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Multiple scattering and few-body Hamiltonian models of nuclear reactions

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The multiple-scattering structure of various scattering operators as well as the integral equations they satisfy are investigated for Hamiltonians consisting of terms of definite but arbitrary connectivities. This is done in order to encompass the effective interactions which arise in few-body models of nuclear reactions. The counterparts of the Watson and other kinds of multiple-scattering expansions are developed under these circumstances. Connected-kernel scattering integral equations which have the unique and practical feature of being both partition labeled and possessing a multiple-scattering structure are obtained for the transition operators. These extended versions of both the multiple-scattering expansions and partition-labeled multiple-scattering integral equations are applied to the specific class of few-body Hamiltonian reaction models found by Polyzou and Redish. The concept of a well-structured reaction mechanism is introduced and it is established that for such a reaction mechanism an appropriately modified multiple-scattering picture carries over to the approximate few-body models. It is shown, for example, that for a well-structured reaction mechanism the analogs of the Born and impulse approximations emerge directly from the extended partition-labeled multiple-scattering equations in contrast to alternative formulations.

[NUCLEAR REACTIONS Multiple-scattering theory with many-body forces. Connected-kernel N-particle equations with multiple-scattering structure. Few-body models for nuclear reactions and related approximations.]

I. INTRODUCTION

Modern formulations of nuclear reaction theory below the pion production threshold involve the use of nonrelativistic many-body integral equations for the description of the N-nucleon dynamics.[1] In such formulations the major problem is the derivation of integral equations for the transition operators which have connected kernels after a finite number of iterations. The connectedness of the kernel is taken to imply its compactness and hence the possibility of the solution of these equations by standard numerical methods. These equations are quite complex in structure and their numerical solution is not feasible at present without further simplifying assumptions for $N > 4$.

Two sets of many-body equations, the Bencze-Redish-Sloan[2] (BRS) and the Baer-Kouri-Levin-Tobocman[3] (BKLT) equations, have featured prominently in recent attempts[4-6] to derive few-cluster models of nuclear reactions. In these models one truncates the space of asymptotic reaction channels to reflect the dominant physical features of the system. Such truncations allow a reduction in the number and dimensionality of the coupled integral equations with which one has to deal.

It should be stressed that neither of these two sets of equations exhibits an explicit multiple-scattering structure. The lack of such structure suppresses certain physical features of the problem and complicates the introduction of related approximations. By way of contrast, Yakubovskii-type theories[1,7,8] have a multiple-scattering structure but involve operators which are only very indirectly related to the physical transition amplitudes. The major objective of this article is to develop scattering integral equations and few-body models of nuclear reactions which combine some of the most attractive features of the BKLT, BRS, and Yakubovskii methods, and, in particular, which possess an explicit multiple-scattering structure.

The more traditional formulations of multiparticle scattering are usually based on very specific nuclear reaction models.[9] Here we refer, for example, to the optical model, distorted-wave Born approximation, the multiple-scattering and R-matrix formalisms. These models have been highly successful in practical applications and have contributed greatly to our present understanding of the structure and interactions of nuclei. However, they suffer some serious limitations which are often overlooked in practice or absorbed into one phenomenological device or another. These limitations can be understood and overcome only within the framework of a many-body theory involving well-defined scattering integral equations such as those referred to previously. In turning to such equations it is important not to ignore those features responsible for the successes of the conventional methods.

In view of this, it is of interest to consider the

relationships among various conventional reaction models[9] and the approximate theories[4-6] based on connected-kernel formalisms. Work in this direction can be found in Ref. 10, where the multiple-scattering structure of the transition operators is considered in terms of the connectivity expansions of the various operators which appear in the usual formulations.[9,11] (See also Ref. 8.) Another major result of Ref. 10 is the derivation of a new set of connected-kernel equations for the transition operators which possess the BRS kernels but also exhibit a multiple-scattering structure. This was carried out in Ref. 10 assuming only pairwise interactions. The present paper extends this work to include the effect of channel-space truncation on the various cluster expansions, in particular, the form of the multiple-scattering series for the two-fragment collision operator, the disconnected structure of the approximate integral equations and related problems. The detailed study of these questions, which involves some aspects of the work of Ref. 10 when arbitrary many-body forces are introduced, raises several important new problems. It is usually assumed that the particles of the system interact only via pairwise forces. This assumption leads to simplifications such as the so-called minimally coupled BRS equations, that is, equations which couple explicitly only the transition operators for two-fragment collisions.[12,13] In contrast, the recently proposed few-body reaction models[4-6] as well as some of the traditional theories, e.g., the optical potential,[14] assume that the many-body system is composed of $n<N$ interacting clusters. The cluster-cluster interaction is a typical many-body force when viewed microscopically. Also, the clusters themselves possess an internal structure which can be modified in the collision process. These features are absent from the usual many-body problem with only pairwise interactions and indicate the type of extensions required to incorporate the few-body reaction models into the theory. The investigation of these problems represents the major portion of this study.

This paper is organized as follows. In Sec. II we introduce the partition notation which greatly simplifies the subsequent discussion. This notation has now become a standard part of the many-body scattering theory[1-7] and its discussion has been abbreviated accordingly. In Sec. III we discuss the various cluster expansions of the elastic transition operator for two-fragment collisions when many-body forces are present. In Sec. IV we derive a new set of N-particle connected-kernel equations which exhibits both a multiple-scattering structure and incorporates many-body forces. The transition operators which enter into these equations have precisely the same off-shell extensions as in the usual theories. These equations are called the partition-labeled equations with multiple-scattering structure, or briefly, the PLMS equations. In Sec. V we review the notion of a reaction mechanism (RM) and the related few-body reaction model as given in Refs. 4 and 6. The reduced form of the integral equations for the RM transition operators is derived. Also, a new form of the imbedding equations, i.e., the equation imbedding the approximation in the full or exact theory, is presented. In Sec. VI we use the results of Secs. III and IV to obtain RM scattering integral equations with a multiple-scattering structure, the RM-PLMS equations. Our results are summarized in Sec. VII. The detailed derivations of the results of this paper employ the full battery of technical machinery of contemporary many-particle scattering theory.[1,4,13,15] Consequently, some of these derivations have been placed in four Appendices in order not to obscure the principal ideas presented in the text.

II. NOTATION

In this section we briefly review the partition notation of many-body scattering theory. We start by introducing the concept of a *partition* a as the grouping of the N particles into n_a distinct clusters. The arrangement of the particles within the clusters and the order of clusters within the partition are irrelevant. The partitions for which $n_a=1$ and $n_a=N$ are unique and will be denoted as 1 and 0, respectively. In the following we use the Latin letters $a, b, c, \ldots$ to denote partitions of the system. The set of all partitions of the N-body system will be denoted $\bar{\mathfrak{a}}_0$. The Greek letters $\alpha, \beta, \gamma, \ldots$ are used to designate the two-cluster partitions and the set of all such partitions will be denoted by $\mathfrak{C}$. A partition which can be identified with a physical asymptotic state of the system (i.e., where all clusters of that partition are bound) is said to be *stable*.

The partition indexing of various operators appearing in many-body scattering theory is facilitated by the introduction of several ideas. We say that a partition b is *contained in* another partition a, written $b\subset a$, if b can be obtained from a by subdividing one or more of its clusters and we write $b\subseteq a$ when we include the possibility of equality. The alternative possibility, where b is *not contained in or equal to* a is denoted by $b\nsubseteq a$. These relationships are conveniently represented by the matrices Δ and $\bar{\Delta}$ with the elements[12]

$$\Delta_{a,b}=1, \quad b\subseteq a \qquad (2.1)$$
$$=0, \quad \text{otherwise}$$

and

$$\bar{\Delta}_{a,b} = 1 - \Delta_{a,b} . \tag{2.2}$$

It can be shown that the inverse of Δ exists both on the entire partition set $\mathfrak{A}_0$ as well as any of its subsets.[4,12,13,15,16]

The N-particle Hamiltonian H is taken to be the sum of the kinetic energy operator H_0 and interaction V:

$$H = H_0 + V . \tag{2.3}$$

In general the Hamiltonian possesses the decomposition[4]

$$H = \sum_a [H]_a , \tag{2.4}$$

where $[H]_a$ is the a-connected part of H. We recall that an operator $\mathcal{O}$ is said to be *a-connected* if it commutes with the generators of translations along the intercluster coordinates of partition a and no others, and if its momentum space matrix elements have the structure

$$\langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_N | \mathcal{O} | \vec{p}_1' \cdots \vec{p}_N' \rangle = \delta(\vec{P}_1 - \vec{P}_1')\delta(\vec{P}_2 - \vec{P}_2') \cdots \delta(\vec{P}_{n_a} - \vec{P}_{n_a}') \mathcal{O}_a(\vec{k}_{int} | \vec{k}_{int}' | \vec{P}_1 \vec{P}_2 \cdots \vec{P}_{n_a}) , \tag{2.5}$$

where $\vec{p}_i$ is the momentum of particle i, P_i refers to the center of mass (c.m.) momenta of the i^{th} cluster, $\vec{k}_{int}$ represents, collectively, the independent internal momenta of the clusters of the partition a, and the functions $\mathcal{O}_a$ do not contain any δ-function singularities. Note that $H_0 = [H]_0$.

