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Explicit formula for hadron-nucleus elastic scattering in the eikonal approximation

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We obtain a closed form asymptotic approximation to the eikonal amplitude in hadron-nucleus scattering which agrees remarkably well with numerical evaluation of the full eikonal amplitude. The characteristic features of the scattering are dominated by the nuclear edge—its radius and thickness. The radius determines the oscillations and the thickness the exponential decrease with momentum transfer of the cross section. By developing systematic corrections to our asymptotic approximation we can separate the effects of "hard" and "soft" collisions in the multiple-scattering sense.

[NUCLEAR REACTIONS Closed form eikonal amplitude for hadron-nucleus scattering.]

I. INTRODUCTION

High energy elastic hadron nucleus scattering is well described by the eikonal approximation.[1] For moderate momentum transfers hadron-nucleus elastic scattering cross sections show regular oscillations superimposed on an exponentially falling average. The minima in the oscillations, though often deep, do not go to zero and the entire oscillatory pattern disappears at sufficiently large momentum transfers. All these features can be accounted for in the eikonal approximation by straightforward numerical evaluation. However, these numerical calculations give little insight into how the various empirical features emerge and on which aspect of the input they depend. Our purpose here is to give a *closed form, analytic, nonperturbative* evaluation of the eikonal amplitude for hadron-nucleus elastic scattering. This cannot be done exactly, but we show that our approximation correctly accounts for all the features of the data mentioned above, agrees with the full numerical evaluation for a wide range of momentum transfers, and is amenable to systematic improvement.

The principal feature of high energy hadron-nucleus elastic scattering, the periodic minima, suggest diffraction scattering from a circular object, but it is not at all clear *a priori* that the other features also have a purely geometric origin. Even the existence of "diffraction minima" is not a certain sign of the dominance of geometry over interaction mechanism. In p-α scattering, for example, the minimum arises from the interference of single and double scattering and therefore depends essentially on the interaction strength. In fact we will show that high energy hadron-nucleus elastic scattering *is* governed primarily by the shape of the nucleus, but that since the nucleus is not "black" and since it does not have sharp edges, that shape reflects itself in more subtle ways than in classical black sphere diffraction theory.

The oscillations in our picture of hadron-nucleus elastic scattering do arise from the nuclear radius c, and they are asymptotically periodic in qc where q is the momentum transfer. Since diffraction from an object with sharp boundaries gives oscillations but a power law decrease in q, we expect the exponential decrease to be related to the diffuse edge of the nucleus. We find that the amplitude falls like $\exp(-\pi\beta q)$ where β is the skin thickness. This agrees with data and provides a simple explanation of the fact that the exponential falloff does not depend on the nucleon number A of the target. We also find the same falloff for the electromagnetic form factor since both are governed by the same "geometric" considerations.[2] This is to be contrasted with p-α and e-α scattering where the strong and electromagnetic scattering have different forms due to the fact that the strong scattering is not dominated by the shape of the alpha particle. In our picture this would be related to the absence of an "edge" in the alpha particle. We also show how a combination of Coulomb scattering, the phase of the elementary hadron-hadron amplitude, and higher order terms in the hadron-nucleus scattering fill in the minima and finally cancel the oscillatory behavior completely for large q.

All of these features arise from the shape of the nucleus, which we describe by a Fermi function. The principal feature of the density that enters is its rapid variation near its surface or edge. Such a rapid variation can be thought of as arising from a singularity of the density function at some nearby complex point. We find that the dominant momentum transfer dependence of the scattering amplitude in the eikonal approximation is given by the contribution of that singularity to

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the eikonal integrals and that that contribution can be evaluated (using the method of steepest descents) explicitly. Systematic corrections to the first, asymptotic contribution can also be evaluated. Although we explicitly use the singularity structure of the Fermi function in our evaluation, it is clear that any function that represents the nuclear shape will have similar features, and that our result therefore does not depend on some explicit and perhaps pathological feature of the Fermi function.

In Sec. II we begin by studying the ordinary form factor of the Fermi function. The features of oscillation in an exponential envelope are already present here, and how they arise from the nearest singularity of the Fermi function is particularly transparent. We emphasize here that although the Fourier transform of the Fermi function eventually has a power law falloff, it has a large region of exponential decay because the power law part has a small coefficient. This is reminiscent of the theory of decaying states. In Sec. III we extend the analysis to hadron-nucleus elastic scattering in the eikonal approximation. We show how, exploiting the nearby singularity of the Fermi density function associated with the rapid density variation at the surface, it is possible to obtain the leading large q behavior of the eikonal amplitude. We then discuss the general features of the result: how the oscillation and exponential falloff develops, how Coulomb scattering and nongeometric terms in the amplitude fill in the minima, and the A dependence of the result. In Sec. IV we present a survey of systematic correction terms to the result of Sec. III. In particular, we see how the result can be interpreted in terms of "hard" or "soft" scattering and what dimensionless parameters control the expansion. Many of the more technical results needed for Secs. III and IV are derived in Appendices. We conclude with Sec. V which summarizes our results. There we show how well our results compare with full numerical evaluation of the eikonal cross section. We also discuss further avenues for investigation suggested by our work.

It should be noted that we have little to say about the interesting question of why these eikonal approximations work so well. Our only contribution to that subject is to have an explicit formual for the eikonal amplitude which at least permits examination of its ingredients.

II. THE FORM FACTOR

The form factor $F(q)$ is the Fourier transform of the single particle ground state density $\rho(r)$. It is defined by

$$F(q)=\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\rho(r) \tag{1}$$

normalized to $F(0)=A$, the number of nucleons. For $\rho(r)$ we take a Fermi distribution

$$\rho_f(r)=\rho_0/\{1+\exp[(r-c)/\beta]\}\,, \tag{2}$$

where c is the radius parameter. We will refer to β as the skin thickness parameter although strictly it is the diffuseness parameter. This gives

$$F(q)=\frac{4\pi}{q}\int_0^\infty r\,dr\sin qr\rho_f(r) \tag{3a}$$

$$=\frac{4\pi}{q}\rho_0\,\mathrm{Im}I\,, \tag{3b}$$

where

$$I=\int_0^\infty e^{iqr}r\,dr/(1+e^{(r-c)/\beta})\,. \tag{4}$$

We evaluate the integral I by deforming the integration path into the first quadrant of the complex r plane.[3] For Re$q>0$, the contour at ∞ makes no contribution but the function $(1+e^{(r-c)/\beta})^{-1}$ has simple poles in the first quadrant at $r=b_n\equiv c+(2n+1)i\pi\beta$, $n=0,1,2,3\ldots.$ The residue of $(1+e^{(r-c)/\beta})^{-1}$ at each of these poles is $-\beta$. We may therefore rotate the contour of integration in (4) to the positive imaginary axis but must add in the residue at the poles passed in that rotation. We find

$$I=-2\pi i\beta\sum_{n=0}^{\infty}b_n e^{iqb_n}+\int_0^{i\infty}e^{iqr}r\,dr\Big/(1+e^{(r-c)/\beta}) \tag{5a}$$

$$=S+I_+\,, \tag{5b}$$

where $b_n=c+(2n+1)i\pi\beta$, $n=0,1,2,3\ldots.$ S of (5) can be summed to give

$$S=-\pi q\frac{d}{dq}\frac{e^{iqc}}{\sinh\pi\beta q}\,. \tag{6}$$

To evaluate I_+ we change variables and get

$$I_+=-\int_0^\infty\frac{e^{-qx}x\,dx}{1+e^{-c/\beta}e^{ix/\beta}}\,, \tag{7}$$

which can be evaluated by expanding the denominator since $|e^{-c/\beta}e^{ix/\beta}|<1$ all x. Recalling that (3b) requires only the imaginary part, we find

$$\mathrm{Im}I_+=2\beta^3q\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-nc/\beta}n}{[(q\beta)^2+n^2]^2}\,. \tag{8}$$

Since S [Eq. (6)] decreases exponentially for large q, while ImI_+ decreases like $1/q^3$, the behavior of $F(q)$ for large q is $1/q^4$ from ImI_+. This power law behavior is associated with the small r behavior of $\rho(r)$. [In general $F(q) \sim q^{-n-3}$ for large q where n is the first nonvanishing odd derivative of $\rho(r)$ at $r=0$.] However, the power behavior of $\mathrm{Im} I_+$ carries a small coefficient $(e^{-c/\beta})$ and hence for a considerable range of q, $F(q)$ will be dominated by the exponential falloff given by S. For small q all the poles contributing to S are important, but for intermediate q, only the closest pole is significant. More explicitly (and up to terms in $\ln q$) if q is in the regime

$$\frac{1}{\pi\beta} \ll q \ll \frac{c}{\pi\beta^2}, \tag{9}$$

the form factor can be represented by

$$F(q) = \frac{8\pi^2\rho_0\beta}{q} e^{-\pi\beta q}(\pi\beta \sin qc - c\cos qc), \tag{10}$$

which is the contribution from the first pole of S only.

