Coulomb interaction in multiple scattering theory

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The treatment of the Coulomb interaction in the multiple scattering theories of Kerman-McManus-Thaler and Watson is examined in detail. By neglecting virtual Coulomb excitations, the lowest order Coulomb term in the Watson optical potential is shown to be a convolution of the point Coulomb interaction with the distributed nuclear charge, while the equivalent Kerman-McManus-Thaler Coulomb potential is obtained from an averaged, single-particle Coulombic T matrix. The Kerman-McManus-Thaler Coulomb potential is expressed as the Watson Coulomb term plus additional Coulomb-nuclear and Coulomb-Coulomb cross terms, and the omission of the extra terms in usual Kerman-McManus-Thaler applications leads to negative infinite total reaction cross section predictions and incorrect pure Coulomb scattering limits. Approximations are presented which eliminate these anomalies. Using the two-potential formula, the full projectile-nucleus T matrix is separated into two terms, one resulting from the distributed nuclear charge and the other being a Coulomb distorted nuclear T matrix. It is shown that the error resulting from the omission of the Kerman-McManus-Thaler Coulomb terms is effectively removed when the pure Coulomb T matrix in Kerman-McManus-Thaler is replaced by the analogous quantity in the Watson approach. Using the various approximations, theoretical angular distributions are obtained for 800 MeV $p + {}^{208}$Pb elastic scattering and compared with experimental data.

[NUCLEAR REACTIONS ^{208}Pb(p,p), $E = 0.8$ GeV, Kerman, McManus, and Thaler, and Watson multiple scattering theories, Coulomb correction terms, high momentum transfer.]

I. INTRODUCTION

At medium energies the theoretical description of single particle projectile elastic scattering from a many-body system of nucleons can be formulated within the framework of the nonrelativistic many-body Schrödinger equation. Two popular approaches are the Watson[1] and the Kerman-McManus-Thaler (KMT)[2] multiple scattering formalisms. However, these and other similar presentations contain no explicit treatment of the Coulomb interaction,[1-4] although for the three-body system Coulomb effects have been studied in detail.[5] For practical microscopic analyses of medium energy data the Watson and KMT approaches are most often used, and the Coulomb interaction is generally included according to some arbitrary prescription.[6,7] Until recently,[6] such prescriptions have lead to adequate descriptions of proton-nucleus elastic scattering data.

However, the recent observation[6] of the unphysical behavior of specific KMT results for 800 MeV $p + {}^{208}$Pb elastic scattering at large momentum transfers was recognized to originate from the inadequacy of the particular prescribed treatment of the Coulomb interaction. This observation[6] provides the motivation for the present study concerning the role of the Coulomb interaction in multiple scattering theory.

In Sec. II the Coulomb contribution to the Watson and KMT optical potentials is examined in detail. Although the lowest order Coulomb term in both optical potentials consists of a convolution of the Coulomb interaction with the nuclear charge distribution, it is found that additional Coulomb-nuclear and Coulomb-Coulomb terms are required for the KMT potential to make it equivalent to that of Watson. Omitting these additional Coulomb terms in KMT is shown to lead to negative infinite total reaction cross sections and incorrect pure Coulomb scattering predictions. Additional approximations are presented which eliminate these anomalies. In Sec. III the results of numerical calculations employing the various approximations are compared to experimental data. A summary of results and some conclusions are given in Sec. IV.

II. THEORY

The goal of multiple scattering theory is to express the complicated projectile-nucleus scattering amplitude in terms of elementary projectile-nucleon T matrices.[1,2] In exact formulations these T matrices are complicated many-body operators and approximations are required to permit calculations. For charged projectiles the Coulomb and

nuclear parts of the projectile-nucleon interaction are mixed when forming the T matrices. The projectile-nucleus optical potentials of KMT and Watson which are constructed thus involve products of Coulomb and nuclear dependent parts, even in first order[4] (i.e., no target nucleon correlations). We now focus on the first order KMT and Watson optical potentials and show that a few reasonable approximations allow the Watson potential to be cleanly separated into Coulomb and nuclear terms, and that additional approximations are required to achieve this separation in the KMT optical potential.

A. The Coulomb interaction in the Watson approach

Initially, we consider the projectile-nucleus optical potential approach developed by Watson.[1] In this formulation one seeks a $U^W(E)$ which satisfies[3]

$$PT(E)P = PU^W(E)P + PU^W(E)PG(E)PT(E)P \ , \quad (1)$$

where P projects onto the target nuclear ground state, $PT(E)P$ is the elastic projectile-nucleus amplitude, and $G(E)$ is given by

$$G(E) = (E - h_0 - H_A + i\eta)^{-1} \ . \quad (2)$$

In Eq. (2), E is the parametric energy, h_0 is the projectile kinetic energy operator, H_A is the target nucleus Hamiltonian, and the $i\eta$ represents the usual outgoing wave prescription. In the lowest order Watson optical potential one approximates $PU^W(E)P$ by

$$PU^W(E)P \simeq PU^{W(1)}(E)P = \sum_i P\tau_i(E)P \ , \quad (3)$$

where

$$\tau_i(E) = v_i + v_i G_0(E) Q \tau_i(E), \quad (4)$$

and $G_0(E) = (E - h_0 + i\eta)^{-1}$; $Q = 1 - P$. The N-N interaction v_i can be expressed as a sum of Coulomb and nuclear parts: $v_i = v_i^C + v_i^N$. Because the Coulomb force is both relatively weak and long ranged, the role of intermediate Coulomb excitation is minimized, and a reasonable approximation is to assume

$$Pv_i^C Q = Qv_i^C P = 0 \quad (5)$$

in Eq. (4). With these approximations, $P\tau_i(E)P$ becomes

$$P\tau_i(E)P = Pv_iP + Pv_iQ[G_0^{-1}(E) - Qv_iQ]^{-1}Qv_iP,$$

$$P\tau_i(E)P \simeq Pv_i^CP + Pv_i^NP + Pv_i^NQ[G_0^{-1}(E) - Qv_iQ]^{-1}Qv_i^NP \ ,$$

$$P\tau_i(E)P \simeq Pv_i^CP + \{Pv_i^NP + Pv_i^NQ\,[G_0^{-1}(E) - Qv_i^NQ]^{-1}Qv_i^NP\} + \{Pv_i^NQ[G_0^{-1}(E) - Qv_i^NQ]^{-1}(Qv_i^CQ)$$
$$\times [G_0^{-1}(E) - Qv_i^NQ - Qv_i^CQ]^{-1}Qv_i^NP\}, \quad (6)$$

$$P\tau_i(E)P \simeq Pv_i^CP + [P\tau_i^N(E)P]$$
$$+ \{Pv_i^NQ[G_0^{-1}(E) - Qv_i^NQ]^{-1}$$
$$\times (Qv_i^CQ)[G_0^{-1}(E) - Qv_iQ$$
$$- Qv_i^CQ]^{-1}Qv_i^NP\} \ . \quad (7)$$

We observe that the first term in square brackets in Eq. (7) is the usual nuclear scattering operator of the Watson prescription without the inclusion of Coulombic effects. It is given by

$$\tau_i^N(E) = v_i^N + v_i^N G_0(E) Q \tau_i^N(E) \ . \quad (8)$$

If, additionally, we ignore the term in the curly braces in Eq. (7) we find that $PU^{W(1)}(E)P$ becomes

$$PU^{W(1)}(E)P \simeq PU^{W(0)}(E)P = P\sum_i v_i^C P + P\sum_i \tau_i^N(E)P$$
$$\equiv PU_C^W P + PU_N^{W(1)}(E)P \ . \quad (9)$$

In this approximation a complete separation between the Coulomb and nuclear parts of the Watson optical potential is obtained. The Coulomb part is just the classical static potential due to the observed nuclear charge distribution. The optical potential $U^{W(0)}$ is distinct from $U^{W(1)}$ by the additional restriction that $v_i^C \simeq Pv_i^C P$. The ability of $U^{W(0)}$ to successfully explain experimental elastic scattering data is demonstrated in Sec. IIC. This serves as justification for the simplified form of the first order Watson optical potential given in Eq. (9).

The structure of the approximation under dis-

cussion is most readily apparent if we write the Watson optical potential as[3]

$$U^W(E) = V + VQ(G(E)^{-1} - VQ)^{-1}V \tag{10}$$

with

$$V = V^C + V^N = Av^C + Av^N . \tag{11}$$

If we then impose $PV^CQ = QV^CP = QV^CQ = 0$ on Eq. (10) we immediately obtain

$$PU^W(E)P \simeq PV^CP + P[V^N + V^NQ(G(E)^{-1} - V^NQ)^{-1}V^N]P$$

$$= PU_C^WP + PU_N^W(E)P . \tag{12}$$

This approximation again leads to a separation of the Coulomb and nuclear contributions to the optical potential. If circumstances warrant the approximation [i.e., the "impulse" approximation or $H_A = 0$ in Eq. (2) and no target nucleon correlations]

$$U_N^W(E) \simeq U_N^{W(1)}(E) , \tag{13}$$

we obtain the first order Watson result displayed in Eq. (9).

