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Nucleon-nucleon interaction as derived from the De Rújula, Georgi, Glashow phenomenological quark-quark potential

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The nucleon-nucleon interaction, originating from the phenomenological quark-quark interaction: one gluon exchange with an *ad hoc* confining potential, is rigorously derived. The coupling constants of this potential and the quark masses—which determine the strengths of the *NN* potentials in the various channels—had been previously reliably determined from the masses of the *s*-wave baryons. Two distinguishing features of this study are the reliable nature of the coupling constants and quark masses used and the inclusion of the detailed radial dynamics of quarks. The main results are as follows: (i) exchange of colored quarks among other things leads to short-range strongly repulsive central *NN* potentials in the odd channels alone, i.e., it yields the saturation property of nuclear forces; (ii) the spin-orbit (tensor) potential between quarks goes over to the spin-orbit (tensor) potential between nucleons; (iii) both these spin-orbit and tensor potentials between nucleons are found to have the correct signatures but they are very weak; (iv) there is insufficient attraction in the *NN* potentials in the intermediate range, so that the colored quark-exchange mechanism is not adequate to keep the deuteron bound. The *NN* potentials derived from one gluon exchange with an *ad hoc* confining potential are compared finally with the corresponding phenomenological ones.

[NUCLEAR STRUCTURE *NN* interaction from nonrelativistic quantum chromodynamics. Quark exchange, spin-orbit and tensor potentials, short-range repulsive core.]

I. INTRODUCTION

The problem of the nucleon-nucleon (*NN*) interaction has engaged the attention of numerous physicists over the last five decades; nevertheless a satisfactory theoretical understanding encompassing all its features has not yet emerged.[1,2] This interaction between two nucleons, at low energies (< 400 MeV projectile energy) can be imagined to be mediated by a nonrelativistic potential. The use of a potential is justified provided only low energy *NN* collisions are considered, and moreover, such a potential description of the interaction is both intuitively appealing and familiar through its long use in classical electrodynamics and gravitation. At higher energies however, such a description—being unable to cope with the phenomenon of creation and annihilation of particles—has been superseded by a field theoretic or a dispersion-theoretic description of the interaction. The field-theoretic description of the interaction is the most general one, and the potential description can be derived from the former, in particular from the constructs of vertex functions and propagators, in the nonrelativistic limit. The *NN* interaction has been described in all three languages mentioned above appropriate to the energy of the colliding nucleons. But in this study attention will be confined solely to the potential description, valid at low energies.

The interaction between nucleons observed at low energies obviously has its genesis in the strong-interaction dynamics of hadrons.[1,3] Novel strides in strong-interaction dynamics have, understandably, given rise to corresponding ones in the *NN* interaction. The meson-exchange *NN* potentials and the reggeon exchanges are striking examples.[3,4] In recent years the spectacular success of the non-Abelian color gauge field theory in explaining hadron spectra (especially the spectra of the charmed mesons) suggests that the successful models of the new strong-interaction theory of quantum chromodynamics (QCD) be brought to bear upon this long standing problem of the *NN* interaction.

The pertinent question now is this: What exactly is lacking and unsatisfactory in the present understanding of the *NN* interaction at low energies? At a phenomenological level, there is a fairly reliable description of the *NN* interaction. The most modern of such descriptions use, for the *NN* interaction, a sum of one-meson–exchange potentials.[3,4] The forms of these meson-exchange potentials, as obtained from the nonrelativistic reduction of meson field theory, are employed in these descriptions, and often the meson-nucleon coupling constants are determined only from a fit to the phase-shift data.[5] Furthermore, all the mesons belonging to the SU(3) octets and singlets are included, thus tremendously complicating the description of the phenomena. If one admits of such a complex description, then a very good χ^2 fit to the phase-shift data is possible.[4,5] The reasons for considering such a phenomenological description unsatisfactory are twofold. Firstly, a description involving numerous mesons, notwithstanding its success, goes counter to the aesthetic principle of simplicity. Secondly, the objection that the meson-exchange potentials—coming as they do from meson field theory—ignore completely the quark constitution of hadrons (which is now well established[6,7]) is unassailable.[3]