One can use the expression (10) for $F(q)$ to transform back and obtain the corresponding r-space density. However, this is not particularly instructive since we are not claiming that that new density is a good approximation to ρ_F of (2). It is not. For example, the new density is not even positive definite. Rather we are claiming that for a wide range of q, (10) is a good approximation to the Fourier transform of ρ_F. In particular, we note the two principal features of (10). It has an exponential falloff with a range governed by $\pi\beta$, where β is the skin thickness, and it has oscillations with a periodicity determined by c, the radius. Precisely those properties will be carried into our discussion of hadronic diffraction scattering. They arise from the dominance of the nearest singularity of $\rho(r)$ associated with its edge. The skin thickness is the imaginary part of the singularity position and the radius its real part. The fact that the density is characterized by such a singularity, or at least a corresponding rapid variation, is a general feature of the shape of the nucleus and not a special or pathological property of the Fermi distribution.

We have also seen that our Fourier transform has a power law falloff for sufficiently large q, but that that falloff has a very small coefficient so that there is exponential behavior for a large range of q. Dominant exponential behavior for functions that are not asymptotically exponential is well known in the theory of decaying states. In that case the familiar exponential time decay law goes over to a power law for sufficiently long times, and for precisely the same reason as in ours. This exponential dominance arises because the imaginary part of the singularity position (the width for decaying states) is small compared to the real part.

III. THE STRONG FORM FACTOR

The hadron-nucleus elastic scattering amplitude A can be expressed in terms of the normalized strong form factor F_s by[1]

$$A(q,k) = i\frac{k\gamma}{2\pi}F_s(q) \tag{11a}$$

with

$$F_s(q) = \frac{2\pi}{\gamma}\int_0^\infty J_0(qb)b\,db(1-e^{-\gamma t(b)}), \tag{11b}$$

where J_0 is the standard Bessel function, q the momentum transfer, k the projectile wave number, and γ is determined by

$$\gamma = \frac{\sigma_{\text{tot}}}{2}(1-ir), \tag{12}$$

where σ_{tot} and r are the elementary hadron-nucleon total cross section and ratio of real to imaginary part respectively. The profile function $t(b)$ is given in the eikonal approximation by

$$t(b) = \int_{-\infty}^{\infty} \rho((z^2+b^2)^{1/2})dz, \tag{13}$$

where ρ is the mass distribution after folding with the elementary interaction. In the following we will assume that ρ is a Fermi distribution as in Eq. (2). F_s is normalized so that it reduces to $F(q)$ of (1) for small γ.

It is not our purpose here to discuss the derivation of (11b) and (13) or its range of validity. Rather we take it as given that (11b) and (13) give a remarkably good account of hadron-nucleus elastic scattering over a wide range of projectile type, incident energy, and momentum transfer. This agreement is usually obtained through detailed and often complicated numerical evaluation and therefore the way in which the answers reflect the physical input is obscure. We will give an approximate *analytic* evaluation of (11b) which will permit the dependence on input parameters to be read off and the physics content of the answer to be explicit. In this section we will obtain only the leading large q approximation to (11b) with (13) using the method of stationary phase, but in Sec. IV and the Appendices we show how correction terms may be systematically obtained. We will see in Sec. V that away from the forward direction, the leading large q approximation with first corrections agrees very well with exact numerical evaluation. We now turn to our evaluation of (11b). For finite q we can drop the "one" term in (11b) (which contributes only in the forward direction) and using the relation between Bessel and Hankel functions we have

$$F_s(q) = -\frac{\pi}{\gamma}\int_0^\infty (H_0^{(1)}(qb) + H_0^{(2)}(qb))\,b\,db\,e^{-\gamma t(b)} \tag{14}$$

$$= \frac{2\pi}{\gamma}(G(q,\gamma) + G^*(q,\gamma^*)), \tag{15}$$

where in going from (14) to (15) we used the fact that $H^{(1)}$ and $H^{(2)}$ are complex conjugates and that γ is the only other complex quantity under the integral in (14). The integral for G can be deformed into the first quadrant of the complex b plane because $H^{(1)}$ provides exponential damping at infinity for $\mathrm{Im}b > 0$. For a Fermi distribution, the profile function $t(b)$ has singularities at $b = b_n = c + (2n+1)i\pi\beta$ and therefore the integral for G may be decomposed into the contribution from these singularities and a background or end point contribution coming from the integral along the imaginary axis. As in the form factor case that end point contribution will lead to a power law falloff in q but with a very small coefficient. We will therefore neglect it for now and return to its calculation in the correction section (IV) where we show that its coefficient is damped considerably over the end point contribution of the form factor by an exponential penetrability factor.

The contribution to G from the singularities of $t(b)$ we call G_s. Since that contribution comes at finite b, we can use the asymptotic form of the Hankel function to evaluate it for large q. We have therefore

$$G_s(q,\gamma) = \frac{e^{-i\pi/4}}{(2\pi q)^{1/2}}\int_C \sqrt{b}\,db\,e^{g(b)} \tag{16}$$

with

$$g(b) = iqb - \gamma t(b), \tag{17}$$

where the contour C in (16) encircles the singularities of $t(b)$ in the first quadrant. We evaluate the leading contribution to (16) by assuming that the singularity of $t(b)$ nearest the real axis, $b = b_0$, dominates. We write

$$t(b) = t_0(b) + \tilde{t}(b), \tag{18}$$

where $t_0(b)$ is the part of t singular at $b = b_0$ and $\tilde{t}(b)$ is analytic at b_0. We now evaluate (16) by deforming the contour to pass through the saddle point or stationary phase point of $g(b)$ associated only with the singularity at b_0. Since that saddle point is near $b = b_0$, we evaluate slowly varying parts of (16) at $b = b_0$ to give

$$G_s(q,\gamma) \cong e^{-i\pi/4 - \gamma\tilde{t}(b_0)}\left(\frac{b_0}{2\pi q}\right)^{1/2}\int_{C_s} db\,e^{g_0(b)} \tag{19}$$

with

$$g_0(b) = iqb - \gamma t_0(b) \tag{20}$$

and where now the contour C_s may be taken to run from $-\infty$ to $+\infty$ and passes through the saddle point. We find this stationary point of g_0, call it b_s $[g_0'(b_s) = 0]$ and expand g_0 around it,

$$g_0(b) \cong g_0(b_s) + \frac{(b-b_s)^2}{2}g_0''(b_s). \tag{21}$$

Using (21) in (19) we must choose our integration path so that $\mathrm{Re}g_0''(b_s) < 0$. We can then integrate (19) with (21) to get

$$G_s(q,\gamma) = -\left(\frac{b_0}{-qg_0''(b_s)}\right)^{1/2} \times \exp\left[-\frac{i\pi}{4} - \gamma\tilde{t}(b_0) + g_0(b_s)\right]. \tag{22}$$

To express this in terms of the parameters of the Fermi distribution and the interaction requires that we find $\tilde{t}(b_0)$, b_s, $g_0(b_s)$, and $g_0''(b_s)$. We now turn to their evaluation, but the reader interested only in the result can skip to Eq. (40) and the subsequent discussion of qualitative features.

We saw in Sec. I that the Fermi distribution $\rho_F(r)$ has simple poles in the first quadrant of the r plane at $r = b_n \equiv c + (2n+1)\pi i\beta$, $n = 0, 1, 2, 3\ldots$ and that the residue of $\rho(r)$ at these poles is $-\rho_0\beta$. The contribution to the pole at b_n to $t(b)$ comes from the integral

$$-\beta\rho_0\int_{-\infty}^{\infty}\frac{dz}{(z^2+b^2)^{1/2} - b_n}, \tag{23}$$

which is singular at $b = b_n$. The singular part of (23) comes from z near zero. It can be evaluated by expanding the square root for $z \ll b$. This gives for the singular part of (23)

$$\frac{-2\pi i\beta\rho_0 b}{(2b(b_n - b))^{1/2}},\quad \mathrm{Im}(2b(b_n-b))^{1/2} > 0. \tag{24}$$

Near $b = b_n$ we can write $2b = b + b_n$ and $b = b_n$ in the numerator and rewrite (24) as

$$\frac{-2\pi i\beta\rho_0 b_n}{(b_n^2 - b^2)^{1/2}},\quad \mathrm{Im}(b_n^2 - b^2)^{1/2} > 0. \tag{25}$$