B. The Coulomb interaction in the KMT approach

The KMT formulation is designed to avoid the double-counting error implicit in the replacement of $\tau_i(E)$ defined in Eq. (4) by

$$t_i(E) = v_i + v_i G_0(E) t_i(E). \tag{14}$$

This is done by finding a pseudo-optical potential $U^K(E)$ such that

$$PT'(E)P = PU^K(E)P + PU^K(E)PG(E)PT'(E)P, \tag{15}$$

where

$$T'(E) = [(A-1)/A]T(E) . \tag{16}$$

In this case the pseudopotential U^K is associated with the Watson optical potential by the relation[3]

$$U^K = [(A-1)/A]U^W + A^{-1}U^WGPU^K , \tag{17}$$

where the explicit energy parameter has been dropped. In the lowest order approximation, $U^W \simeq U^{W(1)} = \sum \tau_i = A\tau$, and Eq. (17) becomes

$$U^K \simeq U^{K(1)} = (A-1)\tau + \tau G_0 PU^{K(1)} = (A-1)(1 - \tau G_0 P)^{-1}\tau$$

$$= (A-1)[1 - (1 - vG_0Q)^{-1}vG_0P]^{-1}$$

$$\times (1 - vG_0Q)^{-1}v$$

$$= (A-1)t . \tag{18}$$

This is the first order KMT approximation and is completely equivalent to the Watson prescription in Eq. (3).

When Coulomb scattering must be taken into account, however, we observe from the nature of Eq. (17) that if $U^{W(1)}$ can be linearly divided into Coulomb and nuclear parts, then $U^{K(1)}$ cannot be similarly separated. Thus, we now seek a method to solve Eqs. (15) and (16) with $U^{K(1)}$ of Eq. (18) when Coulombic effects enter the amplitude t. One approach which uses the two-potential formula to separate Coulomb and nuclear terms is given in Appendix A. Another approach can be obtained through the relation between the pseudo-optical potential U^K and the Watson optical potential U^W given by Eq. (17). Substituting the first order Watson optical potential $U^{W(0)}$ of Eq. (9) into Eq. (17) yields

$$PU^KP \simeq PU^{K(0)}P = [1 - A^{-1}PU^{W(0)}PG_0]^{-1}[(A-1)/A]PU^{W(0)}P$$

$$= \{1 - A^{-1}[PU_C^WP + A(1 + Pt^NPG_0)^{-1}Pt^NP]G_0\}^{-1}[(A-1)/A][PU_C^WP + A(1 + Pt^NPG_0)^{-1}Pt^NP],$$

$$= (A-1)(1 - Pw_cPG_0)^{-1}(Pw_cP + Pt^NP) \equiv (A-1)P\hat{t}P, \tag{19}$$

where

$$Pw_cP = A^{-1}(1 + Pt^NPG_0)PU_C^WP . \tag{20}$$

In obtaining the expressions in Eq. (19) the nuclear T matrix t^N defined by

$$t^N = v^N + v^NG_0t^N \tag{21}$$

is introduced, and the relation between t^N and τ^N,

$$\tau^N = t^N - t^NG_0P\tau^N , \tag{22}$$

is exploited. In the absence of Coulombic effects, Eq. (19) reduces to the familiar lowest order KMT result, viz., $PU^{K(0)}P = (A-1)Pt^NP$. We further note that in the absence of nuclear interactions Eq. (19) becomes

$$PU^{K(0)}P \xrightarrow[t^N \to 0]{} (A-1)(1 - A^{-1}PU_C^WPG_0)^{-1}A^{-1}PU_C^WP$$

$$\equiv (A-1)Pt^CP . \tag{23}$$

Here, Pt^CP is the Coulombic T matrix due to the single particle average Coulomb potential $P\nu_cP \equiv A^{-1}PU_C^WP = A^{-1}P\sum_i v_i^CP$, or

$$Pt^CP = P\nu_cP + P\nu_cPG_0Pt^CP\,. \tag{24}$$

Further manipulation of Eq. (19) yields

$$P\hat{t}P = (1 - Pw_cPG_0)^{-1}(Pw_cP + Pt^NP)$$

$$= Pt^NP + (1 - Pw_cPG_0)^{-1}Pw_cP(1 + G_0Pt^NP) \tag{25a}$$

$$= Pt^CP + (1 - Pw_cPG_0)^{-1}Pt^NP(1 + G_0Pt^CP)\,. \tag{25b}$$

If we are willing to assume that $Pw_cP \simeq P\nu_cP$, based on the argument that the nuclear distortion given by $(1 + Pt^NPG_0)$ is short ranged while the Coulombic force is long ranged, then Eq. (25a) becomes

$$P\hat{t}P \simeq Pt^NP + (1 - P\nu_cPG_0)^{-1}P\nu_cP$$

$$= Pt^NP + Pt^CP\,. \tag{26}$$

The same argument applied to Eq. (25b) gives

$$P\hat{t}P \simeq Pt^CP + (1 + Pt^CPG_0)Pt^NP(1 + G_0Pt^CP), \tag{27}$$

which reduces to Eq. (26) if it is assumed that the long range Coulombic distortions do not significantly alter the nuclear T matrix Pt^NP.

Thus, the lowest order KMT pseudo-optical potential $U^{K(0)}$, which yields a final T matrix equivalent to the lowest order Watson optical potential Eq. (9), contains Coulomb-nuclear interference terms given explicitly in Eq. (25). Additional, reasonable approximations allow one to separate the Coulomb and nuclear parts as in Eq. (26). It is imperative to note, however, that here the lowest order pseudo-optical potential Coulomb term is obtained from the effective, averaged single-particle Coulombic T matrix Pt^CP and *not* from the Coulomb interaction v_i^C, as was the case in Eq. (9). Thus, from Eq. (24) we are compelled to define a Coulombic T matrix of the form

$$t^C = \nu_c + \nu_c G_0 t^C \tag{28}$$

just as we defined a nuclear averaged single-particle T matrix t^N in Eq. (21). Under the assumption $\nu_c \simeq P\nu_cP$, we obtain Eq. (24) from Eq. (28). In contrast one can define τ^C as

$$\tau^C = \nu_c + \nu_c G_0 Q \tau^C \tag{29}$$

and under the assumptions which led to Eq. (9) we obtain

$$P\tau^CP = P\nu_cP + P\nu_cQG_0Q\tau^CP \simeq P\nu_cP\,. \tag{30}$$

Therefore, a proper interpretation of the KMT prescription for Coulomb scattering requires the use of

$$PU_C^{K(0)}P = (A-1)Pt^CP\,, \tag{31}$$

where t^C is obtained from Eq. (28). Using Eq. (28) it follows that

$$U_C^{K(0)} = (A-1)\nu_c + (A-1)\nu_c(G_0^{-1} - \nu_c)^{-1}\nu_c\,. \tag{32}$$

Hence, the use of $(A-1)\nu_c$ for $U_C^{K(0)}$ instead of $(A-1)t^C$ entails a difference, $\Delta U_C^{K(0)} \equiv (A-1)(t^C - \nu_c)$, whose imaginary part is

$$\mathrm{Im}\Delta U_C^{K(0)} = -(A-1)\pi t^C\delta(E - h_0)t^{C\dagger}\,, \tag{33}$$

which contains a familiar singularity because of the infinite range of the Coulomb interaction.[8] Omitting $\Delta U_C^{K(0)}$ will, of course, manifest itself most prominently in any attempt to relate the total reaction cross section to the calculated small angle scattering amplitude. This point will be discussed below.

