we use the convention of Goswami and Pal.[19] The unperturbed hole-particle state is denoted by $|k\rangle$:

$$|k\rangle=[\mathrm{h}^{-1}\mathrm{p}]^{JT}_{MM_T}\,,$$

also to clarify the notation

$$|\bar{k}\rangle=[\mathrm{h}^{-1}\mathrm{p}]^{JT}_{-M-M_T}\,,$$
$$|k^\dagger\rangle=(-1)^{\mathrm{h}+\mathrm{p}-J+1-T}[\mathrm{p}^{-1}\mathrm{h}]^{JT}_{MM_T}\,.$$

The RPA equation is given as

$$\begin{bmatrix}A & B\\ -B & -A\end{bmatrix}\begin{bmatrix}x_k\\ y_k\end{bmatrix}=E\begin{bmatrix}x_k\\ y_k\end{bmatrix},$$

where the matrix elements are

$$A_{kk'}=\langle k|V|k'\rangle+(\epsilon_{\mathrm{p}}-\epsilon_{\mathrm{h}})\delta_{kk'},$$
$$B_{kk'}=\langle k|V|k'^\dagger\rangle\,.$$

The operator $Q^\dagger$ which creates the excited state $|\psi_{JT}\rangle$ when it acts on the ground state $|\psi_0\rangle$ is

$$Q^\dagger=\sum_k[x_kA^\dagger_k-(-1)^{J-M+T-M_T}y_kA_{\bar{k}}]\,.$$

Then the electric multipole transition matrix element (for a $J=0$, $T=0$ ground state) is

$$\langle\psi_{JT}|O^{L\tau}|\psi_0\rangle$$
$$=\delta_{LJ}\delta_{T\tau}\sum_{\mathrm{ph}}[x^*_k\langle[\mathrm{h}^{-1}\mathrm{p}]^{JT}|O^{L\tau}|0\rangle$$
$$+(-1)^{J+T}y^*_k\langle[\mathrm{h}^{-1}\mathrm{p}]^{JT}|O^{L\tau}|0\rangle],$$

where

$$\langle[\mathrm{h}^{-1}\mathrm{p}]^L|O^{L\tau}|0\rangle$$
$$=\left[\frac{2(2\mathrm{p}+1)}{(2L+1)(2T+1)}\right]^{1/2}\langle\mathrm{p}|r^L|\mathrm{h}\rangle\langle\mathrm{p}[y^{L\tau}(\theta,\phi)\mathrm{h}]^{\mathrm{p}}\rangle\,.$$

APPENDIX C

The energy weighted sum rule is given as

$$S^{TL}_{\mathrm{EW}}=\sum_n(E_n-E_0)|\langle n|O^{LT}|0\rangle|^2\,.$$

For $T=0$ we have the following results:

$$S^{T=0,L=0}_{\mathrm{EW}}=\frac{2\hbar^2}{m}A\langle r^2\rangle\,,$$
$$S^{T=0,L=1}_{\mathrm{EW}}=0\,,$$
$$S^{T=0,L\geq 2}_{\mathrm{EW}}=\frac{\hbar^2}{8\pi m}L(2L+1)A\langle r^{2L-2}\rangle\,.$$

Note that the energy weighted sum rule is not satisfied for the cases where the lowest eigenvalue comes out at zero energy. In RPA the closure relation can be written as

$$\sum_{k>0}\left\{\begin{bmatrix}x_k\\ y_k\end{bmatrix}[x^\dagger_k-y^\dagger_k]-\begin{bmatrix}x^*_k\\ y^*_k\end{bmatrix}[y^{*\dagger}_k-x^{*\dagger}_k]\right\}=\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix},$$

where I is the unit matrix and the summation is over positive energy solutions only. For each zero energy eigenvalue the completeness is destroyed and this then affects the sum rule. So the energy weighted sum rule cannot be satisfied whenever eigenvalues have contributions from zero energy.

A more quantitative discussion of eigenvalues at zero energy and their effects on the energy weighted sum rules has recently been carried out by Lane and Martorell.[20]

[1] A. Abbas and L. Zamick, Phys. Rev. C 21, 738 (1980).
[2] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, Reading, Massachusetts, 1975), Vol. II.
[3] T. Suzuki, Nucl. Phys. A217, 182 (1973); Phys. Rev. C 8, 2111 (1973); Nucl. Phys. A220, 569 (1974).
[4] G. E. Brown and M. Bolsterli, Phys. Rev. Lett. 3, 472 (1959).
[5] A. de Shalit and J. D. Walecka, Phys. Rev. 147, 763 (1966).
[6] B. R. Mottelson, *The Many Body Problem, Les Hauches* (Wiley, New York, 1958).
[7] M. Golin and L. Zamick, Nucl. Phys. A249, 320 (1975).
[8] O. Bohigas, A. M. Lane, and J. Martorell, Phys. Rep. 51, 267 (1979); O. Bohigas, J. Martorell, S. Fallieros, and A. M. Lane, Phys. Lett. 64B, 1 (1976).
[9] M. Kohno and K. Ando, Prog. Theor. Phys. 61, 1065 (1979).
[10] M. Golin, Ph.D. thesis, Rutgers University, 1976.
[11] D. Vautherin and D. M. Brink, Phys. Rev. C 5, 626 (1972).
[12] L. Zamick, Phys. Lett. 45B, 313 (1973).
[13] M. Kirson, Nucl. Phys. A317, 388 (1979).
[14] Shiro Yoshida (private communication).
[15] I. S. Towner, Nucl. Phys. A151, 97 (1970); *A Shell Model Description of Light Nuclei* (Clarendon, Oxford, 1977).
[16] J. P. Blaizot and B. Gogny, Nucl. Phys. A284, 429 (1977).
[17] H. Feshbach and F. Iachello, Ann. Phys. (N.Y.) 84, 211 (1974).
[18] G. E. Brown, J. S. Dehesa, and J. Speth, Nucl. Phys. A330, 290 (1979).
[19] A. Goswami and M. K. Pal, Nucl. Phys. 44, 294 (1963).
[20] A. M. Lane and J. Martorell (unpublished).