This has the same singular part at $b = b_n$ and the new singularity introduced at $b = -b_n$ is of no concern to us because we are integrating in the upper half plane. We see from (25) that $t(b)$ has essential singularities at $b = b_n$ and that as we extend b into the first quadrant, the first of these will come at b_0. Therefore we have for $t_0(b)$

$$t_0(b) = \frac{-2\pi i\beta\rho_0 b_0}{(b_0^2 - b^2)^{1/2}},\quad \mathrm{Im}(b_0^2 - b^2)^{1/2} > 0; \tag{26}$$

with (26) for $t_0(b)$ in (20), the condition for a stationary point of g_0 $[g_0'(b_s)=0]$ becomes

$$iq+\frac{i\alpha b_s b_0}{(b_0^2-b_s^2)^{3/2}}=0\,, \tag{27}$$

where we have introduced the dimensionless parameter

$$\alpha=2\pi\gamma\beta\rho_0\,. \tag{28}$$

For large q, (27) requires that b_s be near b_0. We write

$$b_s=b_0+\delta \tag{29}$$

and to lowest order in $1/q$ find

$$\delta=-\frac{b_0}{2}\left(-\frac{\alpha}{qb_0}\right)^{2/3} \tag{30a}$$

$$=-\frac{1}{2}\left(\frac{\alpha^2 b_0}{q^2}\right)^{1/3} e^{(2n+1)\pi i2/3}\,,\quad n=0,1,2\,. \tag{30b}$$

To this same order in $1/q$ we find

$$g_0''(b_s)=\frac{3}{2}\frac{qi}{2\delta}\,. \tag{31}$$

The branch of the $\frac{2}{3}rd$ root, that is, the value of n, to be taken in (30) must satisfy $\mathrm{Im}(b_0^2-b_s^2)^{1/2}>0$ [see (26)], and we must be sure that the path we took in obtaining (22) corresponds to $\mathrm{Re}\,g''(b_s)<0$. For our case these two conditions are equivalent. For most applications, for example proton scattering from a medium or heavy nucleus, only the $n=0$ root will satisfy the condition. In the following we will keep only the $n=0$ root, but it must be kept in mind that the appropriate root of (30) depends on the physical input parameters. Keeping only the $n=0$ root gives, to leading order in q,

$$g_0(b_s)=iqb_0+\tfrac{3}{2}\alpha^{2/3}(qb_0)^{1/3}\,e^{i\pi/6} \tag{32}$$

and

$$g_0''(b_s)=-3q^2\frac{\alpha^{-2/3}e^{-i\pi/6}}{(qb_0)^{1/3}}\,. \tag{33}$$

It remains to find $\tilde{t}(b_0)$, the finite part of the profile function. It should be noted that in (22), $\tilde{t}(b_0)$ enters only through the factor $\exp[-\gamma\tilde{t}(b_0)]$. It introduces no momentum transfer dependence and only a trivial energy dependence via γ. It is important therefore only in setting the overall scale of $G_s(q,\gamma)$.

$\tilde{t}(b_0)$ can be defined in terms of the nonsingular part of $\rho(r)$, $\tilde{\rho}(r)$ as

$$\tilde{t}(b_0)=2\int_0^\infty \tilde{\rho}((z^2+b_0^2)^{1/2})dz\,, \tag{34}$$

where

$$\tilde{\rho}(r)=\rho_0\left(\frac{1}{1+e^{(r-c)/\beta}}+\frac{2b_0\beta}{r^2-b_0^2}\right). \tag{35}$$

The form of the singular term subtracted out on the right in (35) corresponds to going from (24) to (25) and again introduces a singularity at $r=-b_0$ that does not concern us. Making the variable change $(z^2+b_0^2)^{1/2}=y+b_0$ in (34) and using (35) we find

$$\tilde{t}(b_0)=2\rho_0\int_0^\infty\frac{(y+b_0)dy}{(y^2+2b_0y)^{1/2}}\times\left(\frac{1}{1-e^{y/\beta}}+\frac{2b_0\beta}{y^2+2b_0y}\right), \tag{36}$$

where we have explicitly used the fact that $b_0=c+i\pi\beta$ to transform the exponential term. Changing variables again to $z=y/\beta$ we can write

$$\tilde{t}(b_0)=\rho_0(2b_0\beta\pi)^{1/2}f(\tau) \tag{37}$$

with

$$f(\tau)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{dt}{\sqrt{z}}\frac{1+\tau z}{(1+\frac{1}{2}\tau z)^{1/2}}\times\left(\frac{1}{1-e^z}+\frac{1}{z(1+\frac{1}{2}\tau z)}\right) \tag{38}$$

with $\tau=\beta/b_0\cong\beta/c$. Since τ is small, we expand $f(\tau)$ in an asymptotic series in powers of τ (Appendix A). The first term can be evaluated numerically to give

$$f(0)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{dt}{\sqrt{z}}\left(\frac{1}{1-e^z}+\frac{1}{z}\right)=1.460\,, \tag{39}$$

which is a good approximation for most nuclei. In other cases we can use our asymptotic series or numerical evaluation. Since $\tilde{t}(b_0)$ only determines the cross-section scale, and since it depends only on the size of the nuclear target and not on energy or momentum transfer, either of these choices is reasonable and does not significantly compromise our goal of giving an analytic expression for F_s.

In terms of (32) and (33) we have finally for (22)

$$G_s(q,\gamma)=\frac{\alpha^{1/3}q^{-4/3}b_0^{2/3}}{\sqrt{3}}\exp\left[\frac{5\pi}{6}i-\gamma\tilde{t}(b_0)+iqb_0+\tfrac{3}{2}\alpha^{2/3}(qb_0)^{1/3}\,e^{i\pi/6}\right] \tag{40}$$

to leading order in $1/q$.

Let us now turn to a qualitative discussion of (40) and what it predicts for the cross section. In particular, we are interested in the behavior with momentum transfer, interaction strength, and target size or A.

Let us begin by assuming γ is real, that is, that the elementary hadron-hadron amplitude is pure imaginary (12). Then the cross section for hadron-nucleus scattering is proportional to the square of the real part of G (40) via (15). As a function of the momentum transfer q this quantity will oscillate and will in fact have zeros at the points where the cosine of the phase of (40) vanishes. These are given (for real γ) by

$$\frac{5\pi}{6}+qc+\tfrac{3}{2}\alpha^{2/3}\,\mathrm{Im}(qb_0)^{1/3}e^{i\pi/6}-\mathrm{Im}\gamma\tilde{t}(b_0)=\pi/2+n\pi,\quad n=0,1,2,\ldots. \tag{41}$$

With $b_0=c+i\pi\beta$ and $c\gg\pi\beta$ this becomes

$$\frac{5\pi}{6}+qc+\tfrac{3}{4}\alpha^{2/3}(qc)^{1/3}=\frac{\pi}{2}+n\pi. \tag{42}$$

The zeros therefore depend only on qc (recall that c is the nuclear radius). For qc large they are evenly spaced with distance between zeros Δq given by π/c as it is for the electromagnetic form factor. However, the entire pattern is shifted by the constant phase factor $5\pi/6$ and modified by the $\frac{3}{4}\alpha^{2/3}(qc)^{1/3}$ term. For typical nucleon-nucleon amplitudes at a few hundred MeV we take $\sigma_{\mathrm{tot}}=40$ mb. Combining this with $\rho_0=\frac{1}{6}$ fm^{-3} and $\beta=0.54$ fm gives $\alpha\equiv 2\pi\gamma\beta\rho_0=1.13$. Thus α is of order 1 and the $(qc)^{1/3}$ correction term is neither anomalously large nor small. As a function of q, the oscillatory behavior is modulated by an overall exponential factor from the iqb_0 of $\exp(-q\pi\beta)$ again as in the electromagnetic case. These features of oscillatory behavior in an exponentially falling envelope are well known in nucleus scattering. In fact the universal exponential falloff is more clearly seen in the data than one might expect from our result. We see from (40) that the magnitude of G_s has a q dependence determined by the product of $\exp(-\pi\beta q)$ and $q^{-4/3}\exp[(3\sqrt{3}/4)(\alpha^2 qb_0)^{1/3}]$. Remarkably this second factor has very little q variation over the range of q values usually studied since the falling $q^{-4/3}$ is compensated by the rising exponential. This is essentially a numerical accident. It should also be noted that in the electromagnetic form factor case (Sec. II) there is another factor of $1/q$ in the amplitude that makes the falloff slightly faster than $\exp(-\pi\beta q)$.

Both a real part of the elementary amplitude and the Coulomb interaction will fill in the zeros of the cross section. The effect of the Coulomb interaction is especially important for scattering on a heavy nucleus. Within the eikonal approximation the Coulomb interaction is easily included by adding to the strong profile function we have been considering the Coulomb contribution

$$i\eta t^{c}(b), \tag{43}$$

where $\eta=Ze^2/v$, $v=k/(m^2+k^2)^{1/2}$ and $t^c(b)$ is the profile function for the Coulomb interaction folded with the density.[4] Since $t^c(b)$ is a smooth function of b, the stationary point is not modified and we need only evaluate t^c at $b=b_0$, the singular point of the density. In Appendix B we evaluate $t^c(b_0)$.