C. Summary and application to proton elastic scattering at 800 MeV

We now concisely summarize the results of Sec. II A and II B. For clarity we reexpress the lowest order Watson and KMT optical potentials. With $U^{W(0)} = U_C^W + U_N^{W(1)}$ from Eq. (9) and using Eq. (17) we obtain

$$U^{K(0)} = [(A-1)/A]U_C^W + U_N^{K(1)} + A^{-1}$$

$$\times\{[1 - A^{-1}[1 + (A-1)^{-1}U_N^{K(1)}G_0P]U_C^WG_0P\}^{-1}$$

$$\times(U_N^{K(1)}G_0PU_C^W + [1 + (A-1)^{-1}U_N^{K(1)}G_0P]U_C^WG_0P$$

$$\times\{[(A-1)/A]U_C^W + U_N^{K(1)}\})]\,, \tag{34}$$

where

$$U_N^{K(1)} = [(A-1)/A]U_N^{W(1)} + A^{-1}U_N^{W(1)}G_0PU_N^{K(1)}\,. \tag{35}$$

It is to be noted that $U^{K(0)}$ in Eq. (34) is equivalent to that in Eqs. (19) or (25), where we identify

$$U_N^{K(1)} = (A-1)t^N. \tag{36}$$

The lowest order Watson optical potential can be expressed as

$$U^{W(0)} = U_C^W + [A/(A-1)]U_N^{K(1)} - [A/(A-1)^2]U_N^{K(1)}$$

$$\times[1 + (A-1)^{-1}G_0PU_N^{K(1)}]^{-1}G_0PU_N^{K(1)} \tag{37}$$

using Eq. (35). The KMT and Watson optical po-

tentials $U^{K(0)}$ and $U^{W(0)}$ are expressed in terms of U_C^W and $U_N^{K(1)}$ for numerical convenience since the latter terms can be computed more readily than $U_C^{K(0)}$ or $U_N^{W(1)}$. In the limit of no Coulomb interaction, $U_C^W = 0$, Eq. (34) yields

$$\lim_{U_C^W \to 0} U^{K(0)} \to U_N^{K(1)} \tag{38}$$

and in the limit of no nuclear interaction Eq. (34) becomes

$$\lim_{U_N^{K(1)} \to 0} U^{K(0)} \to [(A-1)/A]U_C^W + [(A-1)/A^2]U_C^W G_0 P(1 - A^{-1}U_C^W G_0 P)^{-1} U_C^W \ . \tag{39}$$

This latter expression is just the limit of Eq. (17) with no nuclear forces.

In order to facilitate the discussion that follows, Eqs. (34) and (37) are rewritten in the following abbreviated forms:

$$U^{K(0)} = [(A-1)/A]U_C^W + U_N^{K(1)} + U_{C-N,C}, \tag{40}$$

$$U^{W(0)} = U_C^W + [A/(A-1)]U_N^{K(1)} + U_{N-N} , \tag{41}$$

where the definitions of $U_{C-N,C}$ and U_{N-N} are obvious. Target nucleon correlations have been neglected, the impulse approximation has been imposed and $Pv_i^C Q = Qv_i^C P = Qv_i^C Q = 0$ are assumed in both Eqs. (40) and (41). Both expressions yield equivalent projectile-nucleus scattering amplitudes. From Eq. (34) or Eq. (39) we see that $U^{K(0)}(r)$ does not rapidly approach r^{-1} as $r \to \infty$ so that numerical evaluation of Eq. (40) is complicated by the occurrence of the $U_{C-N,C}$ correction. Similarly U_{N-N} complicates the explicit calculation of $U^{W(0)}$ in Eq. (41).

In order to evaluate Eqs. (1) and (15) using Eqs. (40) and (41), these Lippmann-Schwinger equations are converted to nonlocal Schrödinger equations in the usual way be defining $T'\phi \equiv U^K \Psi'$ and $T\phi \equiv U^W \Psi$, where $G_0^{-1}\phi = 0$. For simplicity we take $U_N^{K(1)}$ and U_C^W to be local potentials in both Eqs. (40) and (41), but $U_{C-N,C}$ and U_{N-N} are allowed to be nonlocal as given by Eqs. (34) and (37). By matching the Schrödinger equation solutions to the appropriate outgoing wave boundary conditions, phase shifts are obtained for all partial waves up to $L_{\max}$. Higher partial waves are assumed to be dominated by the asymptotic (non-nuclear) part of the optical potential. In the Watson approach the full proton-nucleus scattering amplitude is given (ignoring the spin of the projectile) by

$$f(\theta) = f_R + k^{-1} \sum_{l=0}^{L_{\max}} (2l+1) e^{2i\sigma_l} e^{i\delta_l^C} \sin\delta_l^C \, P_l(\cos\theta) \ , \tag{42}$$

where f_R and σ_l are the Rutherford amplitude and Coulomb phase shifts for the charge product Ze^2, k is the projectile-nucleus c.m. wave number, and δ_l^C is the phase shift due to the nuclear potential and the distributed nuclear charge. In the KMT approach, using Eq. (16),

$$f(\theta) = [A/(A-1)]\left[\mathfrak{F}_R + k^{-1} \sum_{l=0}^{L_{\max}} (2l+1) e^{2i\Sigma_l} e^{i\delta_l^{C\prime}} \sin\delta_l^{C\prime} \, P_l(\cos\theta)\right] , \tag{43}$$

where, in general, $\mathfrak{F}_R$ and Σ_l are the scattering amplitude and phase shifts due to the asymptotic part of $U^{K(0)}$ as given by the limit in Eq. (39). If $U_{C-N,C}$ is neglected, then $\mathfrak{F}_R$ and Σ_l could, for example, be the Rutherford amplitude and Coulomb phase shift due to the point charge product $(A-1)Ze^2/A$, or $\mathfrak{F}_R$ and Σ_l could be made equal to the scattering amplitude and the total phase shift resulting from the distributed nuclear charge $[(A-1)/A]U_C^W$ with no nuclear interaction included (see Sec. III). The phase shift due to the short range part of $U^{K(0)}$ is denoted by $\delta_l^{C\prime}$.

Equations (40) and (41) will now be applied to 800 MeV proton elastic scattering from ^{208}Pb where the data have been extended to high momentum transfer, 5.3 fm^{-1}.[6] The effects of $U_{C-N,C}$ and U_{N-N} on the differential cross section will be examined first. The potential $U_N^{K(1)}$ is assumed to be $(A-1)t_{\text{free}}\tilde{\rho}$, where t_{free} is the nuclear part of the free N-N T matrix according to Eq. (21), and $\tilde{\rho}$ is the ground state nuclear density form factor.[7] The Coulomb potential U_C^W is obtained by folding (e/r) with the nuclear charge density. The experimental data, the parameters for t_{free}, the ^{208}Pb matter densities, and further details of the calculations are given in Refs. 6 and 7. For simplicity, the spin dependence of t_{free} is omitted here so that only the spin-independent effects of $U_{C-N,C}$ and U_{N-N} are investigated. Also, because of the high momentum transfer involved, the convergence and numerical accuracy of the computer calculations were carefully checked and verified to be quite adequate.

The calculations of the angular distribution for $p + ^{208}$Pb elastic scattering at 800 MeV using Eqs. (41) and (42) with U_{N-N} omitted is shown by the

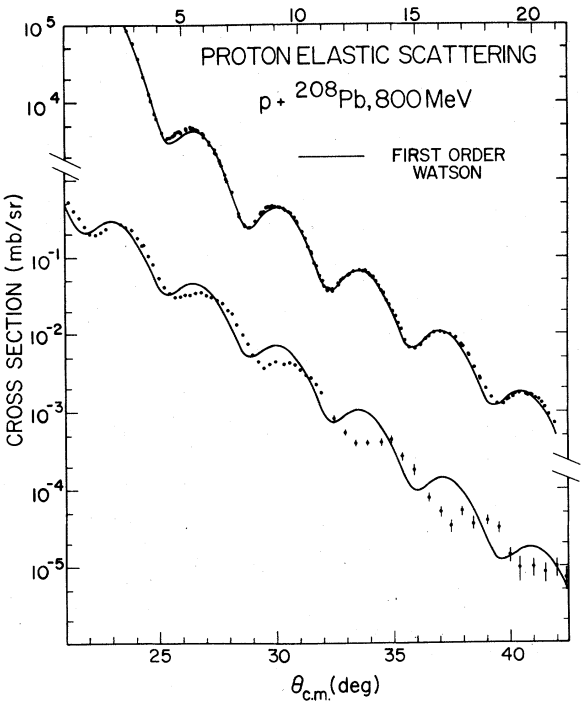

FIG. 1. Predicted and experimental angular distributions for 800 MeV proton elastic scattering from ^{208}Pb. The data are from Ref. 6. The first order Watson calculation (solid curve) corresponds to Eq. (41) with $U_{N-N}=0$ and Eq. (42). The ^{208}Pb neutron matter density has been adjusted to provide the best possible fit to the forward angle data. Note that the angular scale has been divided in two in order to improve the clarity of the figure.