The interaction *internal* to the partition a is

$$V_a \equiv \sum_b \Delta_{a,b}[V]_b , \tag{2.6}$$

and the partition Hamiltonian is

$$H_a = H_0 + V_a . \tag{2.7}$$

Note that for $b \neq 0$, $[H]_b = [V]_b$. If a is a stable partition, H_a is a Hamiltonian describing a physical asymptotic state. The interaction V^a *external* to partition a is

$$V^a = H - H_a = \sum_b \bar{\Delta}_{a,b}[V]_b . \tag{2.8}$$

It should be noted that the partition sums in Eqs. (2.6)–(2.8) are usually severely restricted. For example, if only pair interactions are included, the sums are restricted to the subset of $(N-1)$-cluster partitions.

III. MULTIPLE SCATTERING EXPANSIONS

The multiple-scattering series for various transition operators have been studied extensively for a system with no many-body forces.[8,10,11] In this section we generalize these results to include explicitly the effect of multiparticle forces. The transition operator for the elastic scattering of two bound fragments corresponding to the two-cluster partition α is

$$T_{\alpha,\alpha} = V^\alpha + V^\alpha G V^\alpha , \tag{3.1}$$

where

$$G = (z - H)^{-1} , \tag{3.2}$$

and z is the parametric energy the dependence upon which we suppress.

Since the Hamiltonian possesses the cluster decomposition (2.4), $T_{\alpha,\alpha}$ can be written as the sum of partition-indexed operators

$$T_{\alpha,\alpha} = \sum_a T_a(\alpha) , \tag{3.3}$$

where

$$T_a(\alpha) \equiv [V]_a^\alpha (1 + G V^\alpha) , \tag{3.4}$$

and $[V]_a^b = [V]_a \bar{\Delta}_{b,a}$ for arbitrary partitions a, b. Let

$$t_a(\alpha) = (1 + [V]_a^\alpha G_\alpha [a])[V]_a^\alpha , \tag{3.5a}$$

so that

$$t_a(\alpha) = [1 + t_a(\alpha) G_\alpha][V]_a^\alpha , \tag{3.5b}$$

where

$$G_\alpha = (z - H_\alpha)^{-1} \tag{3.6}$$

and

$$G_\alpha[a] = (z - H_\alpha - [V]_a^\alpha)^{-1} . \tag{3.7}$$

Multiplying Eq. (3.4) by $[1 + t_a(\alpha) G_\alpha]$ on the left, and using Eqs. (3.1) and (3.3) we obtain

$$T_a(\alpha) = t_a(\alpha)\left(1 + \sum_b G_\alpha T_b(\alpha) \bar{\delta}_{a,b}\right), \tag{3.8}$$

where $\bar{\delta}_{a,b} = 1 - \delta_{a,b}$. Equations (3.8) represent the generalization of the Watson[11] multiple-scattering theory to arbitrary two-cluster partitions and potentials. As in the case of pairwise interactions discussed in Ref. 10, Eqs. (3.8) do not represent a set of well-defined integral equations since their kernels may not be connected upon iteration for $N \geq 4$. This means that the numerical solution of Eqs. (3.8) is not possible since the operator inverses needed for the solution are not defined in general. However, in contrast to the case of pair interactions, the existence of many-body forces makes it possible for Eqs. (3.8) to have a connected kernel upon iteration in certain cases. Also, Eqs. (3.8) represent a correct formal relationship between the $t_a(\alpha)$ and $T_a(\alpha)$ operators and yield a generalization of the Watson theory to few-cluster reaction models. We consider this question fur-

ther in Sec. VI.

At this point it is interesting to consider the physical interpretation of the operators $t_a(\alpha)$ and $T_a(\alpha)$. This is difficult to do in general, but in a closure type limit, where, e.g., we replace G_α by G_0 in Eqs. (3.5b) and (3.8), $t_a(\alpha)$ corresponds to scattering of particle internal to the clusters of partition a by the potential $[V]_a$. Then for $a=(12)3\cdots N$, $t_a(\alpha)$ represents the scattering of particles 1 and 2 from each other and for $b=(123)4\cdots N$, $t_b(\alpha)$ corresponds to the scattering of three free particles 1, 2, and 3 by a three-body force $[V]_b$, and so on. It should be noted that although $t_b(\alpha)$ refers to a three-body problem when $G_\alpha=G_0$, Eq. (3.5a) can be solved numerically. This is because $[V]_b$ couples all three particles and the connectivity questions appearing in a three-body problem with pairwise forces do not arise. The physical interpretation of the $T_a(\alpha)$ operator can be inferred from the iterated form of Eq. (3.8). There products of the form $t_a(\alpha)G_\alpha t_b(\alpha)\cdots$ appear and $T_a(\alpha)$ thus represents the part of $T_{\alpha,\alpha}$ in which the interaction $[V]_a$ acted last.

Other operators of a structure similar to $t_a(\alpha)$ can be constructed. Let us consider now the operator[10]

$$t_a^\alpha = V_a^\alpha(1+G_aV_a^\alpha), \tag{3.9}$$

where, in general,

$$V_a^b=\sum_c [V]_c^b\Delta_{a,c} \tag{3.10}$$

and $G_a=(z-H_a)^{-1}$. From Eq. (3.9) and the connectivity expansion of G_a it follows that t_a^α can be written as the sum of operators of definite connectivities, viz.,

$$t_a^\alpha=\sum_d \Delta_{a,d}X_d^\alpha. \tag{3.11}$$

Since $t_1^\alpha=T_{\alpha,\alpha}$, we have from Eq. (3.11)

$$T_{\alpha,\alpha}=\sum_d X_d^\alpha. \tag{3.12}$$

It can be shown that X_d^α is the d-connected part of $T_{\alpha,\alpha}$.[13] Equation (3.11) can be inverted to yield

$$X_d^\alpha=\sum_a (\Delta^{-1})_{d,a}t_a^\alpha. \tag{3.13}$$

An important example of X_d^α obtains when d is an $(N-1)$-cluster partition i'. Then

$$t_{i'}^\alpha=\bar{\Delta}_{\alpha,i'}t_{i'},$$

where $t_{i'}$ is the two-particle transition operator for the scattering of the particle pair i' and $X_{i'}^\alpha$ is proportional to $t_{i'}$.

Evidently, Eqs. (3.9)–(3.12) provide hierarchic expansions of the transition operator $T_{\alpha,\alpha}$ in terms of operators of definite connectivities. Their usefulness lies in the fact that it is possible to derive a connected integral equations for the t_a^α operators, and these equations can be systematically approximated.[10]

We now note that Eq. (3.13) can be used as the definition of the X-operator, given t_a^α. For example, in Ref. 10 it is suggested that one can define, alternatively,

$$\bar{t}_a^\alpha \equiv V_a^\alpha[1+(G_\alpha{}^{-1}-V_a^\alpha)^{-1}V_a^\alpha] \tag{3.14}$$

and

$$\bar{X}_d^\alpha \equiv \sum_a (\Delta^{-1})_{d,a}\bar{t}_a^\alpha. \tag{3.15}$$

In contrast to the X_d^α operators defined above, the $\bar{X}_d^\alpha$ are not necessarily d connected. However, since $\bar{t}_1^\alpha=T_{\alpha,\alpha}$ we obtain as before [cf. Eq. (3.12)]

$$T_{\alpha,\alpha}=\sum_d \bar{X}_d^\alpha. \tag{3.16}$$

As pointed out previously,[10] Eq. (3.16) is the generalization of the correlative expansion of Ernst *et al.*[17] to arbitrary two-cluster partitions. The major difficulty with this expansion is that all of the operators $\bar{t}_a^\alpha$ are N-body operators and it is not clear how one can find connected-kernel equations for them.

This completes our investigation of the various connectivity expansions of the elastic transition operator $T_{\alpha,\alpha}$. Apart from the generalization of the Watson series contained in Eqs. (3.3)–(3.8), the inclusion of many-body forces in the problem is seen to be straightforward.