Including the effect of $t^c(b_0)$ modifies (15) to give

$$F_{s,c}(q)=\frac{2\pi}{\gamma}e^{-i\eta\,\mathrm{Re}t^c(b_0)}\left(G_s(q,\gamma)e^{\eta\,\mathrm{Im}t^c(b_0)}+G_s^*(q,\gamma^*)e^{-\eta\,\mathrm{Im}t^c(b_0)}\right), \tag{44}$$

where the c on F signifies the addition of Coulomb effects. We see from (44) that only the imaginary part of $t^c(b_0)$ concerns us and explicit evaluation yields

$$\mathrm{Im}t^c(b_0)=-2\arctan\pi\beta/c+\mathcal{O}((\beta/c)^{5/2}). \tag{45}$$

If we also include the effect of a real part of the elementary amplitude, which then makes γ and α complex, we get, using (40),

$$F_{s,c}(q)=\frac{2\pi}{\gamma}\frac{\alpha^{1/3}q^{-4/3}c^{2/3}}{\sqrt{2}}\exp\Big[-i\eta\,\mathrm{Re}t^c(b_0)-q\pi\beta-\gamma\,\mathrm{Re}\tilde{t}(b_0)+\tfrac{3}{2}(qc)^{1/3}\cos\frac{\pi}{6}(\mathrm{Re}\alpha^{2/3}+i\,\mathrm{Im}\alpha^{2/3})\Big][e^{i\theta}e^{\rho}+e^{-i\theta}e^{-\rho}], \tag{46a}$$

where

$$\theta=qc+\frac{5\pi}{6}+\tfrac{3}{2}(qc)^{1/3}\sin\frac{\pi}{6}\mathrm{Re}\alpha^{2/3}-\gamma\,\mathrm{Im}\tilde{t}(b_0) \tag{46b}$$

and

$$\rho=\eta\,\mathrm{Im}t^c(b_0)-\tfrac{3}{2}(qc)^{1/3}\sin\frac{\pi}{6}\,\mathrm{Im}\alpha^{2/3}, \tag{46c}$$

and where we have approximated b_0 by c in most places. The factor in the square brackets is responsible for the oscillations in the cross section and its minima are zeros if $\rho=0$. They come when $\theta=\pi/2, 3\pi/2, \ldots$. For ρ small the positions of the minima are not significantly altered but they are filled in. For ρ small the square brackets become 2ρ at a minima. If $\mathrm{Im}\alpha=0$, this filling will be due to the Coulomb force alone and the factor in the square brackets at the minimum becomes

$$2\eta\,\mathrm{Im}\,t^{c}(b_0)\cong -4\eta\pi\beta/c \tag{47}$$

using (45). Hence the Coulomb force produces a constant (q independent) filling of the minimum. Since $\eta\sim Z$ and $c\sim Z^{1/3}$ the filling goes like $Z^{2/3}$ in the amplitude. As is clear from (46c) the effect of a real part in the fundamental amplitude combined with the Coulomb interaction will be to tend to fill in the minima if both correspond to attraction or repulsion, but they will cancel if the forces are opposite. However, the strong interaction effect grows with q like $(gc)^{1/3}$ while the Coulomb part of ρ is q independent. Thus eventually the $\mathrm{Im}\,\alpha^{2/3}$ form will dominate and in fact for sufficiently large q, ρ will no longer be small and the oscillatory behavior of the cross section will disappear entirely.[5] This is seen experimentally but the eikonal formalism is presumably not sufficiently reliable at the large angles in question for our calculation to be more than suggestive.

For discussing the dependence of our result on target size, or nucleon number A, it is convenient to return to the case of $\mathrm{Im}\gamma=0$ and no Coulomb interaction. The A dependence of (40) enters only through $b_0=c+i\pi\beta$ since $c\sim A^{1/3}$. γ and α are A independent and it should be recalled that for proton-nucleus scattering α is of order 1. The A dependence of the oscillatory behavior can be read off from (41). The A dependence of the overall factors for fixed (qc) come from the $b_0^{2/3}$ and the factor of $\exp[-\gamma\,\mathrm{Re}\,\bar{t}(b_0)]$. We take from (28), (37), and (39)

$$\begin{aligned}\gamma\bar{t}(b_0)&=\gamma\rho_0(2b_0\beta\pi)^{1/2}f(\tau)\\&=\alpha\left(\frac{b_0}{2\pi\beta}\right)^{1/2}f(\tau)\cong\alpha\left(\frac{c}{2\pi\beta}\right)^{1/2}\times 1.5\,.\end{aligned} \tag{48}$$

Hence F_s depends on A for fixed (qc) via the quantity

$$c^{2/3}\exp\left[-1.5\alpha\left(\frac{c}{2\pi\beta}\right)^{1/2}\right]. \tag{49}$$

If we write $c=r_0A^{1/3}$ and take $r_0/\beta\sim 2$ this gives an A dependence of the form

$$A^{2/9}\exp(-A^{1/6}), \tag{50}$$

which is a weakly decreasing function of A for all $A>1$. The origin of this decrease is the exponential factor of $-\gamma\rho_0(2c\beta\pi)^{1/2}$. It is an absorption, penetration, or mean free path factor. The larger the nucleus the larger this absorption. It dominates the rising $A^{2/9}$ factor in front that would give growth with A for weak scattering. Perhaps more surprising is the strange radius dependence of the penetration factor in the exponent. One might have expected c or $A^{1/3}$. However, that assumes the projectile is traversing the center of the nucleus. Such central traversals do not yield scattering in the momentum transfer region we are studying. Rather to be scattered to our q the particle must go near the surface. Hence the amount of nucleus it crosses is expressed in a geometric mean of the radius and the skin thickness, which is $(\beta c)^{1/2}$. An A dependence of the sort we obtain here is observed for medium energy proton scattering from complex nuclei.

The strength of the interaction enters the scattering amplitude in a more complicated way. (The $1/\gamma$ in front of F_s is in fact just a normalization. The cross section depends on γF_s and hence has no $1/\gamma$). There is a factor of $\alpha^{1/3}$ in front but from the exponent there is the overall factor

$$\exp\left[-\gamma\,\mathrm{Re}\,\bar{t}(b_0)+\tfrac{3}{2}(qc)^{1/3}\alpha^{2/3}\cos\frac{\pi}{6}\right]. \tag{51}$$

These two terms in the exponent are of the same order of magnitude and opposite sign. Clearly for sufficiently large qc the second term will dominate and the amplitude will grow with interaction strength. For small qc the first term dominates and the amplitude decreases with increasing interaction. This is another manifestation of the absorption or penetration effect carried by the $\gamma\bar{t}(b_0)$ term.

IV. CORRECTION TERMS

In obtaining our result (40) for the asymptotic form of the hadron-hadron nucleus elastic scattering amplitude in eikonal approximation we made a number of approximations. In this section we examine these, and develop systematic corrections for the major approximations. We do this not only to have these corrections but also because the nature of an approximation is best understood in terms of the corrections and in particular of the parameters that control their size.

Our major approximations occurred in going from (18) to (21). We assumed that only the first singularity of the profile function was important and we then evaluated its contribution to the integral by the saddle point method. Much of our attention will be devoted to these two approximations. In addition we made further "kinematic" approximations in obtaining (19). We used the asymptotic form of the Bessel function and we assume that b under the integral can be replaced by its value of the singular point $b=b_0$. It is easy to see that corrections to these approximations are of order $(qb_0)^{-1}$ and can be developed as a systematic series in inverse powers of (qb_0). These corrections are not dynamical in that they do not involve γ. By contrast we will see that the corrections to the saddle point method are of order $(\alpha^2 qb_0)^{-1/3}$ and are therefore both dynamical and more slowly decreasing in q.

Finally in obtaining our asymptotic form (40) we neglected the background or end point contribution coming from the integral along the imaginary axis. At the end of this section we show that it is much smaller than the corresponding (and already small) contribution to the electromagnetic form factor.

To calculate the corrections to the saddle point method we rewrite (19) as

$$G_s(q,\gamma)=e^{-i\pi/4}e^{-\gamma\tilde{t}(b_0)}\left(\frac{b_0}{2\pi q}\right)^{1/2}\times\int_{-\infty}^{\infty}db\, e^{iqb}\left\{1-\exp\left[\frac{i\alpha b_0}{(b_0^{\,2}-b^2)^{1/2}}\right]\right\}, \qquad (52)$$

where we have explicitly used the form of $t_0(b)$ and have reintroduced the 1 from (11b). Expanding the second exponential, the integral can be evaluated as a power series in

$$z=\alpha^2(iqb_0)^{1/2}. \qquad (53)$$

We obtain

$$G_s(q,\gamma)=\frac{2ib_0}{q}e^{-\gamma\tilde{t}(b_0)}\times\sum_{n=1}^{\infty}z^{n/2}\frac{1}{2^{n/2}\Gamma(n/2)n!}K_{(n-1)/2}(-iqb_0). \qquad (54)$$

This can be understood as a multiple scattering expansion of the strong amplitude. We note that these scatterings are separated in (54) into "soft" collisions that are summed into the penetration or renormalization factor $e^{-\gamma\tilde{t}(b_0)}$ and "hard" scatterings. Using the asymptotic form of the Bessel function in (54) we see that these hard scatterings give terms of the sum proportional to $z^n e^{iqb_0}$.