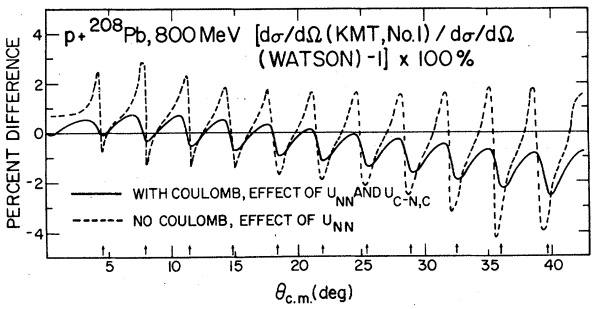

FIG. 2. Percentage difference between the angular distributions of KMT No. 1 and the Watson calculation in Fig. 1 for $p+^{208}$Pb at 800 MeV (solid curve). The same quantity computed with $U_C^W=0$ is indicated by the dashed curve. The solid (dashed) curve thus indicates the effect of U_{N-N} and $U_{C-N,C}$ (U_{N-N} only) on the angular distribution. The arrows denote the angular positions of the diffractive minima in the Watson differential cross section of Fig. 1.

solid curve in Fig. 1. The fit at forward angles is excellent and the smooth diffractive pattern observed over 10 decades in the data is reproduced by the Watson calculation. The theory does however become out of phase with the data at higher momentum transfer. This could be due to omission of any number of additional corrections. Our interest here is in evaluating the effects of $U_{C-N,C}$ and U_{N-N} so these additional, physically important corrections are beyond the scope of the present work.

Since U_{N-N} has been omitted in the calculation of Fig. 1, its effect on the cross section must be ascertained. This is done by setting $U_C^W=0$; then Eqs. (40) and (41) reduce to

$$U^{K(0)}=U_N^{K(1)}\,, \tag{44}$$

$$U^{W(0)}=[A/(A-1)]U_N^{K(1)}+U_{N-N}\,. \tag{45}$$

Calculations were made assuming Eq. (44) and $U^{W(0)}=[A/(A-1)]U_N^{K(1)}$. The difference in the cross sections is therefore due to U_{N-N}. The result is indicated by the dashed curve in Fig. 2, where it is seen that this term affects the angular distribution by only a few percent. The full Watson calculation including U_{N-N} would therefore be very similar to the result shown in Fig. 1. The qualitatively good agreement out to 42° c.m. between experiment and the approximate first order Watson calculation in Fig. 1 partially justifies the assumptions which led to Eq. (9) [i.e., Eq. (5) and ignoring the Qv_i^CQ term in Eq. (7)]. Regarding the discrepancy between theory and data observed at the back angles in Fig. 1, one should reexamine the various approximations leading to Eq. (41) as well as the additional approximations made in the numerical calculation.

In order to ascertain the effect of $U_{C-N,C}$, $U^{K(0)}$ is set equal to $\{[(A-1)/A]U_C^W+U_N^{K(1)}\}$ and the result is compared to the solid curve in Fig. 1. The difference in these two differential cross sections is due to both $U_{C-N,C}$ and U_{N-N}. With $U_{C-N,C}=0$, $U^{K(0)}\to[(A-1)/A]U_C^W$ as $r\to\infty$, and the proton-nucleus scattering amplitude in this approximation, from Eq. (43), is

$$f^{(1)}(\theta)=[A/(A-1)]\Big[f_R'+k^{-1}\sum_{l=0}^{L_{\max}}(2l+1)e^{2i\sigma_l'}e^{i\delta_l^{C\prime}}\sin\delta_l^{C\prime}P_l(\cos\theta)\Big]\,, \tag{46}$$

where f'_R and σ'_l are the Rutherford amplitude and Coulomb phase shifts corresponding to the potential $(A-1)Ze^2/(Ar)$ as explained above. The KMT calculation with $U_{C-N,C}=0$ and $f^{(1)}(\theta)$ computed as in Eq. (46) will be referred to as KMT No. 1. Comparing the cross section prediction using KMT No. 1 to the Watson prediction in Fig. 1 produces the solid curve in Fig. 2. The effect of both U_{N-N} and $U_{C-N,C}$ is seen to be small, about 1–2%. The comparison of the solid and dashed curves convinces us that the effect of $U_{C-N,C}$ alone on the angular distribution is also small, about a few percent.

As mentioned earlier one result of omitting $U_{C-N,C}$ in the KMT approach is that the predicted total reaction cross section obtained from Eq. (46) is negative infinity. This occurs because the infinite total cross section due to $[4\pi A/(A-1)k]\,\mathrm{Im}f'_R(0)$ is not exactly canceled by the infinite elastic total cross section due to $[A/(A-1)]^2\int|f'_R|^2d\Omega$ when evaluating the nuclear total reaction cross section.[8] Omitting $U_{C-N,C}$ disrupts the equality between $U^{K(0)}$ and $U^{W(0)}$, even in the pure Coulombic interaction limit. Because of this the KMT No. 1 approach with $U_N^{K(1)}=0$ will fail to yield the correct scattering amplitude from either a distributed or point charge.

The straightforward way to avoid these difficulties is to include $U_{C-N,C}$ in the KMT calculation and evaluate Eq. (43). The difficulty of actually computing $U_{C-N,C}$ [see Eq. (34)] is readily apparent, and the next section of this paper is devoted to a simpler but less direct approach aimed at accounting for $U_{C-N,C}$. An attempt is made to find some *additional approximation* to the KMT calculation which eliminates the predominant ill effects caused by the omission of $U_{C-N,C}$. The criteria used to select an acceptable additional approximation are that the following be obtained: (1) a finite total reaction cross section, (2) a reasonable reproduction of the Watson differential cross section in Fig. 1, (3) the correct $U_N^{K(1)}=0$ limit, and (4) the point charge, Rutherford scattering limit.

III. ADDITIONAL CORRECTIONS FOR THE KMT CALCULATION

The discussion in the previous section points out the unacceptable problems which arise in approximate KMT calculations which omit $U_{C-N,C}$. In this section an attempt is made to find additional approximations which partially cancel the effects produced by the omission of $U_{C-N,C}$. We therefore wish to modify the projectile-nucleus scattering amplitude constructed from the KMT optical potential in Eq. (40) with $U_{C-N,C}=0$ so that it resembles the exact scattering amplitude as closely as possible. The exact result can be obtained either from Eqs. (15), (16), and (40) or from Eqs. (1) and (41) with $U_{C-N,C}$ and U_{N-N} included, respectively. The exact T matrix referred to here makes use of the full optical potentials in Eqs. (40) or (41) and is not meant to imply a complete evaluation of the entire KMT or Watson optical potentials.[1,2] The Watson approach will be used to express the exact T matrix since Eq. (41) contains no Coulomb correction term. The formal development necessary to this section is the subject of Appendix B.

From Eqs. (1) and (9) the exact projectile-nucleus T matrix is given by

$$T^{\text{exact}}=T_{\text{Coul}}+T^{\text{exact}}_{C-N}=T_R+T^C_{\text{dist}}+T^{\text{exact}}_{C-N}\ , \tag{47}$$

where T_R is the T matrix resulting from a point charge product Ze^2, T^C_{dist} is the projectile-nucleus T matrix which arises from the spatial distribution of the nuclear charge, and T_{Coul} is the full projectile-nucleus T matrix corresponding to Coulomb effects acting alone. The Coulomb distorted Watson nuclear T matrix is given by T^{exact}_{C-N}.

We now turn to the KMT approach. Using Eqs. (15), (16), and (40) with $U_{C-N,C}=0$, the projectile-nucleus T matrix becomes

$$\begin{aligned}T(\text{KMT No. 1})&=[A/(A-1)]T'_{\text{Coul}}+T^{\text{KMT}}_{C-N}\\&=T_{\text{Coul}}+T^{CC}+T^{\text{KMT}}_{C-N}\ ,\end{aligned} \tag{48}$$

where T'_{Coul} is the T matrix resulting from the scaled Coulomb potential $[(A-1)/A]U^W_C$, T^{KMT}_{C-N} is the Coulomb distorted nuclear T matrix in KMT, and T^{CC} is defined by

$$T^{CC}\equiv[A/(A-1)]T'_{\text{Coul}}-T_{\text{Coul}}\ . \tag{49}$$

By comparing Eqs. (47) and (48) it is seen that replacing $[A/(A-1)]T'_{\text{Coul}}$ with T_{Coul} (or dropping T^{CC}) in Eq. (48), will yield a projectile-nucleus T matrix which more closely resembles T^{exact}. Omitting T^{CC} in Eq. (48) therefore yields a new prescription for the KMT projectile-nucleus T matrix, which is denoted as KMT No. 3 and is given by

$$T(\text{KMT No. 3})=T_{\text{Coul}}+T^{\text{KMT}}_{C-N}\ . \tag{50}$$

In the separate limits of no nuclear or no Coulomb interactions T^{exact} and T(KMT No. 3) are equivalent.