At the level of microscopic theories, only the long-range part of the *NN* interaction can be said to have been understood. And this is attributed to the exchange of a single pion.[1,2] To explain the hard core at very short distances, physicists have had to invoke the exchange of the heavier vector mesons ρ and ω and ϕ.[4,8] To account for every specific aspect of the *NN* interaction, a new meson is brought in thereby making the theoretical framework very complex. Indeed, it would be more satisfying to have a simpler theory of the phenomenon, one in which all that which is achieved here by meson nonets is achieved by one or, at the most, two "elementary" quanta or particles. In view of the tremendous success of QCD or the potential models thereof,[6] the question arises as to the possibility of deriving the *NN* potential from this modern theory of strong-interaction dynamics. This question is not an entirely new one and has been in existence for a couple of years.[7,9-11] The aim of this study is to provide a satisfactory answer to this question. Quite apart from these reasons, the new emerging ideology of D. Robson in nuclear structure based on the exchange of colored quarks[12] suggests that the *NN* interaction comes about precisely, through the exchange of constituent quarks. The preliminary success of a phenomenological bond model based on colored quarks[12,13] points to the fact that there is some measure of truth in the concept of quark exchange between nucleons.

Therefore, the objective of this study is to calculate rigorously the *NN* potential which arises from the exchange of colored quarks. The interaction between nucleons which arise in this way can be likened to the van der Waals interaction between atoms. Just as this van der Waals interaction stems from the Coulomb potentials between electrons and protons, the present *NN* interaction originates from the potential (arising from one-gluon exchange with confinement) between colored quarks which constitute colorless (i.e., color singlet) hadrons. Because baryons are colorless, the interaction between them, based on color, can only be a feeble vestige of that between colored quarks themselves, even as the interaction between two He atoms (which are electronically saturated and therefore "colorless") is only a pale vestige of the electrostatic interactions between the constituent electrons and protons.[3]

The theoretical framework employed here can be dubbed the one-gluon exchange with an *ad hoc* confining potential (OGEC) model. The precise meaning of this terminology will become clear in Sec. II A. The starting point is the quark-quark (*qq*) phenomenological interaction, used by De Rújula, Georgi, and Glashow,[6] together with an *ad hoc* confining potential. The calculation of the potential between two clusters of three quarks each is then performed more or less along the lines of the resonating group method of Wheeler.[14] But in order to carry out this calculation, it becomes necessary to determine the quark masses and coupling constants in a reliable way, for the strength of the quark-exchange potential (QEP) in the various channels would depend critically on these values. This determination is analogous to the determination of the pionic mass and the pion-nucleon coupling constant, which are inputs to the meson field-theoretic potentials. As already mentioned, there have been few attempts to calculate the *NN* potential arising from quark exchanges.[7,9-11] The serious drawbacks of these calculations are that the quark masses and the coupling constants have been sloppily determined and the radial dynamics of quarks in nucleons has either been parametrized or treated unrealistically. On account of this, statements concerning the strengths of the potential in various channels cannot be taken to be authentic. The OGEC constants used in this study have been reliably determined from the masses of the *s*-wave baryons, and their consistency with some other *s*-wave baryonic data has also been checked.[15] The space-wave functions which describe the radial dynamics of quarks in nucleons have also been determined from a dynamical approach to *s*-wave baryon spectra.[15] Using these reliable OGEC constants and cluster wave functions (WF's), the *NN* potential arising from quark exchanges has been calculated.

The emergence of the general structure for the NN interaction in the form

$$V = V_c(R) + V_{ls}(R)\vec{L}\cdot(\vec{S}_a + \vec{S}_a) + V_t(R)\frac{\hat{S}_{ab}}{4} \tag{1.1}$$

(though not explicitly shown, all the V's include the various exchanges) *from the qq* OGEC *potential has been here rigorously demonstrated for the first time.* The salient results of this calculation are as follows: (i) The central part of the QEP in the odd channels is strongly repulsive at short distances, *thereby showing that the colored quark-exchange mechanism is capable of yielding the property of saturation of nuclear forces*; (ii) the distance at which strong repulsion abruptly sets in tallies with the *core* radius of the nucleon calculated in a previous study[15]; (iii) the spin-orbit part of the QEP has the "inverted" character, which is observed experimentally, and the tensor part in the various channels has the right signatures; and (iv) the QEP suffers from the serious drawback *of insufficient attraction in the intermediate range*. In view of this limitation, it does not qualify to be the sole mechanism behind all the observed features of the *NN* interactions.