Using the asymptotic form for the Bessel and gamma functions and finding the term of maximum modulus in the series by differentiating with respect to n, we see that the number of terms contributing significantly to (54), and therefore the number of hard scatterings, N, is of order

$$N\sim\mathrm{Re}z^{2/3}=\mathrm{Re}(\alpha^{2/3}(iqb_0)^{1/3}), \qquad (55)$$

which increases with q as one would expect, but very slowly. Even for a nucleus such as Pb and a momentum transfer q of 4 fm^{-1}, we find $N\sim 3$, a rather small number, indicating the rapid convergence of this multiple scattering series in terms of hard collisions. Mathematically this comes about from the $\Gamma(n/2)n!$ in the denominator of (54).

Using the asymptotic form of the Bessel function in (54), converting the sum to an integral, and evaluating that integral again by the saddle point method, we can recover our asymptotic result (40). Corrections to that asymptotic answer are developed in Appendix C where we see that they are given as a series in inverse powers of the parameter $(\alpha^2 iqb_0)^{1/3}$. The first correction term is explicitly given in Eq. (57). It is interesting to conjecture that this weak (cube root) dependence on $qb_0\sim qc$ helps account for the remarkable success of the eikonal approximation at larger momentum transfers than *a priori* might seem appropriate. It should also be noted that this cube root is not a special property of the Fermi distribution but arises from the geometry of cylindrical coordinates.

We now consider corrections to (40) coming from singularities in the profile function beyond the first. The corrections are particularly important in extending the asymptotic form into small momentum transfers since as q decreases, the saddle point moves farther away from b_0 and therefore the other b_n become relatively more important. In going from (18) to (19) we evaluated the part of the profile function coming from the other singularities, $\tilde{t}(b)$ at the singular point of t_0. In fact t has a b dependence and we wish to include it. We take it into account in lowest order by making a Taylor expansion of $\tilde{t}$ around $b=b_0$ and keeping only the first derivative.

$$\tilde{t}(b)\cong\tilde{t}(b_0)+(b-b_0)\frac{d\tilde{t}}{db}\bigg|_{b_0}. \qquad (56)$$

Inserting this in (17) will change both the position of the stationary point and the second derivative. The details of this modification and the value of $d\tilde{t}/db|_{b_0}$ are given in Appendix A. We quote here only the result for the amplitude including both the first correction to the saddle point integral and the first order to dependence of $\tilde{t}(b)$. We call the first order corrected form $G_{sk}(q,\gamma)$ and find for it in terms of the G_s of (40)

$$G_{s,k}(q,\gamma)=G_s(q,\gamma)\left\{\exp\left[\gamma\frac{d\tilde{t}}{db_0}\frac{b_0}{2}\left(\frac{\alpha}{b_0q}\right)^{2/3}e^{2\pi i/3}\right]\times\left(1+\frac{i\gamma}{q}\frac{d\tilde{t}}{db_0}\right)^{-5/6}\times\left[1-\frac{1}{(\alpha^2 iqb_0)^{1/3}}\left(\tfrac{5}{18}+\tfrac{1}{8}\alpha^2\right)\right]+\mathcal{O}\left(\frac{1}{q^{2/3}}\right)\right\}. \qquad (57)$$

Let us now examine the correction terms and what they tell us about the parameters that control our asymptotic approximation. The corrections from the saddle point integration are in the second set of square brackets. They are of two types. The first comes directly from corrections to the asymptotic summation of the series in (54) and requires that

$$\tfrac{18}{5}|(\alpha^2 iqb_0)^{1/3}| \gg 1$$

or approximating b_0 by c and $\frac{18}{5}$ by 3 that $(q_0c) \gg 1/27\alpha^2$, a condition that is easily fulfilled for hadron-nucleus collisions, but is *not* for weak ($\alpha \ll 1$) forces. The second term in the bracket requires that

$$|\,\alpha^2/[8(\alpha^2 iqb_0)^{1/3}]| \ll 1.$$

It comes from the first correction to using the asymptotic form for the Bessel function in (54). It becomes $qc \gg \alpha^4/8^3$. This condition is also easily met by hadron-nucleus collisions, but it is interesting to note that it has the *opposite* dependence on the strength of the elementary interactions. We stress again that for typical proton-nucleus parameters, $\alpha \simeq 1$.

The corrections from the b dependence of the nonsingular part of the profile function are also in two parts. One is in the exponent and comes from corrections to the location of the singular point, and one is a factor coming from corrections to the second derivative. From Appendix A [(A9] we see that we can write for the quantity in the exponent of (57),

$$\gamma \frac{d\tilde{t}}{db_0}\frac{b_0}{2}\left(\frac{\alpha}{b_0 q}\right)^{2/3} = \frac{a_0}{4(2\pi)^{1/2}}\frac{1}{(qb_0)^{2/3}}\left(\frac{b_0}{\beta}\right)^{3/2}\alpha^{5/3}, \qquad (58)$$

where a_0 is a number of order $\frac{1}{2}$. This quantity is of order one except for very small q and does not significantly affect the q dependence of G except again for small q. Notice that for fixed (qb_0) the quantity in (58) grows with b_0, the nuclear size. This reflects the fact that "background" scattering grows in importance as the nucleus grows. Equation (58) also grows with interaction strength since as α grows the nucleus becomes "blacker" and the interesting multiple scatterings occur further out in the region of lower density. For *very* strong forces only the exponential tail of the density will contribute and then we can replace the nuclear density by an exponential. For such a case one finds asymptotically a falloff of $\exp(-\beta\pi q/2)$, which is different from (10) and is no longer dominated by the nuclear "edge" but by its "tail." Thus the correction (58) is a first step to this new behavior. Note that this "strong absorption" limit with $\exp(-\beta\pi q/2)$ falloff is realized only for an intermediate regime of momentum tranfer. As q increases (for fixed α) the region of the nucleus contributing to the scattering will move inward so that this "tail dominance" regime will pass into an "edge dominated" part as in (40) and ultimately it is the end point contribution that dominates. In high energy p-nucleus scattering there seems to be no such intermediate regime with a $e^{-\pi\beta q/2}$ falloff coming from the scattering in the tail region. In heavy-ion scattering by contrast this peripheral scattering seems to be more important due to the combined effect of Coulomb repulsion and strong absorption.[6]

The correction from $d\tilde{t}/db_0$ to the second derivative in (57) is $(i\gamma/q)(d\tilde{t}/db_0)$. It is easily seen that this is (58) divided by $\frac{1}{2}(\alpha^2 qb_0)^{1/3}$. Since we have already argued that this factor is small, the entire correction is small except again at small q. It should be noted that like the correction in the exponent, it grows with b_0 and α for fixed (qb_0).

Finally we turn to the end point contribution to F_s coming from the integral along the imaginary axis. For simplicity we consider the case of real γ (the final result is valid for all γ) and call the end point contribution F_e. We have

$$F_e = \frac{2\pi}{\gamma}\mathrm{Re}\int_0^{i\infty} H_0^{(1)}(qb)b\,db(1-e^{-\gamma t(b)}) \qquad (59a)$$

$$= -\frac{4}{\gamma}\,\mathrm{Im}\int_0^{\infty} K_0(qx)x\,dx(1-e^{-\gamma t(ix)}), \qquad (59b)$$

where in going from (59a) to (59b) we changed variables and expressed $H_0^{(1)}$ of imaginary argument in terms of the purely real function K_0. Since K_0 is purely real, the contribution to F_e comes from $\mathrm{Im}\,t(ix)$. We have from (2) and (13)

$$t(ix) = 2\rho_0\left(\int_0^x \frac{dz}{1+e^{-c/\beta}e^{(i/\beta)(x^2-z^2)^{1/2}}} + \int_x^\infty \frac{dz}{1+e^{-c/\beta}e^{(1/\beta)(z^2-x^2)^{1/2}}}\right) \qquad (60a)$$

$$= 2\rho_0(A+iB+C), \qquad (60b)$$

where C is the second integral and A and B are the real and imaginary parts of the first. We can write

$$F_e = -\frac{4}{q^2\gamma}\int_0^\infty K_0(y)y\,dy\,e^{-2\rho_0\gamma A}e^{-2\rho_0\gamma C}\sin 2\rho_0\gamma B, \qquad (61)$$

where we have changed variables from qx to y so that A, B, and C are now all functions of y/q. Since $K_0(y)$ is exponentially decreasing for large y, we can expand A, B, and C in powers of x or y/q. We see that $A \to 0$, $2\rho C_0 \to t(0)$, and that $B \sim 1/q^2$. Making the expansion we obtain to leading order in $1/q^2$

$$F_e = \frac{2\pi\rho_0 e^{-c/\beta} e^{-\gamma t(0)}}{\beta q^4(1+e^{-c/\beta})^2} \int_0^\infty K_0(y) y^3 dy . \tag{62}$$

The remaining integral is 4 (Ref. 7) and therefore except for the factor of $e^{-\gamma t(0)}$ the result agrees with the leading end point contribution to the electromagnetic form factor (8). It must since we can formally expand (11b) in powers of γ and the first order term is that form factor. The factor of $e^{-\gamma t(0)}$ is a penetrability factor. It reflects the fact that the very large q^2 behavior comes from very small r and that it is much harder to get to small r in the presence of absorption. This means that although our strong form factor ultimately has a power law falloff, the domain of the power law is even more remote than in the electromagnetic case.