Before proceeding further, it is interesting to consider how the various partition sums of this section can be restricted if some terms of the connectivity expansion of the interaction V are allowed to vanish. To do this, define $\hat{\alpha}$, the *interaction set*, to be the set of all partitions $a\in\tilde{\alpha}_0$, $a\neq 0$ such that $[V]_a\neq 0$. The set $\hat{\alpha}$ has a very specific structure. Normally $\hat{\alpha}$ is composed of partitions such that only one of its clusters contains more than one particle. For example, in the case of pairwise interactions, the partitions in $\hat{\alpha}$ are of the form $(i_1i_2)i_3\cdots i_N$, where $\{i_j\}$ denotes any ordering of the integers $1,2,\ldots,N$. However, in the approximate few-body theories,[4-6] the interaction set may also contain partitions in which several clusters contain more than one particle, e.g., as in $(12)345(678)\cdots N$. It is in the latter case that the partition notation of the many-body scattering theory becomes important. Then it follows that the operator $t_a(\alpha)$ defined by Eq. (3.5a), and consequently $T_a(\alpha)$, vanishes unless $a\in\hat{\alpha}$. Similarly, from the definition of V_a it follows that the partition sum in Eq. (3.11) is

restricted to $d \in \mathfrak{a}(a)$. Here $\mathfrak{a}(a)$ is the set

$$\mathfrak{a}(a) = \{d \mid \exists\, b \in \hat{\mathfrak{a}},\ b \subseteq d \subseteq a\}. \tag{3.17}$$

In particular, if $\hat{\mathfrak{a}}$ contain all possible pairwise interactions, $\mathfrak{a}(1)$ contains all partitions from $\tilde{\mathfrak{a}}_0$ excluding the 0-partition.

From the above it is also obvious that *all* partition sums in a theory defined by the interaction set $\hat{\mathfrak{a}}$ must be restricted to the set $\mathfrak{a}(1)$ and the 0-partition. This result has some important practical consequences, which will be discussed in the latter parts of this paper.

IV. SCATTERING EQUATIONS

In the preceding section we have discussed the multiple-scattering structure of the elastic transition operator $T_{\alpha,\alpha}$. We now wish to derive connected-kernel integral equations for the general transition operators

$$T_{a,b} = V^a + V^a G V^b, \tag{4.1}$$

which reflect this structure. In order to do this we require alternative representations of the interaction terms which we develop next.

Let us introduce the diagonal matrix C with the components[13,15]

$$C_a \equiv -(\Delta^{-1})_{1,a}, \quad a \neq 1. \tag{4.2}$$

It can be shown that $C_a = (-1)^{n_a}(n_a - 1)!$. From the definition (4.2) and $\Delta^{-1}\Delta = I$ we obtain the sum rule

$$\sum_a C_a \Delta_{a,b} \bar{\delta}_{a,1} = \bar{\delta}_{b,1}. \tag{4.3}$$

From the definition (2.6) of V_a and Eq. (4.3) it follows that

$$\sum_a \bar{\delta}_{a,1} C_a V_a = \sum_b \bar{\delta}_{b,1}[V]_b = V - [V]_1. \tag{4.4}$$

In a similar manner we find the representation

$$V^a = \sum_b C_b \{V_b^a + \bar{\Delta}_{a,1}[V]_1\} \bar{\delta}_{b,1}. \tag{4.5}$$

Thus, in accommodating the fully connected interaction $[V]_1$ we pick up an effective modification of the V_b^a operator. We then define

$$\mathcal{U}_b^a \equiv C_b \{V_b^a + \bar{\Delta}_{a,1}[V]_1\} \bar{\delta}_{b,1}. \tag{4.6}$$

From the sum rule (4.5) and the resolvent identities for G in terms of the G_a's one obtains from (4.1) the Bencze-Redish-Sloan (BRS) equations in their so-called precursor form[4]

$$T_{a,b} = \sum_c \mathcal{U}_c^a G_c \{G_b^{-1} + T_{c,b}\}, \tag{4.7}$$

or, in matrix form,

$$T = \mathcal{U}\hat{G}\{s\hat{G}^{-1} + T\}, \tag{4.8}$$

where $(\hat{G})_{a,b} = \delta_{a,b} G_a$ and $s_{a,b} = 1$.

Equation (4.6) can be reduced to a form which explicitly contains the subsystem amplitudes.[13] Let us first define the matrix

$$\Gamma \equiv Q_2 \Delta^t C Q_1, \tag{4.9}$$

the operators

$$\mathfrak{M} \equiv \bar{\Delta} P_1 [V]_1 s Q_1 \tag{4.10}$$

and

$$M_{b,a} = [V^b G G_0^{-1}]_a \bar{\delta}_{1,a}, \tag{4.11}$$

where P_i is the projector onto the set of i-cluster partitions, $Q_i = 1 - P_i$, $[\ ]_a$ denotes the a-connected part of the operator in brackets, and Δ^t is the transpose of Δ. We note that Γ relates only partitions with more than two clusters while $M_{b,a}$ can be expressed in terms of the subsystem amplitudes.[2,12,13] It is shown in Appendix A that Eq. (4.8) can be reduced to the compact form

$$T = (M + \mathfrak{M}) G_0 \{Q_1 s \hat{G}^{-1} + (P_2 + \Gamma) T\}, \tag{4.12}$$

which represents the generalization of the results of Refs. 2, 12, and 13 to include an interaction with arbitrary connectivity structure. In Appendix A we also show that Eq. (4.12) has a connected kernel after one iteration. In the absence of multiparticle forces $\mathfrak{M} = 0$ and $MG_0\Gamma T = 0$ so that (4.12) are then essentially the standard minimally coupled BRS equations.[2,12]

Since $\mathfrak{M}$ is connected, its matrix elements possess no δ-function singularities apart from an overall momentum conservation δ function. Furthermore, the second term in the inhomogeneity of Eq. (4.12) vanishes half-on-shell when operating on a two-cluster asymptotic state. This suggests that we introduce the operator

$$\tilde{T} = M G_0 \{Q_1 s \hat{G}^{-1} + (P_2 + \Gamma)\tilde{T}\}, \tag{4.13}$$

which differs from the transition operator T by a connected piece $\tilde{T}_1$ which satisfies

$$\tilde{T}_1 = \mathfrak{M} G_0 \{Q_1 s \hat{G}^{-1} + (P_2 + \Gamma)\tilde{T}\} + (M + \mathfrak{M}) G_0 (P_2 + \Gamma)\tilde{T}_1. \tag{4.14}$$

The fact that all the disconnected substructure of T is contained in $\tilde{T}$ is exploited next.

Consider the operator $[(\bar{\tau})_{a,b} = \bar{\tau}^{a,b}]$

$$\bar{\tau} \equiv \tilde{T} - V_M, \tag{4.15}$$

where $(V_M)_{a,b} = V_b^a$. From Eq. (4.13) it follows that $\bar{\tau}$ satisfies an equation of the form

$$\bar{\tau} = W_{MS} + M G_0 (P_2 + \Gamma)\bar{\tau}. \tag{4.16}$$

It is shown in Appendix B that the inhomogeneous term W_{MS} can be written as

$$W_{\rm MS}^{a,b}=\sum_f W^{a,b}(f), \tag{4.17}$$

where $W^{a,b}(f)$ is the f-connected part of $\bar{\tau}^{a,b}$. Since $\bar{\tau}^{\alpha,\alpha}=\bar{T}_{\alpha,\alpha}$, the results of Sec. III imply that for $a\neq 1$

$$W^{\alpha,\alpha}(a)=X_a^\alpha . \tag{4.18}$$

Alternatively we can define instead of $\bar{\tau}$ the operator[10]

$$\tau=T-V_M . \tag{4.19}$$

It is also shown in Appendix A that τ satisfies

$$\tau=W_{\rm MS}+\mathfrak{M}G_0[Q_1 s\hat{G}^{-1}+(P_2+\Gamma)V_M] + (M+\mathfrak{M})G_0(P_2+\Gamma)\tau . \tag{4.20}$$

Since the first iterate of Eq. (4.20) has a connected kernel, the disconnected structures of τ and $\bar{\tau}$ are the same.

Clearly, the operator $W^{a,b}(f)$ contains within itself a multiple-scattering structure.[10] Also, all the disconnected substructure of the operators τ and $\bar{\tau}$ is contained in the inhomogeneous $W_{\rm MS}$ term. For this reason we call Eqs. (4.16) and (4.20) partition-labeled equations with multiple-scattering structure (PLMS). These equations represent the generalizations of the PLMS equations of Ref. 10 to include many-body forces. It should be noted that the operators τ and $\bar{\tau}$ differ from each other only in the way the fully connected interaction is incorporated. The operators $\bar{\tau}$ and $\bar{T}$ are independent of $[V]_1$ so that $\bar{T}_1$ contains the full effect of $[V]_1$. This lends itself to a possible perturbative treatment of the effects of $[V]_1$ using Eq. (4.14).

The τ operator is more directly related to the T matrix and the development of the PLMS equation given in Ref. 10; in particular $\tau^{\alpha,\alpha}=T_{\alpha,\alpha}$ on-shell. One advantage of the τ operator is the fact that it facilitates various approximation methods, such as the development of the optical potential.[10] Also, because of its direct relationship to the T matrix, Eq. (4.19), comparisons with standard methods in nuclear reaction theory are made more transparent. Finally, we remark that the operators $W^{a,b}(f)$ satisfy important identities which are derived in Appendices B and C, and constitute the generalization of the results of Ref. 10 to include many-body forces.

V. REACTION MECHANISM AND FEW-BODY REACTION MODELS

In this section we consider the applications of the results of the previous two sections to the few-body Hamiltonian formalism of Ref. 4. We use the notation of Ref. 6 in what follows.