The organization of the paper is as follows. In Sec. II, the theoretical framework is set up. Firstly, the theoretical basis for the phenomenological OGEC qq potential is discussed. Subsequently, the six-quark WF is constructed, and a definition is given for the nonrelativistic *NN* QEP. In the third section, the structure of the QEP in the spaces of spin, isospin, and color spin is discussed. The spin-isospin (SI) structure of the spin-spin, spin-orbit, tensor, and central interactions are discussed in that order. In Sec. IV, the radial structure of the QEP is considered. The radial dependence of the central, spin-orbit, and tensor interactions are examined, in that order. Having been assembled, the QEP's central, spin-orbit, and tensor parts are separately compared with the corresponding pieces of the phenomenological potentials in Sec. V. In the same section, the possibility of including the L^2 interaction and the quadratic spin-orbit interaction in the QEP is pointed out. The results obtained here are then compared with those of four other contemporary calculations in Sec. VI. The present work must be considered the most reliable among these in view of the careful determination of the OGEC constants and the inclusion of quark radial dynamics. Lastly, in Sec. VII, a retrospective survey is made.

II. THEORETICAL FRAMEWORK

A. OGEC Potential between quarks

A nucleon is a cluster of three valence quarks, each of them having a distinct color, so that the nucleon WF in color space is a fully antisymmetric 3×3 Slater determinant. In high energy phenomena the interaction between hadrons (which are bound states of quarks) is described by the non-Abelian field theory of QCD. According to this theory, the dynamics of quarks is induced by the color, rather than by the flavor degrees of freedom. As hadrons are color singlets (i.e., they are colorless), it would appear on the face of it that there can be no color interaction between hadrons. It turns out that in OGEC, the direct part of the *NN* color interaction vanishes, as intuitively expected, whereas the exchange part of the color interaction between colorless hadrons is nonvanishing and contributes to the *NN* potential [see Eq. (3.1)]. The two characteristic features of this non-Abelian field theory of QCD are asymptotic freedom and infrared slavery. Asymptotic freedom is tantamount to the logarithmic vanishing of the QCD coupling constant when the quarks are very close together, and infrared slavery is the property by virtue of which quarks are unable to be isolated, i.e., quarks are always confined within hadrons. If the nonrelativistic image of the QCD field theory is to be faithful then it is essential that the OGEC potential between quarks must possess both the aforementioned non-Abelian features. While the artifice of a linear (or harmonic oscillator) confinement potential simulates very well the property of infrared slavery, a corresponding artifice for asymptotic freedom has been generally lacking.[6,7,16] But recently Richardson[17] has considered the question "Can both asymptotic freedom and infrared slavery be incorporated in a $q\bar{q}$ potential in a unified manner?" and has provided a preliminary answer to it. In the light of this investigation it is found that a potential, which incorporates asymptotic freedom, differs only in so far as being softer than a Coulomb potential, near the origin. Consequently the discrepancy between such an asymptotically free potential and the one used here is rather small. In the present analysis, the justification for ignoring asymptotic freedom is this small discrepancy between the two potentials. The question raised by J. L. Richardson deserves further investigation but this falls outside the ambit of this study.

Starting from a *part* of the total interaction Lagrangian of QCD[7];

$$\mathcal{L}'_{\text{int}} = ig\sum_{k=1}^{8} \bar{\psi}\gamma^{\mu}\lambda_k\psi(\mathcal{G}_{\mu})_k\ , \tag{2.1}$$

one can derive the potential between quarks in the nonrelativistic limit be retaining terms in (2.1) to order $1/c^2$. This procedure, being almost parallel to that used in the derivation of the Fermi-Breit interaction in QED,[18] suffices to give the final result. This potential corresponds to a static one-gluon exchange between quarks, which was popularized through the work of De Rújula, Georgi, and Glashow.[6] To get the complete OGEC qq potential, one must add a linear (or a harmonic oscillator) confinement potential. The only other vestige of the non-Abelian nature of QCD is the color-spin scalar $\vec{F}_i \circ \vec{F}_j$. The qq OGEC potential is then