The omission of $U_{C-N,C}$ leads to T(KMT No. 1) in Eq. (48) which contains the long range Coulomb correction amplitude T^{CC} and a nuclear scattering amplitude which differs from that in T^{exact} by the Coulomb distortions. The term T^{CC} is the cause of the negative infinite reaction cross section and the improper pure Coulomb force limit. By omit-

ting this term to obtain T(KMT No. 3) the anomalies caused by the omission of $U_{C-N,C}$ have been partially corrected. T(KMT No. 3) still differs from T^{exact} by the Coulomb distortion effects on the nuclear amplitude but this remaining difference should be inconsequential compared to the difference caused by T^{CC}.

Although the approximation leading to Eq. (50) appears to be sufficient, it is instructive to discuss a different approximation to T(KMT No. 1).[6,7] This involves replacing $[A/(A-1)]T'_R$ with T_R in Eq. (48), where T'_R corresponds to scattering from the reduced point Coulomb potential $[(A-1)/A]Ze^2/r$. This provides an alternate prescription for removing the long range part of T^{CC} from Eq. (48). Hence, T(KMT No. 1) becomes

$$T(\text{KMT No. 1}) = [A/(A-1)](T'_R + T^{C}_{\text{dist}}{}') + T^{\text{KMT}}_{C-N}$$

$$= T_R + T^{RR} + [A/(A-1)]T^{C}_{\text{dist}}{}' + T^{\text{KMT}}_{C-N}, \quad (51)$$

where $T^{C}_{\text{dist}}{}'$ results from a spatially distributed nuclear charge which is reduced in strength by $(A-1)/A$ and $T^{RR} \equiv [A/(A-1)]T'_R - T_R$. The additional approximation suggested here is to omit T^{RR}, yielding

$$T(\text{KMT No. 2}) = T_R + [A/(A-1)]T^{C}_{\text{dist}}{}' + T^{\text{KMT}}_{C-N}. \quad (52)$$

The residual differences between T(KMT Nos. 1, 2, and 3) and T^{exact}, when expressed as

$$T^{\text{exact}} = T(\text{KMT No. } i) + T_i, \quad (53)$$

are given explicitly in Appendix B. Qualitatively the nature of each T_i is as follows. Each of these three KMT prescriptions differs from the exact result in the Coulomb distortions of the nuclear T matrix. However, T_1 contains the long ranged Coulomb squared term T^{CC} of Eq. (49), T_2 contains a strong, short ranged Coulomb squared term, while T_2 contains no further Coulomb dependent terms. The presence of T^{CC} in T_1 prevents KMT No. 1 from yielding finite reaction cross sections or the correct pure Coulomb scattering limits. The strong, short range nature of T_2 permits one to calculate finite reaction cross sections and to recover the correct Rutherford scattering limit with KMT No. 2; however, the pure Coulomb scattering from a distributed charge is not correctly obtained. Also the strong, short ranged part of T_2 can be expected to produce noticeable effects in differential cross section predictions. Clearly, T(KMT No. 3) best approximates T^{exact}.

The partial wave expansions corresponding to T(KMT Nos. 2 and 3) can be obtained from the analogous series for T(KMT No. 1), Eq. (46), by simply replacing $[A/(A-1)]f'_R$ with f_R and $[A/(A-1)]\times f'_{\text{Coul}}$ with f_{Coul}, respectively, where $f_R(f'_R)$ and $f_{\text{Coul}}(f'_{\text{Coul}})$ are the Rutherford and full Coulomb scattering amplitudes corresponding to the unscaled [scaled down by $(A-1)/A$] nuclear charge.

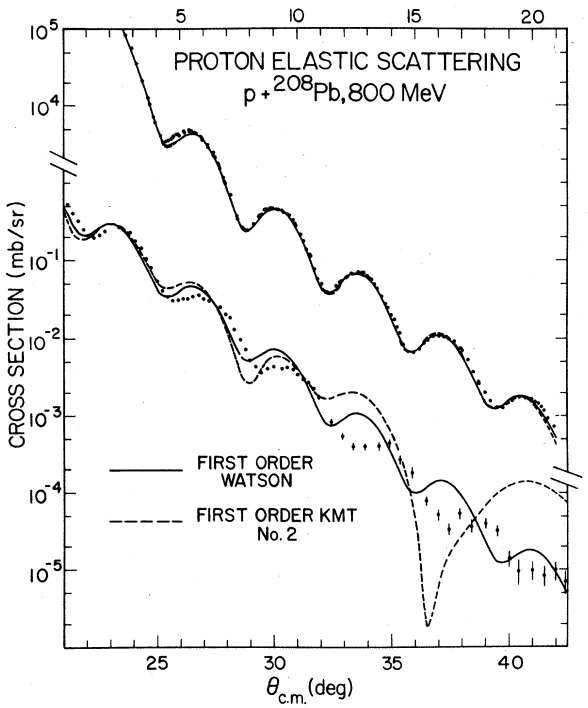

FIG. 3. The 800 MeV $p+{}^{208}$Pb predicted elastic angular distribution of KMT No. 2 (dashed curve) compared with the data and the Watson calculation of Fig. 1 (solid curve).

The angular distributions for $p+{}^{208}$Pb at 800 MeV predicted by KMT Nos. 2 and 3 [see Appendix B, Eqs. (B27) and (B32)], in which the same values of U^W_C and $U^{K(1)}_N$ are assumed as given in Sec. II ($U_{C-N,C}=0$), are shown in Fig. 3 by the dashed curve and in Fig. 4 by the solid curve. The angular distribution prediction of KMT No. 2 is reasonable at forward angles but deteriorates significantly at large momentum transfer near 35° c.m. The KMT No. 3 angular distribution is very similar to the Watson result in Fig. 1, and the difference between the two is shown by the solid curve in Fig. 4. The difference is typically only 1–2%. Recall that the Watson calculation displayed in Fig. 1 omits U_{N-N} so that this curve is not precisely the same as that which would result from T^{exact}. However, from Fig. 2 we see that little difference between the $p+{}^{208}$Pb angular distributions corresponding to $U^{W(0)} = U^W_C + [A/(A-1)]U^{K(1)}_N$ and T^{exact} is expected; hence from Fig. 4 we note that the Coulomb distortion error of KMT No. 3

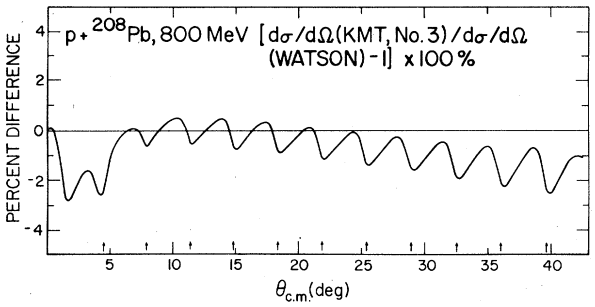

FIG. 4. Percentage difference between the angular distributions of KMT No. 3 and the Watson calculation in Fig. 1 for 800 MeV $p+^{208}$Pb. The solid curve here is indistinguishable from the solid curve in Fig. 2 beyond 10° c.m.; this implies that the long ranged Coulomb squared correction T^{CC} of KMT No. 1 affects the angular distribution only at very forward angles. The arrows indicate the positions of the diffractive minima in the Watson calculation of Fig. 1.

is very small for this particular case. At moderate and back angles > 10° c.m. the angular distribution predicted by KMT No. 3 is indistinguishable from that of KMT No. 1, but recall that this new prescription does not suffer from the defects of KMT No. 1. The KMT calculations Nos. 1 and 3 differ by the long range Coulomb amplitude T^{CC}, Eq. (49) [in Appendix B compare T_1 in Eq. (B22) with T_3 in Eq. (B26)]. Although omitting T^{CC} in KMT No. 3 removes the negative infinite reaction cross section, its effect on the angular distribution is very small and is confined to forward angles (compare the solid curves in Figs. 2 and 4) which is expected because of the long range nature of T^{CC}.

The relative success of each of these KMT methods (with $U_{C-N,C}=0$) in meeting the four criteria listed at the end of Sec. II is given in Table I. In addition, the qualitative nature of the differences between these three KMT scattering amplitudes and the exact amplitude (i.e., T_1, T_2, and T_3) is summarized here also. The long range Coulomb nature of T_1 accounts for the three failures of KMT No. 1 in Table I. Since the infinite Coulomb amplitude in $f^{(2)}$ and $f^{(3)}$ is in the form of an unscaled Rutherford amplitude f_R, KMT Nos. 2 and 3 both yield finite total reaction cross sections. The remaining short range Coulomb correction to KMT No. 2 prevents the $U_N^{K(1)}=0$ limit from being correctly obtained and explains the anomalous structure in the large angle angular distribution. The remaining correction to the Coulomb distortion of the nuclear scattering amplitude is present for KMT Nos. 1, 2, and 3. From Table I, Fig. 4, and the definition of T_3 this remaining Coulomb distortion effect is seen to be of minor importance for this case.