We start by considering the partition Hamiltonian H_a. If a is a stable partition, the *maximally connected eigenstates* $|\phi_a(\nu_a)\rangle$ of H_a (which are those with all n_a clusters in bound configurations) correspond to physical asymptotic states.[4] The *scattering eigenstates* $|\psi_a^{(+)}(\nu_b)\rangle$ evolve from maximally connected eigenstates $|\phi_b(\nu_b)\rangle$, with $b\subset a$, in the infinite past. Here the *channel* ν_a is the collection of the internal quantum numbers of the clusters of a stable partition a. The bound and scattering eigenstates of H_a will be denoted collectively $|\hat{\phi}_a(\nu_b)\rangle$, with $b\subseteq a$. The dependence on the c.m. momenta of each of the n_b clusters will be suppressed.

Let $\bar{\mathfrak{a}}_0$ be the set of all partitions $a\in\bar{\mathfrak{a}}_0$ with $n_a\geq 2$ and $\mathfrak{a}_0$ be the subset of $\bar{\mathfrak{a}}_0$ of stable partitions. Then the set of physical scattering channels is

$$A_0=\{\nu_b,\ \text{all } b\in\bar{\mathfrak{a}}_0\}. \tag{5.1}$$

Following Ref. 4, the reaction mechanism (RM) is then defined as a particular subset A of A_0. The notion of the RM becomes meaningful only if A is identified with the set of dominant reaction channels for the system and this identification has been used to simplify the scattering equations.

It is very useful to introduce the idea of a set A of partitions which is associated with a given RM. The set $\mathfrak{a}$, which we call the *reaction set*,[6] is defined as

$$\mathfrak{a}=\{a\,|\,a\neq 1\ a\supseteq b, \exists\ \nu_b\in A\}, \tag{5.2}$$

and contains all partitions $a\in A(1)$ except $a=1$.

The idea of an RM can be exploited to derive approximate few-body reaction models.[4-6] We next outline the analysis of Refs. 4 and 6 for obtaining approximate scattering equations for the transition operators. Consider the projector

$$\mathcal{P}_a(\nu_b)=\left\{\prod_{i=1}^{n_b}\int d\vec{P}_i\right\}|\hat{\phi}_a(\nu_b)\rangle\langle\hat{\phi}_a(\nu_b)| \tag{5.3}$$

onto the Hilbert space $\mathcal{K}_a(\nu_b)$. The resolution of the identity on the N-particle Hilbert space $\mathcal{K}_N$ can then be written as the direct-sum decomposition

$$I=\sum_b\sum_{\nu_b\in A_0}\Delta_{a,b}\mathcal{P}_a(\nu_b), \tag{5.4}$$

in terms of the $\mathcal{K}_a(\nu_b)$ subspaces.

In the absence of N-body forces the counterpart of Eq. (4.4) for the Hamiltonian H_a is[4]

$$H=\sum_a{}' C_a H_a . \tag{5.5}$$

The prime in the sum in Eq. (5.5) indicates that the $a=1$ term has been omitted. We can then introduce an approximate Hamiltonian $H(A)$ if we

use (5.5) with projected partition Hamiltonians[4]

$$H(A)=\sum_{a\in A} C_a H_a \mathcal{P}_a(A), \tag{5.6}$$

where

$$\mathcal{P}_a(A)=\sum_b \sum_{\nu_b\in A} \Delta_{a,b}\mathcal{P}_a(\nu_b) \tag{5.7}$$

and $\mathcal{P}_a(A)\equiv 0$ for $a\notin \mathfrak{a}$. Equations (5.5) and (5.6) suggest a definition of a projected partition Hamiltonian[4,6]:

$$H_a(A)=\mathcal{P}_a(A)H_a=H_a\mathcal{P}_a(A). \tag{5.8}$$

The model or RM Hamiltonian $H(A)$ possesses the cluster decomposition

$$H(A)=\sum_a [H(A)]_a, \tag{5.9}$$

where $[H(A)]_a$ is the a-connected part of $H(A)$ and $[H(A)]_a\equiv 0$ for $a\notin \mathfrak{a}$. It is shown in Ref. 4 that the RM partition Hamiltonian also has the cluster decomposition

$$H_a(A)=\sum_c \Delta_{a,c}[H(A)_c]. \tag{5.10}$$

Clearly $H_a(A)\equiv 0$ for $a\notin \mathfrak{a}$. Corresponding to Eq. (2.8) we write

$$H(A)=H_a(A)+V^a(A), \tag{5.11}$$

where

$$V^a(A)=\sum_c \bar{\Delta}_{a,c}[H(A)_c]. \tag{5.12}$$

We note that even if $V(A_0)=V$ contains only pair interactions, $V(A)$ has, in general, terms of varying connectivities. In the case of a *few-body reaction mechanism*[4] (i.e., $n_a\leq 4$, all $a\in \mathfrak{a}$), the sums in (5.10) and (5.12) are restricted to a few highly connected terms. Equations (5.9)–(5.12) reduce to the usual partition decomposition (cf. Sec. II) of the full Hamiltonian when $A=A_0$.

The operator $V^a(A)$ represents an (approximate) interaction among the clusters of partition a and generates full N-body dynamics except when $\mathfrak{a}$ is restricted to a single two-cluster partition. Then $V^a(A)\equiv 0$ and the two clusters cannot interact.[4,6] This corresponds to the *trivial reaction mechanism* of Ref. 4.

Some of the essential aspects of the set of partitions associated with an RM can be represented in a concise manner if we make use of the devices of the union and the intersection of partitions.[4,18] The former idea has been defined and used extensively in the Appendices. The *intersection*, $a\cap b$, of two partitions a and b satisfies $a\cap b\supseteq c$ for all partitions c such that $c\subseteq a$, $c\subseteq b$. In other words $a\cap b=b\cap a$ is the partition with the fewest number of clusters which is contained in both a and b. The unions and intersections of several partitions have obvious meanings. Thus, if we call

$$\hat{0}\equiv \bigcap_{a\in A} a,$$

$$\hat{1}\equiv \bigcup_{a\in A} a,$$

then a *nontrivial* RM is one for which $\hat{1}=1$ and a *well-structured* RM is one for which $\hat{0}\in \mathfrak{a}$.

The partitions $\hat{0}$ and $\hat{1}$ can be called the *lower* and *upper bounds* of the set $\mathfrak{a}$. We note that we need not have $\hat{0}, \hat{1}\in \mathfrak{a}$. The partition $\hat{0}$ has been introduced and called the *dominant partition* in Ref. 19 in a somewhat different context. The meaning of the partitions $\hat{0}$ and $\hat{1}$ becomes more evident if we realize that $[H(A)]_a\equiv 0$ if either $\hat{0}\not\subseteq a$ or $a\not\subseteq \hat{1}$. Thus, $\hat{0}$ represents the most disconnected partition of the model; its clusters behave as indivisible objects within the framework of this approximation. In the same sense, $\hat{1}$ represents the most connected partition. If $\hat{1}\neq 1$, the system can be regarded as several independent smaller systems; this corresponds to the idea of the trivial reaction mechanism.[4,6] Reasons for confining oneself to a nontrivial, well-structured RM are discussed in Sec. VI. Also, it is important to note that, in contrast to the exact problem, the structure of the Hamiltonian $H(A)$ is such that $[H(A)]_a\neq 0$ for partitions a composed of several distinct particle clusters. This difference between the exact and approximate problems is illustrated in Fig. 1 and Fig. 2.

In what follows we always assume that the RM in question is nontrivial. Also, for the sake of simplicity, we suppose that there are no N-body forces. This restriction is easily removed.[4]

In analogy with the exact problem we define the RM Green's functions[4]

$$G(A)=[z-H(A)]^{-1}, \tag{5.13}$$

$$G_a(A)=[z-H_a(A)]^{-1}. \tag{5.14}$$

It is important to realize that $G_a(A)$ and $G(A)$ are operators in the *full Hilbert space*. Specifically,

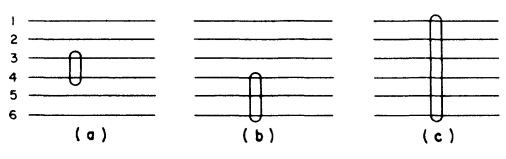

FIG. 1. *Exact Problem*. Horizontal lines represent particles and the blobs denote the interaction of the enclosed particles. (a) A pair interaction of connectivity 1 2 (3 4) 5 6. (b) A 3-particle force of connectivity 1 2 3 (4 5 6). (c) A fully connected force of connectivity (1 2 3 4 5 6).