$$V(ij)=\alpha_s\vec{F}_i\cdot\vec{F}_j\Big[b+\frac{1}{r}\Big(1-\frac{r^3}{a^3}\Big)-\frac{1}{2m^2c^2}\Big(\frac{\vec{P}_i\cdot\vec{P}_j}{r}+\frac{\vec{r}\cdot(\vec{r}\cdot\vec{P}_i)\vec{P}_j}{r^3}\Big)-\frac{\pi\hbar^2}{m^2c^2}\delta^3(\vec{r})(1+\tfrac{8}{3}\vec{S}_i\cdot\vec{S}_j)$$
$$-\frac{\hbar}{2m^2c^2r^3}[\vec{r}\times(\vec{P}_i-2\vec{P}_j)\cdot\vec{S}_i-\vec{r}\times(\vec{P}_j-2\vec{P}_i)\cdot\vec{S}_j]+\frac{\hbar^2}{m^2c^2}\Big(\frac{\vec{S}_i\cdot\vec{S}_j-3(\vec{S}_i\cdot\hat{n})(\vec{S}_j\cdot\hat{n})}{r^3}\Big)\Big]. \qquad (2.2)$$

Several remarks about this potential are necessary. Here α_s represents the strong-coupling constant and is independent of the energy (logarithmic dependence of α_s on the energy is tantamount to asymptotic freedom). If this energy dependence is included before Fourier transforming the momentum space interaction from (2.1), then one arrives at a coordinate space potential which obeys asymptotic freedom.[17] In the present analysis however, with a view to avoid complications in the radial matrix element this energy dependence has been overlooked. b is a constant which is included because the confinement potential is asymptotically nonvanishing. a is the confinement constant; it is the distance at which the sum of the Coulomb and confinement potentials in (2.1) vanish. $\vec{r}=\vec{r}_i-\vec{r}_j$ and $\vec{P}_i(\vec{S}_i)$ is the momentum (spin) vector of the ith quark. Since the nucleons are composed of only the nonstrange n and p quarks (which have the same mass since the assumption is made that the electromagnetic interaction between quarks is switched off), the mass m is that which corresponds to nonstrange quarks.

It is important to note that in the OGEC potential the dominant short-range interaction is the Coulomb potential arising from the exchange of a massless gluon of the color gauge field. The spin-orbit, tensor, spin-spin, and momentum dependent potentials are only corrections to this dominant term, as borne out by their multiplicative factors $1/c^2$. The long-range dominant part is the confinement potential, which has been taken to be quadratic. Lattice gauge theories give some justification for the linear choice of a confinement potential, whereas the oscillator confinement potential has only a phenomenological basis. From a previous, careful study[15] of baryon spectroscopy based on OGEC however, it has been found that these two forms of confinement potentials are equally reliable.

The distinguishing feature of this investigation is that the OGEC constants, namely, m, α_s, a, and b, have been carefully determined from a dynamical approach to baryon spectroscopy.[15] In that approach both the linear and oscillator confinement potentials were used along with the one-gluon–exchange qq potential. The radial wave functions which were variationally determined were of two types: Exponential $[\psi\propto R^{5/2}\exp(-\beta R/2)]$ and Gaussian $[\psi\propto R^{5/2}\exp(-\frac{1}{2}\beta^2R^2)]$. The arguments R in these were again chosen to be of the following two types: $R=r_{12}+r_{23}+r_{31}$ and $R=(r_{12}{}^2+r_{23}{}^2+r_{31}{}^2)^{1/2}$. Thus, in all, eight cases were studied. These eight cases can be denoted by VCT, with V standing for the variational wave function [$E(G)$ for exponential (Gaussian)], C for the confinement potential [$L(H)$ for linear (harmonic oscillator) types], and T for the arguments of type I and type II indicated above. Table I exhibits the values of the OGEC constants that go in as inputs into this theory of the NN interaction.

B. The six-quark wave function

Having defined the OGEC potential between constituent colored quarks, the ground must next be prepared for the definition of the NN QEP. The

TABLE I. The four OGEC constants (α_s, a, b, m_p) determined from a dynamical approach to baryon spectroscopy (Ref. 15). The meaning of VCT is explained in Sec. II A. The constants in the qq potential (2.2) are thus accurately known. The somewhat larger value for the nonstrange quark mass (m_p=575 MeV) is vindicated in Ref. 15. The quark mass has the same value for all the eight cases studied. The quantity β is a property of the quark radial wave function, i.e., of the dynamics; even though it varies with the cases, the rms charge radius of the proton is roughly constant for all the eight cases (Ref. 15). The case relevant to this study is GH II.