From Table I we see that KMT No. 3 alone meets all four criteria successfully, and hence is the best prescription for canceling the effects due to the omission of $U_{C-N,C}$ in Eq. (40). While this is true for $p+^{208}$Pb at 800 MeV, the success seen here might be merely fortuitous. Tests similar to those conducted here should be carried out on a case by case basis for other applications of KMT at high momentum transfer.

The peculiar KMT result of Ref. 6 may now be explained. The calculation in Ref. 6 is very similar to KMT No. 2 and uses a prescription which is given by[9]

TABLE I. The relative success of KMT Nos. 1, 2, and 3 in meeting various criteria and the qualitative nature of the differences between T(KMT Nos. 1, 2, and 3) and the exact scattering amplitude T^{exact}.

	KMT No.		
	1	2	3
σ_{REAC} finite	No	Yes	Yes
Correct $U_N^{K(1)}=0$ limit	No	No	Yes
Correct point charge limit	No	Yes	Yes
Reasonable $d\sigma/d\Omega$	Yes	No	Yes
Qualitative nature of T_i corrections	Alteration of Coulomb distortion of nuclear amplitude + long range Coulomb squared, moderate short range dependence	Alteration of Coulomb distortion of nuclear amplitude + sizable short range Coulomb squared dependence	Alteration of Coulomb distortion of nuclear amplitude

$$f(\text{Ref. 6}) = f_R + [A/(A-1)]\, k^{-1} \sum_{l=0}^{L_{\max}} (2l+1) e^{2i\sigma_l} e^{i\delta_l^{C\prime\prime}} \sin\delta_l^{C\prime\prime} P_l(\cos\theta) \,, \tag{54}$$

where $\delta_l^{C\prime\prime}$ is the phase shift obtained from matching the Schrödinger equation solution for the optical potential $(U_C^W + U_N^{K(1)})$ to Coulomb wave functions which are based on the point Coulomb potential Ze^2/r. Explicit calculations demonstrate that KMT No. 2 and Eq. (54) yield very similar angular distributions (see Fig. 3) when the same U_C^W and $U_N^{K(1)}$ are used. Thus the physically unreasonable structures in the large angle $p + {}^{208}\text{Pb}$ elastic angular distribution of Ref. 6 and Fig. 3 for KMT No. 2 have a common origin. The problem is due to the omission of $U_{C-N,C}$ in Eq. (40) and an improper cancellation of this effect in going from Eq. (46) to KMT No. 2 or Eq. (54). In particular, the residual short range Coulomb correction in T_2 would have to be added to the amplitudes in KMT No. 2 and Eq. (54) or in Ref. 6 in order to significantly change the back angle cross section. The actual calculation in Ref. 6 includes a spin-dependent, second order KMT optical potential,[7] but these additional refinements do not affect the large angle irregularity.

While KMT No. 3 removes the gross problems which originate from the omission of $U_{C-N,C}$ it is clear from Eq. (53) for T_3 and Fig. 4 that the full $U^{K(0)}$ in Eq. (40) is not being computed exactly. One possible way to insure that T^{exact} is being evaluated accurately is to compute U_{N-N} in Eq. (41) and use the Watson partial wave expansion Eq. (42) to obtain the proton-nucleus scattering amplitude. If one assumes $U_C^W = 0$ such that $U^{K(0)} = U_N^{K(1)}$ then the equivalent Watson potential $U_N^{W(1)}$ from Eq. (17) is

$$(-A^{-1}U_N^{W(1)}) = [-(A-1)^{-1}U_N^{K(1)}] + [-(A-1)^{-1}U_N^{K(1)}]\, G_0 P (-A^{-1}U_N^{W(1)}) \,, \tag{55}$$

where we have multiplied through by $(-A^{-1})$. Equation (55) is of the form $T = V + VG_0PT$, so that a Lippmann-Schwinger or Schrödinger equation solution for the "T matrix" with the "optical potential" $[-(A-1)^{-1}U_N^{K(1)}]$ yields the equivalent Watson potential upon multiplication by $(-A)$. Since $U_N^{W(1)} = [A/(A-1)]U_N^{K(1)} + U_{N-N}$, U_{N-N} has thus been included. The resulting proton-nucleus scattering amplitude can be readily evaluated from $U^{W(0)} = U_C^W + U_N^{W(1)}$ and Eq. (42).

Finally, since analyses which examine nuclear matter density distributions are predominantly affected by the forward angle elastic scattering data ($q \lesssim 2.5 - 3.0$ fm^{-1}),[6,7,10,11] the U_{N-N} and $U_{C-N,C}$ corrections are not significant in such analyses, at least for this particular case. Furthermore, the forward angle calculation and any deduced matter density distributions will be unaffected by the various additional approximations represented by KMT Nos. 2 and 3. The full significance of the Coulomb effects discussed here becomes apparent only for multiple scattering descriptions of charged projectile scattering at high momentum transfer.

IV. SUMMARY AND CONCLUSIONS

The approximations necessary to separate the first order Watson optical potential into Coulomb and nuclear terms have been specified and the equivalent KMT optical potential derived. It was shown that Coulomb-nuclear and Coulomb-Coulomb cross correction terms are obtained in KMT. Although this KMT, Coulomb correction term $U_{C-N,C}$ has a minor effect on angular distributions, its omission results in negative infinite total reaction cross sections. Two further approximations which are motivated by comparing the final proton-nucleus T matrices evaluated with and without $U_{C-N,C}$ are presented. Both avoid the infinite reaction cross section anomaly, but one produces a physically unreasonable angular distribution at high momentum transfer. The prescription which more effectively recovers the full T matrix produces reasonable angular distributions, gives the correct Coulomb scattering limit, recovers the Rutherford scattering amplitude in the extreme limit of a point charge target, and gives a finite total reaction cross section. This prescription KMT No. 3 expresses the full proton-nucleus scattering amplitude as a sum of two amplitudes which alternately vanish when either $U_C^W = 0$ or when $U_N^{K(1)} = 0$. The Coulomb amplitude is constructed to be the scattering amplitude which would result from the potential U_C^W with no $(A-1)/A$ scaling. The final projectile-nucleus T matrix of KMT No. 3 continues to differ from the exact T matrix with respect to the Coulomb distortion of the nuclear T matrix. While this effect is quite small for the particular case considered here one should carefully examine the importance of this remaining Coulomb distortion error for other targets and energies.

Our attempt to find two approximations [i.e., setting $U_{C-N,C} = 0$ and replacing $[A/(A-1)]f'_{\text{Coul}}$ with f_{Coul}] which will cancel one another provides a

satisfactory alternative to the difficult task of evaluating $U_{C-N,C}$ directly, at least for the particular case examined here. Another practical approach would be to evaluate U_{N-N} directly as demonstrated in Eq. (55) and calculate the projectile-nucleus scattering using the Watson optical potential and Eq. (42). Finally the main discussion in this work is relevant to any charged projectile-nucleus (protons, $\pi^{\pm}$, kaons, etc.) elastic scattering at high momentum transfer. However, the Coulomb related effects discussed here are of minor importance at forward angles where recent nuclear matter distribution studies have concentrated.[6,7,10,11]

ACKNOWLEDGMENT

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APPENDIX A

An alternate approach to that presented in Secs. II and III for the inclusion of Coulomb effects in the KMT prescription is given here. The separation of Coulomb and nuclear scattering is best accomplished by means of the so-called two-potential formula. One familiar form of that relation is

$$T = T_c + \Omega^{(-)\dagger} V^N \omega_c \ , \tag{A1}$$

where the external interaction potential V is separated into

$$V = V^C + V^N \tag{A2}$$

with

$$T_c = V^C \omega_c \ , \tag{A3}$$

$$\omega_c = (1 - GV^C)^{-1} \ , \tag{A4}$$

and

$$\Omega^{(-)\dagger} = 1 + \Omega^{(-)\dagger} VG \ . \tag{A5}$$

After some straightforward manipulation, Eq. (A1) may be reexpressed as

$$T = T_c + \bar{\omega}_c^{(-)\dagger} T_N \omega_c \ , \tag{A6}$$

where

$$T_N = V^N + V^N G T_N = V^N \omega_N = \omega_N^{(-)\dagger} V^N, \tag{A7}$$

$$\omega_N^{(-)\dagger} = (1 - V^N G)^{-1} \ , \tag{A8}$$

and

$$\bar{\omega}_c^{(-)\dagger} = (1 - \omega_N^{(-)\dagger} V^C G)^{-1} \ . \tag{A9}$$

The standard approximation for the treatment of the Coulomb term is given by

$$V^C \simeq PV^C P = P \sum_{i=1}^{Z} v_i^C P = PU_C^W P \ . \tag{A10}$$

In this case the elastic scattering as given by Eq. (A6) becomes

$$\begin{aligned} PTP \simeq P\bar{T}P = P\bar{T}_c P + (1 - P\omega_N^{(-)\dagger} PV^C PG_0)^{-1} \\ \times PT_N P(1 - G_0 PV^C P)^{-1} \\ = P\bar{T}_c P + P\bar{\omega}_c^{(-)\dagger} PT_N P\bar{\omega}_c P \ , \end{aligned} \tag{A11}$$

where

$$P\bar{T}_c P = PV^C P + PV^C PG_0 P\bar{T}_c P = PV^C P\bar{\omega}_c P, \tag{A12}$$

$$\bar{\omega}_c^{(-)\dagger} = (1 - \omega_N^{(-)\dagger} PV^C PG_0)^{-1} \ , \tag{A13}$$

and

$$\bar{\omega}_c = (1 - G_0 PV^C P)^{-1} \ . \tag{A14}$$

We note that Eq. (A6) is exact, and that Eq. (A11) is exact to within the Coulomb approximation explicitly presented in Eq. (A10) with the replacement of G by G_0.