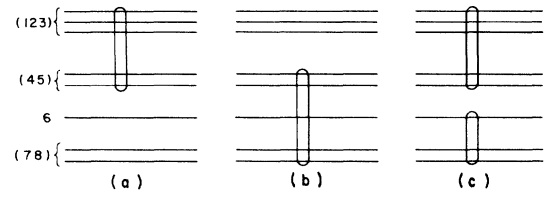

FIG. 2. *Approximate Problem*. The line groupings correspond to bound-particle clusters. The $\hat{0}$ partition is (1 2 3) (4 5) 6 (7 8). (a) The blob represents an effective two-body, 5-particle force of connectivity (1 2 3 4 5) 6 (7 8). (b) An effective three-body force. (c) A force of the type not found in the standard exact problem in that it represents a simultaneous action of a pair of two-body interactions. Diagrams of this type arise from the projectors $P_a(\nu_a)$, $a \supset \hat{0}$.

we see that

$$G_a(A) = \mathcal{P}_a(A)G_a(A_0) + \frac{Q_a(A)}{z}, \tag{5.15}$$

where $Q_a(A) = 1 - \mathcal{P}_a(A)$.

From Eqs. (5.13) and (5.14) we obtain the resolvent identity

$$G(A) = G_a(A) + G_a(A)V^a(A)G(A). \tag{5.16}$$

The RM transition operators $T_{a,b}(A)$ are defined in a manner consistent with the interpretation of $G(A)$, namely,[4,6]

$$T_{a,b}(A) = V^a(A) + V^a(A)G(A)V^b(A), \tag{5.17}$$

in analogy with Eq. (4.1). The operators $T_{a,b}(A)$ satisfy the RM-BRS equation[4]

$$T_{a,b}(A) = \sum_c \mathcal{V}_c^a(A)G_c(A)[G_b^{-1}(A) + T_{c,b}(A)], \tag{5.18a}$$

or in matrix form

$$T(A) = \mathcal{V}(A)G(A)[s\hat{G}^{-1}(A) + T(A)], \tag{5.18b}$$

where

$$\mathcal{V}_c^a(A) = V_c^a(A)C_c \tag{5.19}$$

and

$$V_c^a(A) \equiv \sum_b{}' \bar{\Delta}_{a,b}[H(A)]_b\Delta_{c,b}. \tag{5.20}$$

The main result of Ref. 4 is that Eqs. (5.18) are connected-kernel equations and their solutions satisfy a unitarity condition consistent with the RM.

The number of equations (5.18) which are coupled together in practice depends upon the property

$$V_c^a(A) = 0, \text{ if } c \notin \mathfrak{a}, \tag{5.21}$$

which follows directly from definition (5.20). Because of (5.21) it is necessary to consider only that subset of Eqs. (5.18) which couple together those $T_{a,b}(A)$ operators with $a, b \in \mathfrak{a}$. Evidently only the transition operators indexed by reaction set partitions are of physical relevance in the model defined by $H(A)$. These remarks can be made manifest by introducing a projector $P(A)$ onto the reaction set $\mathfrak{a}$ of partitions. By (5.19) and (5.21) we infer

$$\mathcal{V}(A)P(A) = \mathcal{V}(A). \tag{5.22}$$

Thus using (5.22) the set of integral equations (5.18) can be reduced to a set for the reaction set operators $P(A)T(A)P(A)$:

$$P(A)T(A)P(A) = P(A)\mathcal{V}(A)\hat{G}(A) \times [s\hat{G}^{-1}(A)P(A) + P(A)T(A)P(A)]. \tag{5.23}$$

The major results of Sec. IV depend upon the reduction of (4.8) to the form (4.12). We next investigate the possibility of performing a similar reduction of Eqs. (5.18). An immediate difficulty is the definition of the counterparts of M and G_0. (We recall that $\mathfrak{M} \equiv 0$ in the present instance.) Unless $\mathfrak{a} = \mathfrak{a}_0$ we have $0 \notin \mathfrak{a}$ and the introduction of the quantity $G_0(A) = z^{-1}$, e.g., is rather contrived.

The partition symbol 0 plays a unique role in the exact theory as the most disconnected partition corresponding to a physical channel. In a well-structured RM this role is taken over by the partition $\hat{0}$. In this case we define

$$M_{b,a}(A) \equiv [V^b(A)G(A)G_{\hat{0}}^{-1}(A)]_a, \tag{5.24}$$

and then we obtain for a well-structured RM

$$T(A) = M(A)G_{\hat{0}}(A)\{s\hat{G}^{-1}(A) + (P_2 + \Gamma)T(A)\}. \tag{5.25}$$

The derivation of Eq. (5.24) is given in Appendix D, where it is also shown that (5.25) has a connected kernel after one iteration if A is nontrivial.

It is easy to show that the quasiminimal coupling property (5.23) is preserved in (5.25). Since

$$M_{b,a}(A)G_{\hat{0}}(A) = \bar{\Delta}_{b,a}\sum_{c,d} \delta_{a,c\cup d}[V(A)]_c[G(A)]_d, \tag{5.26}$$

we see that

$$M_{b,a}(A)G_{\hat{0}}(A) = 0, \text{ if } a \notin \mathfrak{a}, \tag{5.27}$$

so

$$M(A)G_{\hat{0}}(A)P(A) = M(A)G_{\hat{0}}(A). \tag{5.28}$$

In order to deduce the counterpart of the form (5.23) for (5.25) one needs besides (5.28) the identities $P_2P(A) = P(A)P_2$ and $P(A)\Gamma = P(A)\Gamma P(A)$.

These are easily proved and we see that (5.25) implies a set of integral equations which couple together only reaction set transition operators.

The approximate transition operator $T(A)$ is related to the exact one via an *imbedding equation*. One form of the imbedding equation found in Ref. 4 is given by

$$T_{a,b}=[G_b^{-1}+G_a^{-1}G(A)G_b^{-1}] + G_a^{-1}G(A)\hat{T}G(A)G_b^{-1}, \tag{5.29}$$

where $\hat{T}$ is the solution of

$$\hat{T}=T_{0,0}(A')+H(A')G(A)H(A)G(A')\hat{T}, \tag{5.30}$$

and $A'=A_0-A$ is the RM *complementary* to A. The term in brackets in Eq. (5.29) is on-shell equivalent to $T_{a,b}(A)$.

An alternative form of the imbedding equation can be obtained by a procedure analogous to that used in Refs. 5 and 6. We define an auxiliary operator $J(A)$ as the solution of

$$J(A)=1+M(A)G_{\emptyset}(A)(P_2+\Gamma)J(A), \tag{5.31}$$

so

$$T(A)=J(A)M(A)G_{\emptyset}(A)s\hat{G}^{-1}(A). \tag{5.32}$$

The operator $J(A)$ is related to its exact counterpart, $J(A_0)=J$, by the imbedding equation

$$J=J(A)+J(A)[MG_0-M(A)G_{\emptyset}(A)](P_2+\Gamma)J. \tag{5.33}$$

T is obtained from J, using (5.32) for $A=A_0$. The imbedding relation (5.33) is analogous to the two-potential formula and may be simpler to apply in practical calculations than (5.29) because of this resemblance.

Polyzou and Redish[4] prove that (5.30) is a connected-kernel integral equation and thus this imbedding algorithm is well defined in that no disconnected-kernel integral equations have crept in surreptitiously. It is a rather straightforward exercise in the techniques used in Appendices A and D to show, using the form of $J(A)$ implied by (5.31), that the kernel of the imbedding equation (5.23) becomes connected after a single iteration. Thus the imbedding algorithm (5.31)–(5.33) is also well defined and allows one, in principle, to calculate in a constructive way corrections to a model problem defined by an RM A. This feature is probably the one which is most markedly different from conventional approximations in nuclear reaction theories where one often has no systematic procedure for calculating corrections to low-order approximations.

VI. MULTIPLE SCATTERING IN FEW-BODY MODELS

We now wish to consider the multiple-scattering structure of the various scattering operators in Sec. V. We start by considering the operator

$$\tau^{a,b}(A)=T_{a,b}(A)-V_b^a(A), \tag{6.1}$$

and restrict ourselves to a nontrivial, well-structured RM. Following the development of Secs. IV and V, we find in Appendix D the RM equivalent of the PLMS equations:

$$\tau(A)=W_{\rm MS}(A)+M(A)G_{\emptyset}(A)(P_2+\Gamma)\tau(A), \tag{6.2}$$

where

$$W_{\rm MS}^{a,b}(A)=\sum_c{}' W^{a,b}(c|A), \tag{6.3}$$

and $W^{a,b}(c|A)$ is the c-connected part of $\tau^{a,b}(A)$. By the arguments of Sec. IV it is clear that $W^{a,b}(c|A)$ corresponds to scattering within the clusters of the partition c, and vanishes unless $c\in\mathcal{a}$.

Equations (6.1)–(6.3) comprise that version of the reaction theory of Ref. 4 which possesses an explicit multiple-scattering structure, and, as such, represent one of our principal results. In the remainder of this section we explore the properties and the physical interpretation of various facets of this structure. Before doing this we observe that for a nontrivial RM the kernel of (6.2) becomes connected upon a single iteration. Also in the same way that we were led to Eq. (5.23) it follows that Eqs. (6.2) can be reduced to a set which couples together only the reaction set of operators $P(A)\tau(A)P(A)$. Finally, in Ref. 10 the significant advantages that the PLMS equations offer over the original BRS equations are explored at length. These advantages are particularly striking in applications such as the optical model where a succession of multiple-scattering type approximations are often useful. For very similar reasons these advantages are also realized for RM-type few-body reaction theories as well.