Case: VCT	$\frac{\alpha_2}{\hbar c}$	a (fm)	$-2\alpha_s b$ (MeV)	β (fm^{-1})
GH I	0.4584	0.4332	−1886.52	1.14
GL I	0.4672	0.2748	−2431.82	1.12
GH II	0.5918	0.4655	−1710.68	2.02
GL II	0.5932	0.3100	−2251.43	2.00
EH I	0.3241	0.3998	−2077.32	3.82
EL I	0.3786	0.2372	−2609.23	3.72
EH II	0.4424	0.4417	−1894.77	6.69
EL II	0.4569	0.2757	−2420.43	6.60

intuitive concept of this potential is the following. Imagine two nucleons separated by a distance $\vec{R}$. By virtue of the qq interaction defined in (2.2), this configuration of the two nucleons corresponds to a certain potential energy, and when we change $\vec{R}$, this potential energy follows this change, and the NN QEP is just this potential energy. Mathematically, if $\hat{V}$ is the potential energy between six quarks and $|\Psi\rangle$ is the state of six quarks, then the potential energy is $\propto\langle\Psi|\hat{V}|\Psi\rangle$. Essentially, if we sharpen this definition, we would have the NN QEP, which can then be subsequently calculated. The above argument suggests that first of all the six quark wave function be constructed, corresponding to the two-nucleon spatial configuration specified above.

Let $\phi_a(123)$, $\phi_b(456)$ be the completely antisymmetric WF's for the two-nucleon clusters a and b situated at a distance $\vec{R}$ apart. The completely antisymmetric wave function Ψ for six quarks (i.e., for the two nucleon system) can be built up) from the basis function ϕ_a, ϕ_b for three-quark subsystems,

$$\Psi=\frac{1}{\sqrt{20}}\sum_{\alpha=1}^{20}(-1)^{\alpha}\hat{P}_{\alpha}[\phi_a(123)\phi_b(456)]_{ST}\times F_{ST}\left(\frac{\vec{r}_1+\vec{r}_2+\vec{r}_3-\vec{r}_4-\vec{r}_5-\vec{r}_6}{3}\right). \tag{2.3}$$

In (2.3) the cluster WF's ϕ_a, ϕ_b are coupled to total spin (isospin) $S(T)$ so that Ψ is a state of sharp spin (isospin) $S(T)$. And F_{ST} is the WF describing the relative motion of the clusters a and b. The operators $\{\hat{P}_\alpha\}$ are all possible *intercluster* permutations, and the sign $(-1)^\alpha$ is the signature of the permutation $\hat{P}_\alpha$. This six-quark WF can be cast in a more convenient "cluster-antisymmetrized" form

$$\Psi=\frac{1}{\sqrt{10}}\Bigg(1-\sum_{\substack{i\in a\\ j\in b}}\hat{P}_{ij}\Bigg)[\phi_a(123)\phi_b(456)]_{ST}\times\psi_{ST}\left(\frac{\vec{r}_1+\vec{r}_2+\vec{r}_3-\vec{r}_4-\vec{r}_5-\vec{r}_6}{3}\right), \tag{2.4}$$

where

$$\psi_{ST}(\vec{R})=\frac{1}{\sqrt{2}}[F_{ST}(\vec{R})+(-1)^{S+T+1}F_{ST}(-\vec{R})],$$

and

$$a(b)=\{1,2,3\}(\{4,5,6\}).$$

An advantage of the cluster-antisymmetrized form is that it is an illustration of the validity of the spin-statistics theorem of Ehrenfest and Oppenheimer.[19]

C. Definition of the NN quark exchange potential

Let H be a six-quark Hamiltonian defined through the two-body interactions (2.2), i.e.,

$$H=\sum_{i=1}^{6}T_i+\tfrac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}{}'V(ij). \tag{2.5}$$

Considering the sum of the kinetic energies (KE) of the three quarks in nucleon $a(b)$, this sum is equal to the KE of these three quarks *about* the center of mass of nucleon $a(b)$ together with the KE of the center of mass of nucleon $a(b)$.[20] Calling the latter parts T_a, T_b, their sum can again be split up into two parts, namely, the KE of the reduced mass particle of the nucleons a and b and the KE of the center of mass of nucleons a and b. If all calculations are performed in the center-of-mass frame of the nucleons a and b, then the latter term in the KE drops out giving

$$\sum_{i=1}^{6}T_i=t_a+t_b+\frac{\vec{p}^2}{3m}. \tag{2.6}$$

Here $t_a(t_b)$ denote the KE of three quarks *about* the center of mass of their respective nucleon $a(b)$, i.e., it is the internal KE while $\vec{p}^2/3m$ represents the KE of the reduced mass particle of nucleons a and b each having a mass $3m$ (m being the quark mass).