V. CONCLUSIONS

We have seen that the eikonal amplitude for elastic hadron-nucleus scattering can be evaluated in closed form away from the forward direction, and that correction to the leading asymptotic representation of the amplitude can be developed systematically. The major physical ingredient of this approximation is the shape of the nuclear density, which we take to be a Fermi function. It is the radius of this distribution that determines the oscillations of the cross section, as is well known from classical diffraction theory, and it is the diffuse boundary that gives the exponential falloff of the large cross section. We find that falloff to be given, in the amplitude, by $e^{-\pi\beta q}$ where q is the momentum transfer and β the skin thickness parameter of the Fermi function. This exponentially damped oscillatory behavior is seen both in the Fourier transform of the Fermi function (Sec. II) and in the full eikonal amplitude (Sec. IV). It arises from the rapid variation of the distribution at the surface. Mathematically such rapid variation implies the existence of a nearby singularity (in the complex plane), and we evaluate the Fourier transform and the eikonal form by taking advantage of the dominance of that singularity. It is easy to see that any other distribution function having a similar shape will give essentially the same result. However, for a Gaussian or exponential distribution, there is no longer rapid variation at the surface and no corresponding nearby singularity. As a result for these forms there is no simple connection between the Fourier transform of the density (electron scattering) and the eikonal form (proton scattering). This is well known in the case of the alpha particle where the density is Gaussian.

By studying the eikonal form in our case of rapidly varying density we find that the scattering series can be separated into soft scatterings due to the slowly varying parts of the density and hard scatterings due to the rapidly varying parts. The soft scatterings sum into a penetration factor while the momentum transfer is built up from the hard scatterings. Their effect is controlled by the parameter $(\alpha^2 qc)^{1/3}$ where α is a dimensionless measure of the primary interaction strength and c the nuclear radius. We see that this parameter grows very slowly with momentum transfer and helps to explain the wide range of validity of the eikonal approximation.

To see how well our approximate form works we wish to compare it with an exact (numerical) evaluation of the eikonal cross section. In Fig. 1 we show our lowest order [from (46)] and first order corrected [from (57)] cross section compared with the full calculation for proton scattering on ^{16}O, ^{40}Ca, and ^{208}Pb in the vicinity of 1 GeV. By studying $(1/k^2)(d\sigma/d\Omega)$ we remove most of the energy dependence of the cross section [see (11a)]. The remaining energy dependence is in γ [(12)], but around 1 GeV proton-nucleon total cross sections have very little energy dependence.

In the calculation we have used $\sigma_{tot} = 40$ mb and r in Eq. (12) of -0.275. For all nuclei we used a k corresponding to 1.04 GeV of kinetic energy (i.e., no center of mass effect). The radius and density parameters we used were

$$c = 6.624 \text{ fm}, \quad \beta = 0.549 \text{ fm},$$
$$\rho_0 = 0.16 \text{ fm}^{-3} \text{ for } ^{208}\text{Pb},$$
$$c = 3.725 \text{ fm}, \quad \beta = 0.591 \text{ fm},$$
$$\rho_0 = 0.148 \text{ fm}^{-3} \text{ for } ^{40}\text{Ca},$$

and

$$c = 2.381 \text{ fm}, \quad \beta = 0.671 \text{ fm}, \quad \rho_0 = 0.21 \text{ fm}^{-3} \text{ for } ^{16}\text{O}.$$

We make no claim for the precise physical reality of these parameters, in particular for ^{16}O a two parameter Fermi distribution is not realistic. Rather we are interested in comparing our approximate forms with exact evaluation of the eikonal amplitude since it is well known that the full eikonal calculation agrees very well with experiment in this region for these cases. We see from Fig. 1 that for moderate momentum transfer even the zero order eikonal formula agrees essentially perfectly with the full calculation, and the first order correction extends that agreement into quite small values of q. It should be emphasized that there are no free parameters in this comparison and it is an absolute comparison over some six orders of magnitude. It can

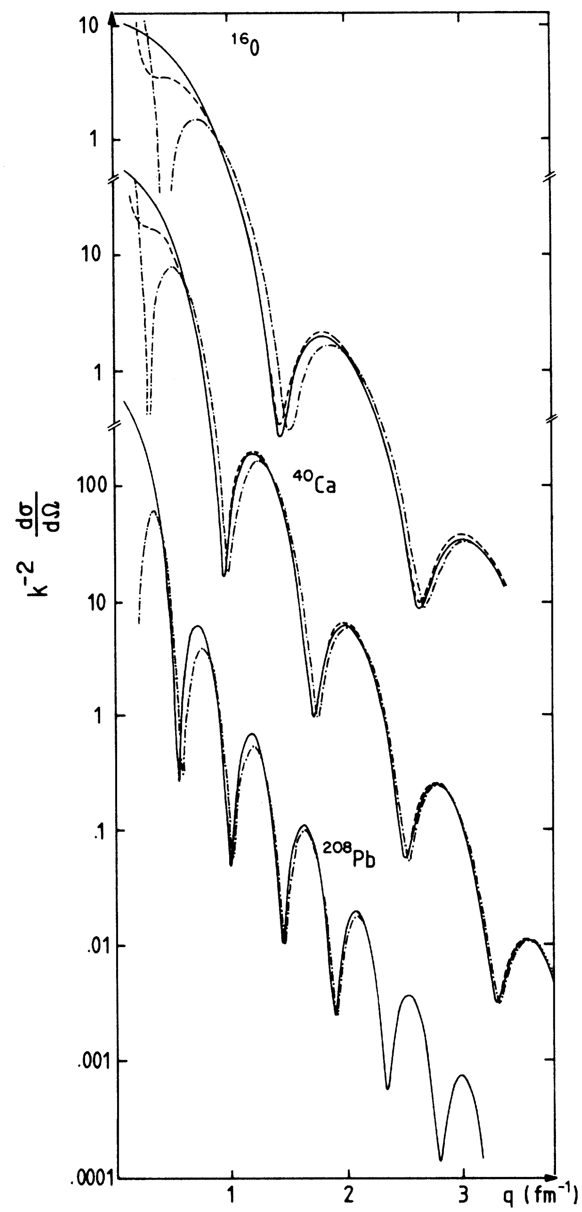

FIG. 1. Full numerical evaluation (solid line) of the eikonal cross section for proton nucleus elastic scattering compared with our leading asymptotic form (dash-dot line) and the form with first order corrections (dashed lines) as a function of momentum transfer. The vertical scale is $(1/k^2)\ (d\sigma/d\Omega)$ in arbitrary units, but for each nucleus there is no arbitrary normalization among the three curves. The scattering parameters are appropriate to 1 GeV and are given in the text. For Pb we do not show the first corrected results since they are identical to the asymptotic form except in the forward peak.

be seen in Fig. 1 that the mechanics for filling in the diffraction minima as well as the A dependence of our result is correct, as discussed in Sec. III.

The numerical success of our approximate forms even for relatively light nuclei such as ^{16}O and ^{40}Ca and into small values of momentum transfer ($q \sim 1\ \text{fm}^{-1}$) may at first seem remarkable since that success is based on the zero order or first order asymptotic expansion of the Bessel or Hankel function. It should be recalled, however, that the first correction to the asymptotic expansion of any zero order Bessel or Hankel function $Z_0(x)$ is of order $1/8x$ and the next correction is $0.07/x^2$. It is the existence of these very small numerical coefficients that makes the asymptotic expansion converge much more rapidly than one might expect. In evaluating the correction term we have also neglected the skin thickness term $\pi\beta$ compared with the radius c. Again this might not seem valid for light nuclei, but recall that $\pi\beta$ is the imaginary part of b_0 and c its real part. Therefore the error made in neglecting $\pi\beta$ is of order $(\pi\beta/c)^2$ which is negligible in the correction terms.

Beside giving insight into the major physical content of the eikonal amplitude, our result (40) or (57) can also be viewed as an analytic, unitary nonperturbative model of diffraction scattering which contains a great deal of physics. As such, it is of interest beyond the assumption used to obtain it. It will be interesting to see whether it can be usefully applied in other parts of physics in which diffraction scattering is important. These include heavy ion reactions, electron atom scattering, and particle physics.

Further afield there are other reactions and other kinds of measurements which should be amenable to our techniques. Polarization in hadron-nucleus scattering is an obvious candidate and we plan to study it. Other cases are inelastic and inclusive reactions in which "off-shell" amplitudes may also be needed. These require further study.