We now note that for $\alpha\in\mathcal{a}$ we have, for the special case of the nonrearrangement scattering of two fragments,

$$W^{\alpha,\alpha}(c|A)=X_c^{\alpha}(A), \tag{6.4}$$

where

$$X_c^{\alpha}(A)=\sum_{a\in\mathcal{a}}(\Delta^{-1})_{c,a}t_a^{\alpha}(A), \tag{6.5}$$

and

$$t_a^{\alpha}(A)=V_a^{\alpha}(A)[1+G_a(A)V_a^{\alpha}(A)]. \tag{6.6}$$

Clearly $X_c^{\alpha}(A)$ is the c-connected part of the model elastic scattering operator $T_{\alpha,\alpha}(A)$. Equations

(6.4)–(6.6) are the RM equivalents of the multiple-scattering expansions of Sec. III.

To illustrate the meaning of the various terms in the expansion (6.3), we note that any partition $a \in \mathfrak{a}$ can be constructed from $\hat{0}$ by joining two or more of the clusters of the partition $\hat{0}$. Furthermore, since there exists no partition $c \in \mathfrak{a}$, $c \subset \hat{0}$, the clusters of $\hat{0}$ cannot be further divided in a reaction model described by the set $\mathfrak{a}$, and therefore they behave as particles with internal structure in the scattering process. This is evident if we recall that $H_{\hat{0}}(A) = P_{\hat{0}}(A)H_{\hat{0}}$, and $P_{\hat{0}}(A)$ projects onto the space of the maximally connected eigenstates $|\phi_{\hat{0}}(\nu_{\hat{0}})\rangle$ of $H_{\hat{0}}$. Thus the spectrum of $H_{\hat{0}}(A)$ includes only those states in which the clusters of $\hat{0}$ are in bound configurations. For a few-body reaction mechanism, $n_{\hat{0}} \leq 4$. Now $t^{\alpha}_{\hat{0}}(A) \equiv 0$, and for a partition $a \supset \hat{0}$, $t^{\alpha}_{a}(A)$ corresponds to the scattering of two (ij), three $(ijk)\cdots n_{\hat{0}} - n_a(ijk\cdots)$ clusters of the partition $\hat{0}$:

$$T_{\alpha,\alpha}(A) = \sum_{i<j}\left[t^{\alpha}_{ij}(A) + \sum_{j<k} t^{\alpha}_{ijk}(A) + \cdots\right]. \quad (6.7)$$

In the "impulse approximation" to (6.7) we have

$$T_{\alpha,\alpha}(A) \cong \sum_{i'} t_{i'}(A)\bar{\Delta}_{\alpha,i'}, \quad (6.8)$$

where $i' = (ij)$ refers to a pair of clusters and

$$t_{i'}(A) = V_{i'}(A)\{1 + [z - H_{\hat{0}}(A) - V_{i'}(A)]^{-1}V_{i'}(A)\}. \quad (6.9)$$

Clearly $t_{i'}(A)$ is an effective two-body transition operator.

The development leading to Eqs. (6.7)–(6.9) closely resembles the formulation of an exact $n_{\hat{0}}$-body problem. However, there are several significant differences. First, we note that the free propagator G_0 of the exact problem is replaced by $G_{\hat{0}}$. Since $H_{\hat{0}}$ in our model possesses a spectrum more complicated than that of the ordinary kinetic energy operator, the "two-body" operators $t_{i'}(A)$ have a rather complex pole-cut structure corresponding to the bound states of the individual clusters of partition $\hat{0}$. Such a structure would be entirely absent if the $n_{\hat{0}}$ bodies really were elementary. It is in this sense that the model problem resembles the problem of $n_{\hat{0}}$ particles *with internal structure*.

This feature of the model problem requires the reinterpretation of the impulse-closure type approximations. Normally what is referred to as the closure approximation amounts to the replacement of the two-cluster propagator G_α by, e.g., the approximate propagators $(\hat{z} - H_0)^{-1}$ or $(\hat{z} - H_{i'})^{-1}$, where $\hat{z}$ is a parametric energy, and, in general, $\hat{z} \neq z$ and i' refers to a pair of particles. The difficulty with the literal transcription of this to an RM model is that $H_0(A)$ and $H_{i'}(A)$ may vanish for a particular choice of A so that G_α is replaced by $\hat{z}^{-1}$. This clearly is not consistent with the picture of the RM model as an $n_{\hat{0}}$-body system. A correct equivalent of the $(\hat{z} - H_0)^{-1}$ propagator in our model then is $[\hat{z} - H_{\hat{0}}(A)]^{-1}$, the equivalent of the two-particle propagator is $[\hat{z} - H_{\hat{0}}(A) - V_{i'}(A)]^{-1}$, and so on. These remarks can also be understood in a somewhat different context. We recall that the RM assumption is that there exists a dominant set of channels. In contrast, the replacement of G_α by G_0 assumes that the various eigenstates of H_α are equivalent in the scattering process. Clearly, these two assumptions are not compatible with each other. This point is important, e.g., in the development of a consistent theory of an optical potential.

The above discussion illustrates that various rather trivial approximations often made in the exact problem must be treated with some caution in the RM cluster model in order to ensure a consistent approximation scheme. Various other features of the model problem are absent from the exact one. In particular, in the model problem all interactions are nonlocal, energy-dependent and multiparticle. Thus, even if the model leads to an effective three-body problem, its solution is considerably more difficult than that of the "standard" three-body problem.

An interesting possibility arises if $n_{\hat{0}} = 3$. In this case it is possible to write the RM equivalent of the Watson equations of Sec. II, viz.,

$$T_{i'}(\alpha|A) = \tilde{t}_{i'}(A)\left\{1 + \sum_{j' \neq i'} G_\alpha(A)T_{j'}(\alpha|A)\right\}, \quad (6.10)$$

where the cluster pairs $i', j' \not\subset \alpha$, and

$$\tilde{t}_{i'}(A) = t_{i'}(\alpha|A) = V_{i'}(A)\{1 + G_\alpha(A)\tilde{t}_{i'}(A)\}. \quad (6.11)$$

Equations (6.10) and (6.11) represent the cluster generalizations of the Watson equations[10] for the three-body problem and thus have connected kernels after one iteration. As in the exact case, $\tilde{t}_{i'}(A)$ is a many-body operator. Various impulse-closure type approximations on the intercluster degrees of freedom allow us to replace $\tilde{t}_{i'}(A)$, e.g., by the operator $t_{i'}(A)$ given by (6.9). We have

$$t_{i'}(A) = V_{i'}(A)[1 + G_{\hat{0}}(A)t_{i'}(A)], \quad (6.12)$$

and $t_{i'}(A)$ is recognized as the RM equivalent of the two-particle scattering operator of the exact problem. We note that in order to determine $t_{i'}(A)$ it is necessary to calculate the (microscopic) interaction $V_{i'}(A)$. An alternative to this is to

replace the (nonlocal energy dependent) potential $V_{i'}(A)$ by a local optical potential $U_{i'}$ for cluster-cluster scattering. Unfortunately, it is not clear how to do this consistently so as to obtain a unitary theory. Finally, in analogy with the results of Sec. III, it is also possible to write down the RM correlative expansion. Since this is quite straightforward we will not do so here.

VII. SUMMARY

In this paper we investigate the multiple-scattering structure of various scattering operators and the integral equations they satisfy for many-body Hamiltonians consisting of terms of definite but arbitrary connectivities. This refers to those situations which involve various kinds of ordinary multiparticle forces but more importantly, it also includes the effective interactions which arise in few-body models of nuclear reactions. In these models interactions with somewhat unusual structural characteristics are encountered. The principal objective of our work is to extend the applicability of multiple-scattering approximations under certain physical circumstances to few-body models of nuclear reactions.

Our results fall into two main groups. First, we find the modifications of multiparticle scattering theory, with particular regard to multiple scattering, which arise because of the presence of multiparticle interactions. This extends previous work which was confined to only pairwise forces.[10] This first group of results represents a nontrivial extension of a wide variety of important aspects of multiparticle scattering theory as it applies to nuclear reactions. These aspects include the Watson theory, the Bencze-Redish-Sloan equations, and various resummations of the multiple-scattering series.

Our second group of results consists in the application of the methods which were used to obtain the preceding results to the particular few-body model of nuclear reactions proposed by Polyzou and Redish.[4] One of the major features of this is the refinement in the idea of a reaction mechanism which is required in order to obtain a consistent description of multiple scattering within these models. The particular versions of the impulse-closure approximations which emerge in this few-body model of nuclear reactions are found, and their physical significance is pointed out. Several new properties of the models proposed in Ref. 5 are investigated, including a new method of imbedding the approximate theory within the exact theory and a new set of connected-kernel equations for the model transition operators which possesses a multiple-scattering substructure, unlike the corresponding equations found in Ref. 4.