Before the potential energy term in (2.5) is considered, we observe a result which will be used in the sections to follow. If $\hat{P}$ is the sum of the nine permutation operators $\hat{P}_{ij}$ in (2.4), then

$$\langle\Psi|\Psi\rangle=\tfrac{1}{10}\langle\phi_a\overrightarrow{(123)}\phi_b^{S,T}(456)\psi_{ST}|(1-\hat{P})(1-\hat{P})|\phi_a\overrightarrow{(123)}\phi_b^{S,T}(456)\psi_{ST}\rangle$$
$$=\langle\phi_a\overrightarrow{(123)}\phi_b^{S,T}(456)\psi_{ST}|(1-\hat{P})|\phi_a\overrightarrow{(123)}\phi_b^{S,T}(456)\psi_{ST}\rangle. \tag{2.7}$$

This norm $\mathfrak{N}$ is invariant under the two sets of permutations of the indices i,j; $i,\ j\in a$ and of the indices k,l; $k,l\in b$. Use of these two symmetries reduces (2.7) to the form

$$\mathfrak{N}=\langle\phi_a\phi_b\psi_{ST}|(1-9\hat{P}_{36})|\phi_a\phi_b\psi_{ST}\rangle. \tag{2.8}$$

To prevent the notation from becoming clumsy, from this point onwards, the arguments of ϕ_a and

ϕ_b can be suppressed, and also the coupling symbol $\overset{S,T}{\rightarrow}$ —which indicates that the nucleonic cluster functions have been coupled to have a total spin (isospin) of $S(T)$—can be dropped. For the expectation value (EV) of the potential energy operator in (2.5) in the state Ψ, one has

$$\left\langle\Psi\left|\tfrac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}{}' V(ij)\right|\Psi\right\rangle = \tfrac{1}{10}\left\langle\phi_a\phi_b\psi_{ST}\left|(1-\hat{P})\tfrac{1}{2}\sum_{ij}{}' V(ij)(1-\hat{P})\right| \times\phi_a\phi_b\psi_{ST}\right\rangle .$$

So long as the indices i, j in the above sum span all the six quarks, $\hat{P}$ commutes with the total potential energy operator. In this way, the operator in the EV becomes $\frac{1}{2}\sum'_{ij}V(ij)(1-\hat{P})$; using again the two symmetries of the indices pertaining to the two clusters, this EV can be reduced to the form

$$\left\langle\phi_a\phi_b\psi_{ST}\left|\tfrac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}{}' V(ij)(1-9\hat{P}_{36})\right|\phi_a\phi_b\psi_{ST}\right\rangle . \quad (2.9)$$

The qq interactions in the above sum can be divided into two sets: intercluster interactions and intracluster interactions. These intracluster qq interactions together with the internal KE's $t_a + t_b$ in (2.6) contribute to the rest mass of the two nucleon clusters, namely, $2M_N$. Subtracting out this rest mass from the EV $\langle\Psi|H|\Psi\rangle$, one has

$$\begin{aligned}\langle\Psi|H|\Psi\rangle - \varepsilon\langle\Psi|\Psi\rangle &= \left\langle\phi_a\phi_b\psi_{ST}\left|\frac{\vec{p}^2}{3m}(1-9\hat{P}_{36})\right|\phi_a\phi_b\psi_{ST}\right\rangle \\ &+\left\langle\phi_a\phi_b\psi_{ST}\left|\sum_{\substack{i\in a\\ j\in b}} V(ij)(1-9\hat{P}_{36})\right|\phi_a\phi_b\psi_{ST}\right\rangle \\ &-\varepsilon\langle\phi_a\phi_b\psi_{ST}|(1-9\hat{P}_{36})|\phi_a\phi_b\psi_{ST}\rangle . \quad (2.10)\end{aligned}$$

Functional variation of this expression with respect to Ψ_{ST} yields the following Schrödinger equation for the relative motion of the two nucleons:

$$\frac{\vec{p}^2}{3m}\psi_{ST} + \left\langle\phi_a\phi_b\left|\sum_{\substack{i\in a\\ j\in b}} V(ij)(1-9\hat{P}_{36})\right|\phi_a\phi_b\psi_{ST}\right\rangle\Big/\mathfrak{N} = \varepsilon\psi_{ST} \quad (2.11)$$