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APPENDIX A

We evaluate in this appendix the nonsingular part of the profile function defined in (34), (35) and its derivative at $b = b_0$. According to (37) $t(b_0)$ is written as

$$\bar{t}(b_0)=\rho_0(2\pi\beta b_0)^{1/2}f(\tau) \tag{A1}$$

with

$$f(\tau)=\frac{1}{\pi^{1/2}}\int_0^\infty \frac{dz(1+\tau z)}{z^{1/2}(1+\frac{1}{2}\tau z)^{1/2}}\left(\frac{1}{1-e^z}+\frac{1}{z(1+\frac{1}{2}\tau z)}\right). \tag{A2}$$

In order to treat the two terms in the integral of (A2) separately, we introduce a finite lower limit ϵ in the integral and do some manipulations before we take the limit $\epsilon\to 0$. The second term can be integrated and we find

$$f(\tau)\cong\frac{1}{\pi^{1/2}}\lim_{\epsilon\to 0}\left\{\frac{2}{(\epsilon+\frac{1}{2}\tau\epsilon^2)^{1/2}}+\int_\epsilon^\infty\frac{dz}{z^{1/2}}[1+\tfrac{3}{4}\tau z-\tfrac{5}{32}(\tau z)^2]\frac{1}{1-e^z}\right\},$$

where in the first term of the integral in (A2) we have expanded $(1+tz/2)^{-1/2}$. In the remaining integral we add and subtract the singular term $z^{-3/2}$ and find

$$f(\tau)\cong\frac{1}{\pi^{1/2}}\lim_{\epsilon\to 0}\left\{\frac{2}{(\epsilon+\frac{1}{2}\tau\epsilon^2)^{1/2}}-\frac{2}{\epsilon^{1/2}}+\int_\epsilon^\infty\frac{dz}{z^{1/2}}\left(\frac{1}{1-e^z}+\frac{1}{z}\right)+\int_\epsilon^\infty\frac{dz}{z^{1/2}}[\tfrac{3}{4}\tau z-\tfrac{5}{32}(\tau z)^2]\frac{1}{1-e^z}\right\}.$$

The limit $\epsilon\to 0$ can now be performed; the singular pieces in ϵ cancel and we find the following asymptotic expansion in τ:

$$f(\tau)=a_1+a_2\tau+a_2\tau^2+\cdots, \tag{A3}$$

with

$$a_1=\frac{1}{\pi^{1/2}}\int_0^\infty\frac{dz}{z^{1/2}}\left(\frac{1}{1-e^z}+\frac{1}{z}\right)=1.460, \tag{A4a}$$

$$a_2=\tfrac{3}{8}\zeta(\tfrac{3}{2})=0.980, \tag{A4b}$$

$$a_3=-\tfrac{15}{128}\zeta(\tfrac{5}{2})=-0.157. \tag{A4c}$$

The same procedure is applied for calculating the derivative of $\bar{t}(b_0)$,

$$\left.\frac{d\bar{t}}{db}\right|_{b=b_0}=2\lim_{\epsilon\to 0}\frac{d}{db}\int_{b+\epsilon\beta}^\infty\frac{r}{(r^2-b^2)^{1/2}}\bar{\rho}(r)dr\,\bigg|_{b=b_0}. \tag{A5}$$

One obtains directly, using the definition of $\bar{\rho}(r)$ in (35) and $\tau=\beta/b_0$,

$$\left.\frac{d\bar{t}}{db}\right|_{b=b_0}=\lim_{\epsilon\to 0}\left[\frac{\rho_0}{2\beta}(2b_0\beta)^{1/2}\epsilon^{-1/2}(\tau-1)+2\rho_0 b_0\int_{b_0+\epsilon\beta}^\infty\frac{r}{(r^2-b_0^2)^{3/2}}\bar{\rho}(r)dr\right]. \tag{A6}$$

Introducing in (A6) the integration variable

$$z=(r-b_0)/\beta$$

we have

$$\left.\frac{d\bar{t}}{db}\right|_{b=b_0}=\frac{\rho_0}{2\beta}(2b_0\beta)^{1/2}\lim_{\epsilon\to 0}\left\{\epsilon^{-1/2}(\tau-1)+\int_\epsilon^\infty dz\frac{1+\tau z}{z^{3/2}(1+\frac{1}{2}\tau z)^{3/2}}\left(\frac{1}{1-e^z}+\frac{1}{z(1+\frac{1}{2}\tau z)}\right)\right\}. \tag{A7}$$

The second term in the integral of (A7) can be directly integrated. In the first term we again expand the term $(1+\tau z/2)^{-3/2}$ and find

$$\left.\frac{d\bar{t}}{db}\right|_{b=b_0}\cong\frac{\rho_0}{2\beta}(2b_0\beta)^{1/2}\lim_{\epsilon\to 0}\left\{\epsilon^{-1/2}(\tau-1)+\frac{2}{3}\frac{1}{\epsilon^{3/2}(1+\frac{1}{2}\epsilon\tau)^{3/2}}+\int_\epsilon^\infty\frac{dz}{z^{3/2}}[1+\tfrac{1}{4}\tau z-\tfrac{9}{32}(\tau z)^2]\frac{1}{1-e^z}\right\}. \tag{A8}$$

After adding and subtracting the singular pieces in ϵ of the last integral, the limit $\epsilon\to 0$ can be taken with the result

$$\left.\frac{d\bar{t}}{db}\right|_{b=b_0}=\frac{1}{2\beta}\rho_0(2\pi\beta b_0)^{1/2}\times(a_0+\tfrac{1}{4}a_1\tau-\tfrac{3}{8}a_2\tau^2+\cdots) \tag{A9}$$

with

$$a_0=\frac{1}{\pi^{1/2}}\int_0^\infty\frac{dz}{z^{3/2}}\left(\frac{1}{1-e^z}+\frac{1}{z}-\frac{1}{2}\right)=-0.416. \tag{A10}$$

The length $(2\pi\beta b_0)^{1/2}\cong(2\pi\beta c)^{1/2}$ is characteristic of the profile function for values around $b=c$.

To see this we evaluate the profile function at the nuclear radius with the same method,

$$t(c)=2\rho_0\int_c^\infty \frac{r\,dr}{(r^2-c^2)^{1/2}}\frac{1}{1+e^{(r-c)/\beta}}, \tag{A11}$$

which, after introducing $z=(r-c)/\beta$ as integration variable, becomes

$$\begin{aligned} t(c) &= (2\pi\beta c)^{1/2}\frac{1}{(\pi)^{1/2}}\int_0^\infty \frac{dz(1+\tau z)}{z^{1/2}(1+\frac{1}{2}\tau z)^{1/2}}\frac{1}{1+e^z} \\ &\cong (2\pi\beta c)^{1/2}\frac{1}{(\pi)^{1/2}}\int_0^\infty \frac{dz}{z^{1/2}}(1+\tfrac{3}{4}\tau z)\frac{1}{1+e^z} \\ &= (2\pi\beta c)^{1/2}(\alpha_1+\alpha_2\tau) \end{aligned} \tag{A12}$$

with

$$\alpha_1=(1-\sqrt{2})\zeta(\tfrac{1}{2})=0.605, \tag{A13a}$$

$$\alpha_2=\frac{3}{8}\left(1-\frac{1}{\sqrt{2}}\right)\zeta(\tfrac{3}{2})=0.287. \tag{A13b}$$

By contrast for small impact parameters, the profile function becomes proportional to the nuclear radius

$$\begin{aligned} t(0) &= 2\rho_0\int_0^\infty \frac{dr}{1+e^{(r-c)/\beta}} \\ &= 2\rho_0\left(c+\int_c^\infty \frac{dr}{1+e^{r/\beta}}\right) \\ &\cong 2\rho_0(c+\beta e^{-c/\beta}). \end{aligned} \tag{A14}$$

APPENDIX B

We calculate the profile function $t^c(b)$ of (43) for the Coulomb potential $-Ze^2V_c(r)$ of the nuclear density. We want $t^c(b)$ at $b=b_0$, the singular point of the strong interaction profile function $t(b)$. We have

$$t^c(b)=2\int_0^\infty V_c((z^2+b^2)^{1/2})dz=z\int_b^R \frac{r}{(r^2-b^2)^{1/2}}V_c(r)dr, \tag{B1}$$

where R is a Coulomb cutoff radius and $V_c(r)$ is given by

$$V_c(r)=\frac{1}{r}\int_0^r \rho(r')r'^2dr'+\int_r^\infty \rho(r')r'dr', \tag{B2}$$

where the density is now normalized to one. After one integration by parts $t^c(b)$ becomes

$$t^c(b)=2\left\{\ln 2R-\ln b\int_0^b \rho(r)r^2dr-\int_b^R \ln[r+(r^2-b^2)^{1/2}]\rho(r)r^2dr+\int_b^R (r^2-b^2)^{1/2}\rho(r)r\,dr\right\}. \tag{B3}$$