The results obtained in this paper suggest several important avenues of practical applications to nuclear reactions. One of these applications is the development of the optical potential and we intend to explore this topic elsewhere.

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APPENDIX A: REDUCED SCATTERING EQUATIONS

In this appendix we show that the Eq. (4.8) can be placed in the reduced form (4.12). We also review several technical devices which are useful in multiparticle scattering theory.

Let

$$\hat{V}_M = V_M + \bar{\Delta} P_1 [V]_1 \mathcal{S} = \bar{\Delta}[\hat{V}]\Delta^t, \tag{A1}$$

where $(V_M)_{a,b} = V_b^a$ and

$$[\hat{V}] \equiv Q_1[V] + P_1[V]_1 \mathcal{S} P_0 . \tag{A2}$$

We note that $\mathcal{S}P_0\Delta^t = \mathcal{S}$ and since $[\hat{V}]Q_1 = [\hat{V}]$ we have

$$\hat{V}_M = \bar{\Delta}[\hat{V}]Q_1\Delta^t . \tag{A3}$$

Equation (4.6) can then be rewritten as

$$\mathcal{U} = \hat{V}_M C Q_1 , \tag{A4}$$

and thus Eq. (4.7) becomes

$$T = \hat{V}_M Q_1 C \hat{G}(\mathcal{S}\hat{G}^{-1} + T) . \tag{A5}$$

It follows from (A3) that

$$\hat{V}_M Q_1 \hat{G} = \bar{\Delta}[\hat{V}\cdot\hat{G}]Q_1\Delta^t Q_1 , \tag{A6}$$

where

$$[\hat{V}\cdot\hat{G}]_{a,b} \equiv \sum_{e,d} \delta_{b,e\cup d}[\hat{V}]_{a,d}[G]_e . \tag{A7}$$

In Eq. (A7) the notation $e\cup d = d\cup e$ refers to the *union* of partitions e and d.[4,18] The union of e and d is itself a partition such that $e\cup d \supseteq e, d$ and if a is such that $a \supseteq e, d$, then $a \supseteq e\cup d$ as well. Thus $e\cup d$ is the partition with the greatest number of clusters which contains both a and d. The power of this concept in multiparticle scattering theory derives from the circumstance that the product of two operators of well-defined connectivities a and b, respectively, has connectivity $a\cup b$.[4,15] We see then that (A7) is a sum of b-connected operators. In obtaining (A7) we use the fact that by (A2) the operator $[\hat{V}]_{a,e}$ is a connected and the decomposition

$$G_a = \sum_a \Delta_{a,d}[G]_d , \tag{A8}$$

of G_a into the components $[G]_d$ which are the d-

connected parts of the full Green's function.

If we call [Eq. (4.11)]

$$M_{a,b} \equiv [V^a G G_0^{-1}]_b \bar{\delta}_{b,1}, \tag{A9}$$

one can show using (2.8), (A7), and (A8) that

$$\bar{\Delta}[\hat{V}\cdot\hat{G}]Q_1 G_0^{-1} = M + \bar{\Delta}P_1[V]_1 \mathcal{S} Q_1 . \tag{A10}$$

The square of the kernel $K = \hat{V}_M Q_1 \hat{G} C$ of the integral equation (A5) is

$$K^2 = \bar{\Delta}[\hat{V}\cdot\hat{G}]Q_1 \mathfrak{u} [\hat{V}\cdot\hat{G}]Q_1 \Delta^t C Q_1 , \tag{A11}$$

where

$$(Q_1\mathfrak{u})_{f,d} \equiv \delta_{1,f\cup d} . \tag{A12}$$

From Eq. (A12) we see that K^2 is a fully connected operator. In deriving (A10) we have made use of the sum rule[13]

$$\Delta^t C \bar{\Delta} = Q_1 \mathfrak{u} . \tag{A13}$$

If we call

$$\mathfrak{M} \equiv \bar{\Delta} P_1 [V]_1 \mathcal{S} Q_1 , \tag{A14}$$

and make use of the decomposition

$$Q_1\mathfrak{u} = P_2\bar{\Delta} + Q_2(Q_1\mathfrak{u}) \tag{A15}$$

and the product rule (A13), we see by referring to (A10) that K^2 assumes the compact form

$$K^2 = (M + \mathfrak{M}) G_0 (P_2 + \Gamma) K , \tag{A16}$$

where

$$\Gamma \equiv Q_2 \Delta^t C Q_1 . \tag{A17}$$

We recall that P_2 is the projector on the set of two-cluster partitions and that $Q_2 = I - P_2$. In deriving (A16) we have also used the elementary properties

$$P_2 \Delta^t Q_1 = P_2 C = P_2 .$$

Equation (A5) can be written as

$$T = K \mathcal{S} \hat{G}^{-1} + KT . \tag{A18}$$

Thus, using (A16) we see that

$$KT = (M + \mathfrak{M}) G_0 (P_2 + \Gamma) T . \tag{A19}$$

Also by the same sort of manipulations which were used to obtain (A16), it follows that

$$K \mathcal{S} \hat{G}^{-1} = (M + \mathfrak{M}) G_0 . \tag{A20}$$

Equation (4.13) follows from Eqs. (A16)–(A20). If we call $\tau = T - V_M$, then

$$\tau = (M + \mathfrak{M}) G_0 \{ Q_1 \mathcal{S} \hat{G}^{-1} + (P_2 + \Gamma) V_M - V_M + (P_2 + \Gamma)\tau \} . \tag{A21}$$

We are only interested in $Q_1 \tau Q_1$ and on the Q_1 subset of partitions

$$W_{MS} = M G_0 \mathcal{S} G^{-1} - V_M + M G_0 (P_2 + \Gamma) V_M \equiv \sum_f{}' W(f) , \tag{A22}$$

as shown in Appendix B. Equations (A20) and (A21) then yield Eq. (4.20). We also note that in (A22) there is no dependence on the fully connected interaction $[V]_1$. Since $\tilde{\tau}$ satisfies the same equation as τ if $[V]_1 = 0$, Eqs. (A20) and (A21) lead directly to Eq. (4.16) when we set $\mathfrak{M} \equiv 0$.

APPENDIX B: *W* IDENTITY

The operator $\tau = T - V_M$ has the components

$$\tau^{b,a} = V^{b,a} + V^b G V^a ,$$

or

$$\tau^{b,a} = V^b G G_a^{-1} - V_a^b . \tag{B1}$$

We call $W^{b,a}(f)$ the f-connected part of $\tau^{b,a}$, i.e.,

$$W^{b,a}(f) = \bar{\Delta}_{b,f}[V]_f \bar{\Delta}_{a,f} + \sum_{e,d,c} \bar{\Delta}_{b,e}[V]_e [G]_c [V]_d \bar{\Delta}_{a,d} \delta_{e\cup d\cup c,f} . \tag{B2}$$

By definition, $W^{b,a}(f)$ vanishes unless $f \in \mathfrak{a}(1)$, $f \not\subset a$, b, and we have

$$\tau^{b,a} = \sum_f W^{b,a}(f) . \tag{B3}$$

The partition sums in (B2), (B3) and in the equations to follow are over the set $\mathfrak{a}(1)$.

From (B1) it follows that

$$\tau^{b,a} = \tau^{b,0} G_0 G_a^{-1} - V_a^b . \tag{B4}$$

If we use (B3) in (B4) we obtain the identity

$$\sum_f W^{b,a}(f) = \sum_f W^{b,0}(f) G_0 G_a^{-1} - V_a^b . \tag{B5}$$

After a few manipulations the disconnected part of (B5) can be written as

$$\sum_f{}' W^{b,a}(f) = \sum_f{}' W^{b,0}(f) G_0 G_a^{-1} + \left[\sum_f{}' W^{b,0}(f) G_0 V_a \right]_{\text{conn}} - V_a^b , \tag{B6}$$

where the prime indicates that the 1 partition has been omitted from the sum.

Let us consider the sum

$$\sum_f{}' W^{b,0}(f) G_0 V_a = \sum_f W^{b,0}(f) G_0 V_a \bar{\delta}_{n_f,2} + \sum_\gamma W^{b,0}(\gamma)(G_0(V_{a,\gamma} + V_a^\alpha) , \tag{B7}$$

where γ denotes a two-cluster partition. The connected part of (B7) is then

$$\left[\sum_f{}' W^{b,0}(f)G_0V_a\bar{\delta}_{n_f,2}\right]_{\text{conn}}+\sum_\gamma W^{b,0}(\gamma)G_0V_a^\gamma$$
$$=\sum_f \delta_{1,f\cup a}W^{b,0}(f)G_0[V]_a\bar{\delta}_{n_f,2}\Delta_{a,a}$$
$$+\sum_\gamma W^{b,0}(\gamma)G_0V_a^\gamma. \tag{B8}$$

A comparison of the definition (A9) of M with Eqs. (B1) and (B3) reveals that

$$M_{b,f}=W^{b,0}(f). \tag{B9}$$

We also note that ($Q_{1,0}=I-P_1-P_0$)

$$Q_{1,0}[V]_a\Delta_{a,a}=[V]_a(\Delta^t)_{a,a}. \tag{B10}$$

Using (B8)–(B10) and (A14) we obtain ($a\neq 1$)

$$\left\{\sum_f{}' W^{b,0}(f)G_0V_a\bar{\delta}_{n_f,2}\right\}_{\text{conn}}=(MG_0\Gamma V_M)_{b,a}, \tag{B11}$$

where Γ and V_M are defined in Appendix A. Combining (B6) and (B11) we get

$$\sum_f{}' W^{b,a}(f)=\sum_f{}' W^{b,0}(f)G_0G_a^{-1}+(MG_0\Gamma V_M)_{b,a}$$
$$+\sum_\gamma W^{b,0}(\gamma)G_0V_a^\gamma-V_a^b. \tag{B12}$$

The term in (B7) containing $V_{a,\gamma}$ has been dropped since it contains no fully connected parts.