In this formulation, the QEP is defined by the second term on the left hand side of (2.11). The identification of the QEP with this term would have been exact had the bare reduced mass $3m/2$ been equal to $\frac{1}{2}M_N$. This equality can come about only if mass renormalization is performed, but this is outside the scope of the present nonrelativistic framework. In any case, the NN interaction can be defined by the second term on the left-hand side of (2.11). The remainder of this paper is concerned with the calculation of this EV. Robson's definition of the QEP[12] is a particular case of the definition given here, and it suffers from the drawback of ignoring all but the $V(36)$ term in the summation in (2.11). The error involved in this omission will be discussed in subsequent sections.

III. GENERAL STRUCTURE OF THE QEP

The evaluation of the QEP in the subspaces of color, spin-isospin (SI) and ordinary space is considered now in that order. The EV defining the QEP is invariant under two sets of symmetry operations: permutations of any pair of indices $\in$ nucleon a and permutations of any pair of indices $\in$ nucleon b. Using these two symmetries, the direct part of the QEP can be brought to the form

$$\mathcal{V}_D = 9\langle\phi_a\phi_b|V(36)|\phi_a\phi_b\rangle .$$

This direct term vanishes on account of ϕ_a, ϕ_b being color singlets.

Thus the NN interaction arises solely from the antisymmetrization of the six-quark WF, i.e., only when quarks are exchanged between the two nucleons. Using the two symmetries specified above and the additional $(123) \longleftrightarrow (456)$ symmetry of the EV, the exchange part of the QEP can be reduced to

$$\begin{aligned}\mathcal{V} &= -9\left\langle\phi_a\phi_b\left|\sum_{\substack{i\in a\\ j\in b}} V(ij)\hat{P}_{36}\right|\phi_a\phi_b\right\rangle \\ &= -9\langle\phi_a\phi_b|[4V(14)+4V(16)+V(36)]\hat{P}_{36}|\phi_a\phi_b\rangle . \quad (3.1)\end{aligned}$$

A. Evaluation of the color EV's

The permutation operator in (3.1) acts on all the subspaces: color, SI, and ordinary space,

$$\hat{P}_{36} = \hat{P}^C_{36}\hat{P}^\sigma_{36}\hat{P}^\tau_{36}\hat{P}^x_{36} . \quad (3.2)$$

For the group $SU_C(3)$, the permutation operator is

$$\hat{P}^C_{36} = \tfrac{1}{3} + 2\vec{F}_3\cdot\vec{F}_6 . \quad (3.3)$$

The form given in Robson's paper[12] is incorrect. Considering first the 14 term in (3.1) as an illustration, the EV in color subspace is (suppressing the arguments of ζ_a, ζ_b),

$$\Gamma(14) = \langle\zeta_a\zeta_b|\vec{F}_1\cdot\vec{F}_4(\tfrac{1}{3}+2\vec{F}_3\cdot\vec{F}_6)|\zeta_a\zeta_b\rangle .$$

Direct calculations gives

$$\Gamma(14) = \tfrac{2}{9}\sum_{\alpha,\beta}\tfrac{1}{4}\,\mathrm{Tr}^2(F(\alpha)F(\beta)) = \tfrac{1}{9} ,$$

since[21]

$$\mathrm{Tr}(F(\alpha)F(\beta)) = \tfrac{1}{2}\delta_{\alpha\beta} . \quad (3.4)$$

Thus the QEP, after completing the color algebra for the remaining two cases, acquires the form

$$\mathcal{V} = -4\langle \phi_a\phi_b | [V(14) + V(36) - 2V(16)] \times \hat{P}_{36}^{\sigma\tau x} | \phi_a\phi_b\rangle \,. \qquad (3.5)$$

B. Evaluation of the spin-isospin EV's

The first step will consist in the calculation of the norm defined in (2.8). It will be helpful to remember that the cluster functions therein are coupled to total spin (isospin) $S(T)$. Color algebra (namely, the EV $\langle \zeta_a\zeta_b | \hat{P}_{36}^{C} | \zeta_a\zeta_b\rangle$) gives a factor $\frac{1}{3}$, and disentangling the space part from the SI part, one has

$$\mathfrak{N} = 1 - 3\langle \phi_a\phi_b | \hat{P}_{36}^{x} | \phi_a\phi_b\rangle \langle \phi_a\phi_b | \hat{P}_{36}^{\sigma\tau} | \phi_a\phi_b\rangle \,. \qquad (3.6)$$