The first order Watson approximation for $PT_N P$, given by

$$PT_N P \simeq PT_N^{(1)} P = PU_N^{W(1)} P + PU_N^{W(1)} PG_0 PT_N^{(1)} P \tag{A15}$$

with

$$PU_N^{W(1)} P = P \sum_i \tau_i^N P = AP\tau^N P \ , \tag{A16}$$

when inserted into Eq. (A11) yields the identical result given by Eq. (9).

We may calculate the quantity $PT_N P$ which appears in Eq. (A11) according to the KMT prescription as discussed in the main text. The first order KMT nuclear optical potential

$$PU_N^{K(1)} P = (A - 1) Pt^N P \tag{A17}$$

may be used to calculate the scaled T matrix

$$PT'_N P = [(A - 1)/A] PT_N P \ , \tag{A18}$$

from which the quantity $PT_N P$ follows immediately. Similarly the Watson Coulomb optical potential

$$PU_C^W P = PV^C P \tag{A19}$$

may be used to calculate the Coulomb T matrix $P\bar{T}_c P$. The Coulomb distorted wave function oper-

ator $P\bar{\omega}_c P$ can be readily evaluated using Eq. (A12), $PV^C P$, and the solution for $P\bar{T}_c P$ implied by $PV^C P$. The distorted wave function operator $P\bar{\omega}_c^{(-)\dagger} P$ can be evaluated in an analogous manner with the modified Coulomb optical potential $P\omega_N^{(-)\dagger} PV^C P$. Then the distorted wave matrix element of $PT_N P$ as given in Eq. (A11) may be calculated and added to $P\bar{T}_c P$ to yield $P\bar{T}P$. This prescription represents an appropriate modification of the KMT procedure in the presence of the Coulomb field. The calculations implied here are of course equivalent to that demanded in Eqs. (9), (25), (40), or (41). The value of the expression in Eq. (A11) is that approximations are readily suggested. For instance, it is probable that in most circumstances of interest $P\bar{\omega}_c P$ can safely be replaced by $P\bar{\omega}_c P$. In fact, over a very wide range of energy and momentum transfer it may be sufficient to approximate both $P\bar{\omega}_c P$ and $P\bar{\omega}_c P$ by unity, in which case the prescription of Eq. (A11) becomes particularly uncomplicated.

APPENDIX B

The formal development of the KMT Coulomb prescriptions of Sec. III, including the relationships between KMT Nos. 1, 2, and 3 is given here. This appendix demonstrates that an additional approximation (to the Coulomb amplitude) can be made which, when imposed on the projectile-nucleus T matrix constructed from the KMT optical potential of Eq. (40) with $U_{C-N,C}=0$, yields a better KMT approximation of the exact T matrix.

The exact projectile-nucleus T matrix is constructed using Eqs. (1) and (9) and is given by (elastic channel projections of the operator relations in this appendix can be readily made)

$$T^{\text{exact}} = [1-(U_C^W + U_N^{W(1)})G_0 P]^{-1}(U_C^W + U_N^{W(1)}) . \quad \text{(B1)}$$

A more convenient form is obtained by invoking the two-potential formula. Then T^{exact} becomes

$$T^{\text{exact}} = T_{\text{Coul}} + T_{C-N}^{\text{exact}} , \quad \text{(B2)}$$

where T_{Coul} is the projectile-nucleus T matrix resulting from U_C^W alone. It is given by

$$T_{\text{Coul}} = [1-U_C^W G_0 P]^{-1} U_C^W . \quad \text{(B3)}$$

Solving for T_{C-N}^{exact} yields

$$T_{C-N}^{\text{exact}} = [1-(U_C^W + U_N^{W(1)})G_0 P]^{-1} U_N^{W(1)} [1-G_0 P U_C^W]^{-1} , \quad \text{(B4)}$$

which is the Coulomb distorted nuclear T matrix. The Coulomb T matrix can be further separated into a point Coulomb part (Rutherford amplitude) and a term which depends on the distributed nuclear charge. Thus the exact T matrix may be written as

$$T^{\text{exact}} = T_R + T_{\text{dist}}^C + T_{C-N}^{\text{exact}} , \quad \text{(B5)}$$

where

$$T_R = (1-U_{\text{pt}}^C G_0 P)^{-1} U_{\text{pt}}^C , \quad \text{(B6)}$$

$$T_{\text{dist}}^C = (1-U_C^W G_0 P)^{-1}\ U_C^{W\prime}(1-G_0 P U_{\text{pt}}^C)^{-1} . \quad \text{(B7)}$$

On-shell matrix elements of the operator T_R in Eq. (B6) yield the usual Rutherford amplitude. In these equations the Coulomb potential U_C^W is divided into point and distributed terms,

$$U_C^W = U_{\text{pt}}^C + U_C^{W\prime} , \quad \text{(B8)}$$

where $U_{\text{pt}}^C = Ze^2/r$. Note that $U_C^{W\prime}$ vanishes for $r \gg R_{\text{nuc}}$, R_{nuc} being the nuclear radius. Equation (B8) divides the Coulomb potential into long and short range parts.

The projectile-nucleus T matrix for KMT No. 1 is obtained with Eqs. (15), (16), and (40) with $U_{C-N,C}=0$ and is

$$T(\text{KMT No. 1}) = (\{1-[(A-1)/A]U_C^W + U_N^{K(1)}\}G_0 P)^{-1} \times \{U_C^W + [A/(A-1)]U_N^{K(1)}\} . \quad \text{(B9)}$$

As before, a more convenient expression can be obtained with the aid of the two-potential formula. If T'_{Coul} is defined to be the T matrix resulting from $[(A-1)/A]U_C^W$, then

$$[A/(A-1)]T'_{\text{Coul}} = \{1-[(A-1)/A]U_C^W G_0 P\}^{-1} U_C^W . \quad \text{(B10)}$$

Writing the full T matrix in Eq. (B9) as

$$T(\text{KMT No. 1}) = [A/(A-1)]T'_{\text{Coul}} + T_{C-N}^{\text{KMT}} \quad \text{(B11)}$$

yields

$$T_{C-N}^{\text{KMT}} = (1-\{[(A-1)/A]U_C^W + U_N^{K(1)}\}G_0 P)^{-1} \times [A/(A-1)]U_N^{K(1)}\{1-[(A-1)/A]G_0 P U_C^W\}^{-1} \quad \text{(B12)}$$

which is the Coulomb distorted nuclear T matrix in KMT. Finally, defining T^{CC} to be $\{[A/(A-1)]T'_{\text{Coul}} - T_{\text{Coul}}\}$ the projectile-nucleus T matrix for KMT No. 1 becomes

$$T(\text{KMT No. 1}) = T_{\text{Coul}} + T^{CC} + T_{C-N}^{\text{KMT}} , \quad \text{(B13)}$$

where

$$T^{CC} = -A^{-1}\{1-[(A-1)/A]U_C^W G_0 P\}^{-1} U_C^W G_0 \times P U_C^W [1-G_0 P U_C^W]^{-1} . \quad \text{(B14)}$$

The KMT No. 3 prescription is obtained by dropping the T^{CC} term in Eq. (B13) yielding T(KMT No. 3) in Eq. (50). To obtain T(KMT No. 2) we start

with Eq. (B11) and define

$$T'_{\rm Coul} \equiv T'_R + T^{C}_{\rm dist}{}' \ , \tag{B15}$$

then

$$T(\text{KMT No. 1}) = [A/(A-1)]T'_R + [A/(A-1)]T^{C}_{\rm dist}{}' + T^{\rm KMT}_{C-N} \ , \tag{B16}$$

where

$$[A/(A-1)]T'_R = \{1 - [(A-1)/A]U^{C}_{\rm pt}G_0P\}^{-1}U^{C}_{\rm pt} \ , \tag{B17}$$

$$[A/(A-1)]T^{C}_{\rm dist}{}' = \{1 - [(A-1)/A]U^{W}_{C}G_0P\}^{-1}U^{W}_{C}{}' \times \{1 - [(A-1)/A]G_0PU^{C}_{\rm pt}\}^{-1} \ . \tag{B18}$$