Equation (B12) can be written in a matrix form as

$$W_{\text{MS}}=MG_0(s\hat{G}^{-1}+P_2+\Gamma)V_M-V_M. \tag{B13}$$

Equation (B13) is the basic identity needed to derive the PLMS equations of Sec. IV. We remark that the considerations of this appendix are valid whether or not $[V]_1=0$.

APPENDIX C: HALF-ON-SHELL EQUIVALENCE THEOREMS

Let us consider the operator

$$\tau_a^b\equiv V_a^b+V_a^bG_aV_a=V_a^bG_aG_0^{-1}. \tag{C1}$$

We note that $\tau_1^b=\tau^{b,0}$, so that from (B3) we have

$$\tau_a^b=\sum_f W^{b,0}(f)\Delta_{a,f}, \tag{C2}$$

where again the sum is over $\mathfrak{a}(1)$. Thus if $f\neq 1$,

$$\sum_a(\Delta^{-1})_{f,a}\tau_a^b=W^{b,0}(f)=(MG_0)_{b,f}G_0^{-1}, \tag{C3}$$

and using (C1), we obtain

$$(MG_0)_{b,f}=\sum_a{}'(\Delta^{-1})_{f,a}V_a^bG_a. \tag{C4}$$

Let α be a stable two-cluster partition. From (C4) we have on-shell with the aid of Lippmann identity[20]

$$(MG_0)_{b,f}G_\alpha^{-1}|\phi_\alpha\rangle=\sum_a{}'(\Delta^{-1})_{f,a}V_a^b\{G_aG_\alpha^{-1}|\phi_\alpha\rangle\}$$
$$=\delta_{f,\alpha}V_\alpha^b|\phi_\alpha\rangle, \tag{C5}$$

where again we require $f\neq 1$. From Eqs. (B13) and (C5) we conclude that half-on-shell

$$W_{\text{MS}}^{b,\alpha}|\phi_\alpha\rangle=\{MG_0(P_2+\Gamma)V_M\}_{b,\alpha}|\phi_\alpha\rangle. \tag{C6}$$

Let us refer to Eq. (4.20). Since $\mathfrak{M}$ does not possess any δ-function singularities, $\mathfrak{M}G_0s\hat{G}^{-1}$ vanishes on-shell and we find from Eq. (C6)

$$(\tau)_{b,\alpha}|\phi_\alpha\rangle=\{(M+\mathfrak{M})G_0(P_2+\Gamma)V_M\}_{b,\alpha}|\phi_\alpha\rangle$$
$$+\text{terms containing }\tau. \tag{C7}$$

The significance of Eqs. (C6) and (C7) is that they explain the lack of multiple-scattering structure in the half-on-shell version of the integral equation (4.12) for T. If we make use of our previous results, especially (C5), we find that T is half-on-shell equivalent to the integral equation

$$T_{\text{BRS}}=V_M+(\mathfrak{M}+M)G_0(P_2+\Gamma)T_{\text{BRS}}. \tag{C8}$$

Equation (C8) is the general version of the BRS equation.[2] We then conclude as in Ref. 10 that the multiple-scattering structure of T_{BRS} is recovered in the second Born approximation to T_{BRS} as obtained from (C8). This is hardly a trivial observation since from (C8) it is not at all obvious how one could recover the *first* Born approximation

$$T_{b,a}\simeq V^b,$$

let alone the impulse approximation. These matters are investigated further in detail in Ref. 10.

APPENDIX D: RM SCATTERING EQUATIONS

We present here the details underlying the derivation of Eqs. (5.25) and (6.2) for a well-structured RM A. Since $\hat{0}\in\mathfrak{a}$ in such a case we have from (5.11) as the analog of (2.3)

$$H(A)=H_{\hat{0}}(A)+V(A), \tag{D1}$$

where

$$V(A)=V^{\hat{0}}(A). \tag{D2}$$

We note that $V_{\hat{0}}^b(A)=0$ for all $b\in\mathfrak{a}$.

Let [cf. (6.1)]

$$\tau^{a,b}(A)=T_{a,b}(A)-V_b^a(A)$$
$$=V^{a,b}(A)+V^a(A)G(A)V^b(A). \tag{D3}$$

Clearly

$$\tau^{a,\hat{0}}(A) = V^b(A)G(A)G_{\hat{0}}(A)^{-1}, \tag{D4}$$

and we are led to the identification

$$M_{a,b}(A) = [\tau^{a,\hat{0}}(A)]_b . \tag{D5}$$

Since $b \cup \hat{0} = b$ for $b \in \mathfrak{a}$ we have

$$M_{a,b}(A)G_{\hat{0}}(A) = [V^a(A)G(A)]_b . \tag{D6}$$

It should be noted that $M_{a,\hat{0}}(A) \equiv 0$.

The kernel of Eq. (5.18) is

$$K(A) = \mathcal{U}(A)\hat{G}(A) . \tag{D7}$$

By analogy with the development in Appendix A we find that

$$K(A)^2 = M(A)Q_{\hat{0},1}G_{\hat{0}}(A)(Q_1\mathcal{U})[V(A)\cdot\hat{G}(A)]Q_1\Delta^t C Q_1 , \tag{D8}$$

where

$$[V(A)\cdot\hat{G}(A)]_a = \sum_{c\cup d} \delta_{a,c\cup d}[V(A)]_c[G(A)]_d . \tag{D9}$$

Also

$$K(A)s\hat{G}(A)^{-1} = M(A)G_{\hat{0}}(A)Q_1 s\hat{G}(A)^{-1} . \tag{D10}$$

With (D8) and (D10) we obtain Eq. (5.25) from (5.18).

The derivation of the RM version of the PLMS equations also follows the development in Appendix A. We set

$$W^{a,b}(f\,|A) \equiv [\tau^{a,b}(A)]_f , \quad \hat{0} \subset f \tag{D11}$$

and we note that $[\tau^{a,b}(A)]_{\hat{0}} = 0$ and

$$M_{a,f}(A) = W^{a,\hat{0}}(f\,|A) . \tag{D12}$$

Since

$$\sum_f W^{a,b}(f\,|A) = \sum_f W^{a,\hat{0}}(f\,|A)G_{\hat{0}}(A)G_a(A)^{-1} - V_a^b(A) ,$$

we can follow the same analysis as in Appendix B to obtain, in matrix form

$$W_{MS}(A) = M(A)G_{\hat{0}}^{\wedge}(A)s\hat{G}(A)^{-1} - V_M(A) + M(A)G_{\hat{0}}^{\wedge}(A)(P_2+\Gamma)V_M(A) , \tag{D13}$$

where

$$W_{MS}^{a,b}(A) \equiv \sum_f{}' W^{a,b}(f\,|A) . \tag{D14}$$

Equations (D3) and (D13) when used with (5.24) yield the RM-PLMS equations (6.2).

The appropriate specializations of the half-on-shell equivalence theorems of Appendix C are straightforward to obtain. Let $\alpha \in \mathfrak{a}$ be a stable two-cluster partition. Then one finds that half-on-shell

$$[M(A)G_{\hat{0}}^{\wedge}(A)]_{b,f}G_\alpha(A)^{-1}\,|\,\phi_\alpha\rangle = \delta_{f,\alpha}V_\alpha^b(A)\,|\,\phi_\alpha\rangle \tag{D15}$$

and

$$W_{MS}^{b,\alpha}(A)\,|\,\phi_\alpha\rangle = \{M(A)G_{\hat{0}}^{\wedge}(A)(P_2+\Gamma)V_M(A)\}_{b,\alpha}|\,\phi_\alpha\rangle . \tag{D16}$$

We also deduce that

$$[\tau(A)]_{b,\alpha}\,|\,\phi_\alpha\rangle = [M(A)G_{\hat{0}}^{\wedge}(A)(P_2+\Gamma)V_M(A)]_{b,\alpha}\,|\phi_\alpha\rangle . \tag{D17}$$

The significance of Eqs. (D14)–(D15) is similar to their counterparts (C5)–(C8) and we refer to the discussion in Appendix C.

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