As long as the integration path and the real axis do not include a singularity [we need $t^c(b)$ for values of $b=b_0-i|\epsilon|$] we can use

$$\int_0^b \rho(r)r^2dr=1-\int_b^R \rho(r)r^2dr \tag{B4}$$

and therefore (B3) can be written as

$$t^c(b)=2\left\{\ln\frac{2R}{b}-\int_b^R \ln\left[\frac{r}{b}+\frac{1}{b}(r^2-b^2)^{1/2}\right]\rho(r)r^2dr+\int_b^R (r^2-b^2)^{1/2}\rho(r)r\,dr\right\}. \tag{B5}$$

In the integrals of (B5) we can replace R by ∞ and since the integrals are finite at $b=b_0$ we can calculate $t^c(b_0)$. Changing variables to $y=(r-b_0)/\beta$ and with $\rho(r)=\rho_0/1+e^{(r-c)/\beta}$ and $\tau=\beta/b_0$ we have

$$\begin{aligned} t^c(b_0)=2\Bigg(\ln\frac{2R}{b_0}-\rho_0 b_0{}^2\beta\Bigg\{&\int_0^\infty dy\,\ln[1+\tau y+(2\tau y+\tau^2y^2)^{1/2}]\frac{(1+\tau y)^2}{1-e^y} \\ &-\int_0^\infty (1+\tau y)(2\tau y+\tau^2y^2)^{1/2}\frac{dy}{1-e^y}\Bigg\}\Bigg). \end{aligned} \tag{B6}$$

The integrals of (B6) can be expanded in powers of τ and the final result reads

$$t^c(b_0)=2\left\{\ln\frac{2R}{b_0}-\rho_0\beta^2(2\pi\beta b_0)^{1/2}\left[\tfrac{1}{2}\zeta(\tfrac{5}{2})+\mathcal{O}\left(\frac{\beta}{b_0}\right)\right]\right\}. \tag{B7}$$

For the imaginary part we find for the leading term in β/c

$$\mathrm{Im}\,t^c(b_0)=-2\arctan\frac{\pi\beta}{c}+\mathcal{O}\left(\left(\frac{\beta}{c}\right)^{7/2}\right), \tag{B8}$$

where the first term comes from the log term in (B7).

APPENDIX C

In this appendix we evaluate the various corrections to our asymptotic formula (40) for the strong form factor. First we consider corrections to the approximate evaluation of the integral (19). We consider

$$G_{s_1}(q\gamma)=e^{-i\pi/4-\gamma\bar{t}(b_0)}\left(\frac{b_0}{2\pi q}\right)^{1/2}\times\int_{-\infty}^{\infty}db\,e^{iqb}\left(1-e^{i\alpha b_0/(b_0^2-b^2)^{1/2}}\right). \quad \text{(C1)}$$

In the integral we can replace e^{iqb} by $\cos qb$. Expanding $e^{i\alpha b_0/(b_0^2-b^2)^{1/2}}$, the power series can be integrated term by term (R. G. p. 426) and we obtain the following "multiple-scattering expansion":

$$G_{s_1}(q,\gamma)=2i\frac{b_0}{q}e^{-\gamma\bar{t}(b_0)}\sum_{n=1}^{\infty}\frac{(\alpha^2 iqb_0)^{n/2}K_{(n-1)/2}(-iqb_0)}{2^{n/2}\Gamma(n/2)n!}. \quad \text{(C2)}$$

In the asymptotic expansions of the Bessel function (R. G. p. 963) we keep the first two terms

$$K_{(n-1)/2}(-iqb_0)=e^{i\pi/4}e^{iqb_0}\left(\frac{\pi}{2qb_0}\right)^{1/2}\times\left[1+\frac{i}{8qb_0}n(n-2)\right]. \quad \text{(C3)}$$

With

$$f(z)=\sum_{n=1}^{\infty}\frac{z^n}{2^{n/2}n!\,\Gamma(n/2)} \quad \text{(C4)}$$

and

$$z=(\alpha^2 iqb_0)^{1/2}, \quad \text{(C5)}$$

(C2) can be rewritten as

$$G_{s_1}(q,\gamma)=\frac{1}{q}e^{3i\pi/4-\gamma\bar{t}(b_0)}\left(\frac{2\pi b_0}{q}\right)^{1/2}\times e^{iqb_0}\left[f(z)+\frac{i}{8qb_0}\left(z^2\frac{d^2}{dz^2}-z\frac{d}{dz}\right)f(z)\right]. \quad \text{(C6)}$$

One verifies from the power series expansion (C4) that $f(z)$ is a solution of the differential equation

$$z\frac{d^3f}{dz^3}=f(z). \quad \text{(C7)}$$

We now define the function $g(z)$ by factorizing out the known leading term [cf. Eq. (40)] of the asymptotic expansion of $f(z)$,

$$f(z)=z^{1/3}e^{(3/2)z^{2/3}}g(z). \quad \text{(C8)}$$

In the variable

$$y=z^{2/3} \quad \text{(C9)}$$

the function

$$h(y)=g(z) \quad \text{(C10)}$$

can be shown from (C7) to satisfy the differential equation

$$y^{-3}h-\tfrac{1}{2}y^{-2}(3h+2h')+\tfrac{1}{5}(4h'''+18h'+27h')=0. \quad \text{(C11)}$$

The ansatz for the asymptotic solution of (C11),

$$h=\sum_{n=0}^{N}a_n y^{-n},$$

leads to the 3-term recursion relation

$$a_{n+1}=\frac{3(12n(n+1)-5)a_n+2(9-4n^2)na_{n-1}}{54(n+1)}, \quad \text{(C12)}$$

and therefore the asymptotic expansion to (C8) reads

$$f(z)=z^{1/3}e^{(3/2)z^{2/3}}\sum_{n=0}^{N}a_n z^{-2n/3}. \quad \text{(C13)}$$

By comparing (C13) with Eq. (40), the normalization is obtained to

$$a_0=(6\pi)^{-1/2}. \quad \text{(C14)}$$

Inserting the result (C13) into (C6), $G_s(q,\gamma)$ reads including the lowest order correlation to the stationary method

$$G_{s_1}(q,\gamma)=G_s(q,\gamma)\left[1-\frac{1}{(\alpha^2 iqb_0)^{1/3}}\left(\tfrac{5}{18}+\tfrac{1}{8}\alpha^2\right)\right]+\mathcal{O}((q^{-2/3})). \quad \text{(C15)}$$

We now consider the corrections arising from the singularities in the profile function which are further away from the real axis. These singularities lead to an additional b dependence of the profile function at $b=b_s$. The lowest order correction term is obtained by replacing $\bar{t}(b)$ in Eq. (16) [cf. Eq. (18)],

$$\bar{t}(b)\cong\bar{t}(b_0)+(b-b_0)\frac{d\bar{t}}{db}\bigg|_{b=b_0},$$

which amounts to replacing $g_0(b)$ by $g_1(b)$ in (19),

$$g_1(b)=g_0(b)+i\delta q(b-b_0)=-i\delta qb_0+i(q+\delta q)b-\gamma t_0(b) \quad \text{(C16)}$$

with

$$\delta q=i\gamma\frac{d\bar{t}}{db}\bigg|_{b=b_0}. \quad \text{(C17)}$$

The evaluation of the b integral in (19) can now be repeated by simply replacing q by $q+\delta q$. Expanding in (32) and (33) in the quantity $\delta q/q$ one obtains Eq. (57).

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[1]For a review of the eikonal approximation and references to the earlier literature see C. Quigg and C. J. Joachain, Rev. Mod. Phys. 46, 279 (1974). For areview of recent work on applications to p-nucleus scattering see C. Bruge, in *Proceedings of the Seventh International Conference on High-Energy Physics and Nuclear Structure*, edited by M. P. Locher (Birkhauser, Basel and Stuttgart, 1977), p. 69, ff.

[2]B. Schrempp and F. Schrempp, Report No. Th. 2573-Cern. Our geometrical interpretation of the exponential falloff in p-nucleus scattering and its relation to the electromagnetic form factor is in contradiction to the results of these authors. They find, by starting with a sharp cutoff density, both an energy and A-dependent fall off parameters ($\sim A^{1/9}/k^{2/3}$, k-incident momentum) which is of purely dynamical origin.

[3]I. Zh. Pethov, V. K. Luk-yanov, and Yu. S. Pol, Yad. Fiz. 4, 57 (1966) [Sov. J. Nucl. Phys. 4, 41 (1967)]; W. Czyz, L. Lesnick, and A. Matecki, Ann. Phys. (N.Y.) 42, 97 (1967). These authors have obtained many of the same results as we report for the electromagnetic form factor. We repeat them in order to emphasize their relation to the singularities in the nuclear density, which is the pivotal step to our understanding of the strong form factor.

[4]W. Czyz, L. Lesniak, and H. Wotek, Nucl. Phys. B19, 125 (1970).

[5]G. S. Blanpied *et al.*, Phys. Rev. Lett. 39, 1447 (1977). The reported measurements of p-^{208}Pb scattering at 800 MeV display the smooth diffractive pattern at large momentum transfers.

[6]J. Knoll and R. Schaeffer, Ann. Phys. (N.Y.) 97, 307 (1976).

[7]I.S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965), referred to here as R. G.