Evaluating $\{[A/(A-1)]T'_R - T_R\}$ with Eqs. (B6) and (B17) yields

$$T^{RR} \equiv [A/(A-1)]T'_R - T_R = \{1 - [(A-1)/A]U^{C}_{\rm pt}G_0P\}^{-1}(-A^{-1}U^{C}_{\rm pt}G_0PU^{C}_{\rm pt}) \times [1 - G_0PU^{C}_{\rm pt}]^{-1} \ . \tag{B19}$$

The expression for T(KMT No. 1) becomes

$$T(\text{KMT No. 1}) = T_R + T^{RR} + [A/(A-1)]T^{C}_{\rm dist}{}' + T^{\rm KMT}_{C-N} \ . \tag{B20}$$

The correction term T^{RR} has been constructed to yield the long range limit of T^{CC}. Omitting T^{RR} in Eq. (B20) yields T(KMT No. 2) of Eq. (52).

The differences between T(KMT Nos. 1, 2, and 3) and $T^{\rm exact}$ can be evaluated by solving (where $i = 1$, 2, and 3)

$$T^{\rm exact} = T(\text{KMT No. } i) + T_i \tag{B21}$$

for each T_i. The difference corresponding to KMT No. 1 is

$$T_1 = T^{\rm exact}_{C-N} - T^{\rm KMT}_{C-N} - T^{CC} \ . \tag{B22}$$

The operator T_1 becomes 0 and $-T^{CC}$ in the limit of no Coulomb and no nuclear potentials, respectively. The long range part of T_1 is contained in T^{CC}. Solving for T_2 yields

$$T_2 = T^{\rm exact}_{C-N} - T^{\rm KMT}_{C-N} + T^{C}_{\rm dist} - [A/(A-1)]T^{C}_{\rm dist}{}' \tag{B23}$$

which becomes, with the aid of Eqs. (B7) and (B18),

$$T_2 = T^{\rm exact}_{C-N} - T^{\rm KMT}_{C-N} + \omega^{W}_{C}\,\omega^{W}_{C}{}' \times \{A^{-1}(U^{W}_{C}G_0PU^{W}_{C}{}' + U^{W}_{C}{}'G_0PU^{C}_{\rm pt}) - [(2A-1)/A^2]\,U^{W}_{C}G_0PU^{W}_{C}{}'\,G_0PU^{C}_{\rm pt}\}\omega'_{\rm pt}\,\omega_{\rm pt} \ . \tag{B24}$$

The operators in Eq. (B24) are given by

$$\omega^{W}_{C} = (1 - U^{W}_{C}G_0P)^{-1} \ , \tag{B25a}$$

$$\omega^{W}_{C}{}' = \{1 - [(A-1)/A]U^{W}_{C}G_0P\}^{-1} \ , \tag{B25b}$$

$$\omega_{\rm pt} = (1 - G_0PU^{C}_{\rm pt})^{-1} \ , \tag{B25c}$$

$$\omega'_{\rm pt} = \{1 - [(A-1)/A]G_0PU^{C}_{\rm pt}\}^{-1} \ . \tag{B25d}$$

Because the third term in T_2 in Eq. (B24) is proportional to $U^{W}_{C}{}'$, T_2 is short range in character. From the definition of $U^{W}_{C}{}'$ in Eq. (B8) we see that $U^{W}_{C}{}' \to -\infty$ as $r \to 0$ so that this part of T_2 could very likely have important consequences. In the limit of no Coulomb potential T_2 becomes 0 but if the nuclear potential is neglected, T_2 becomes equal to the third term in Eq. (B24). In the point charge limit $U^{W}_{C}{}' \to 0$ and $T_2 \to 0$ so that T(KMT No. 2) recovers the Rutherford scattering result. Finally, T_3 is given by

$$T_3 = T^{\rm exact}_{C-N} - T^{\rm KMT}_{C-N} \ , \tag{B26}$$

which contains no long range parts and vanishes if either U^{W}_{C} or $U^{K(1)}_{N}$ are set to zero.

The partial wave expansion corresponding to KMT No. 2 can be obtained by replacing $[A/(A-1)] \times f'_R$ with f_R in Eq. (46). The result is

$$f^{(2)}(\theta) = f_R + [A/(A-1)]k^{-1}\sum_{l=0}^{L_{\rm max}}(2l+1)e^{2i\sigma'_l}e^{i\delta^{C}_{l}{}'} \times \sin\delta^{C}_{l}{}'P_l(\cos\theta) \ , \tag{B27}$$

where f_R is the Rutherford scattering amplitude evaluated from the operator relation in Eq. (B6). The prescription used to generate KMT No. 3 from KMT No. 1 is to replace $[A/(A-1)]f'_{\rm Coul}$ with $f_{\rm Coul}$ where these two amplitudes correspond to on-shell matrix elements of the operators in Eqs. (B10) and (B3), respectively. To evaluate the partial wave expansion for KMT No. 3 we start with the partial wave series for KMT No. 1, Eq. (46), which is rewritten as

$$f^{(1)}(\theta) = [A/(A-1)](2ik)^{-1}\left[\sum_{l=0}^{L_{\rm max}}(2l+1)(e^{2i\delta^{T}_{l}{}'}-1)P_l(\cos\theta) + \sum_{l=L_{\rm max}+1}^{\infty}(2l+1)(e^{2i\sigma'_l}-1)P_l(\cos\theta)\right] \ , \tag{B28}$$

where $\delta^{T'}_{l} = \delta^{C}_{l}{}' + \sigma'_l$ is the total phase shift, and it is assumed that $\delta^{T'}_{l} = \sigma'_l$ for $l > L_{\rm max}$. To this partial wave series is added and subtracted the quantity

$$[A/(A-1)](2ik)^{-1}\sum_{l=0}^{L_{\max}}(2l+1)(e^{2i\sigma_l^{C\prime}}-1)P_l(\cos\theta)\ ,$$

where $\sigma_l^{C\prime}$ is the total phase shift due to $[(A-1)/A]U_C^W$ acting alone. If we assume that $\sigma_l^{C\prime}=\sigma_l'$ for $l>L_{\max}$ then the amplitude of Eq. (46) becomes

$$f^{(1)}(\theta)=[A/(A-1)]\left\{f'_{\text{Coul}}(\theta)+(2ik)^{-1}\sum_{l=0}^{L_{\max}}(2l+1)e^{2i\sigma_l^{C}\prime}\,[e^{2i(\delta_l^{T\prime}-\sigma_l^{C\prime})}-1]P_l(\cos\theta)\right\}\ , \tag{B29}$$

which is in the form of Eq. (43). In Eq. (B29), $[A/(A-1)]f'_{\text{Coul}}(\theta)$ is that amplitude obtained from Eq. (46) with $U^{K(0)}=[(A-1)/A]U_C^W$ only, or

$$f'_{\text{Coul}}(\theta)=f'_R+(2ik)^{-1}\sum_{l=0}^{L_{\max}}(2l+1)e^{2i\sigma_l'}(e^{2i\sigma_l^{CC\prime}}-1)P_l(\cos\theta)\ , \tag{B30}$$

where $\sigma_l^{C\prime}=\sigma_l^{CC\prime}+\sigma_l'$, and $\sigma_l^{CC\prime}$ is the phase shift due to the distributed nuclear charge. We now replace $[A/(A-1)]f'_{\text{Coul}}$ with f_{Coul}, given by

$$f_{\text{Coul}}(\theta)=f_R+(2ik)^{-1}\sum_{l=0}^{L_{\max}}(2l+1)e^{2i\sigma_l}(e^{2i\sigma_l^{CC}}-1)P_l(\cos\theta)\ , \tag{B31}$$

where σ_l^{CC} is the phase shift due to the distributed nuclear charge given by U_C^W only. With this additional approximation the amplitude of Eq. (B29) becomes,

$$f^{(3)}(\theta)=f_{\text{Coul}}(\theta)+[A/(A-1)](2ik)^{-1}\sum_{l=0}^{L_{\max}}(2l+1)e^{2i(\sigma_l^{CC\prime}+\sigma_l')}[e^{2i(\delta_l^{C\prime}-\sigma_l^{CC\prime})}-1]P_l(\cos\theta)\ . \tag{B32}$$

Equations (B31) and (B32) are the partial wave expansions for T(KMT No. 3) in Eq. (50).

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