In Appendix A, it is shown that the SI EV in (3.6) possesses a factorization property, and calculating the various $9-j$ symbols therein, one has

$$\mathfrak{N} = 1 - 3\langle \phi_a\phi_b | \hat{P}_{36}^{x} | \phi_a\phi_b\rangle \times [(S-\tfrac{1}{2})(T-\tfrac{1}{2}) + 2(\tfrac{1}{2}-S/3)(\tfrac{1}{2}-T/3) + \tfrac{1}{9}(\tfrac{1}{2}+S/3)(\tfrac{1}{2}+T/3)] \,. \qquad (3.7)$$

This expression is symmetric in S and T, as it ought to be, and the factorization of the EV into space and SI parts, being a property common to all the EV's in the six-quark state, will be frequently encountered in the following paragraphs. It will be worthwhile to remember that all spin-independent qq interactions in (2.2), when averaged in the six-quark state Ψ, will have their SI structure identical to that of the norm in (3.7).

The interactions between quarks, which depend on their spins, are of the spin-spin, spin-orbit, and tensor types [see Eq. (2.2)]. Each of these types is considered separately now.

1. The spin-spin interaction

The starting point will be always the "color independent" form of the NN QEP, namely (3.5),

$$\mathcal{V}_{SS} = -4\langle \phi_a\phi_b | [V_{SS}(14) + V_{SS}(36) \times -2V_{SS}(16)] \hat{P}_{36}^{\sigma\tau x} | \phi_a\phi_b\rangle \,.$$

By inserting the qq interaction (2.2) in the above expression,

$$V_{SS}(ij) = -\frac{8\pi\hbar^2\alpha_s}{3m^2c^2}\delta^3(\vec{r}_{ij})\vec{S}_i\cdot\vec{S}_j \,, \qquad (3.8)$$

and evaluating the SI EV using the result of Appendix A, one has

$$\mathcal{V}_{SS}(16) = 8\left(-\frac{8\pi\hbar^2\alpha_s}{3m^2c^2}\right) \times \langle \phi_a\phi_b | \delta^3(\vec{r}_{16})\hat{P}_{36}^{x} | \phi_a\phi_b\rangle f_{16}(S,T) \,, \qquad (3.9)$$

with

$$f_{16}(S,T) = \left(\frac{3-2T}{72}\right)[S(S+1)-3] - \left(\frac{3+2T}{36}\right)\left(\frac{S(S+1)+3}{18}\right). \qquad (3.10)$$

Repeating this calculation for the 14 and 36 cases, one has

$$\mathcal{V}_{SS}(14) = -4\left(-\frac{8\pi\hbar^2\alpha_s}{3m^2c^2}\right) \times \langle \phi_a\phi_b | \delta^3(\vec{r}_{14})\hat{P}_{36}^{x} | \phi_a\phi_b\rangle f_{14}(S,T) \,, \qquad (3.11)$$

with

$$f_{14}(S,T) = -\frac{(3+2S)}{36}\frac{(3+2T)}{36} + \frac{5S(3+2T)}{6(81)} + \frac{(-1)^S(3+2T)}{12(81)}(\tfrac{3}{2}-S(S+1)^2 \,, \qquad (3.12)$$

and

$$\mathcal{V}_{SS}(36) = -4\left(-\frac{8\pi\hbar^2\alpha_s}{3m^2c^2}\right) \times \langle \phi_a\phi_b | \delta^3(\vec{r}_{36})\hat{P}_{36}^{x} | \phi_a\phi_b\rangle f_{36}(S,T) \,, \qquad (3.13)$$

with

$$f_{36}(S,T) = \frac{3+2T}{9} - 2\left(\frac{(3+2T)(2-S)(S+3)}{36\times 36} + \frac{S(S+1)(3-2T)}{72} + \frac{(2T-1)(1-S)}{8}\right) - 3\left(\frac{(2S-1)(2T-1)}{16} + \frac{(3-2S)(3-2T)}{72} + \frac{(3+2S)(3+2T)}{36\times 36}\right). \qquad (3.14)$$

The spin-spin interaction between nucleons a and b can now be cast in the canonical form of a central potential with the Wigner, Bartlett, Heisenberg, and Majorana exchanges.[22] Inserting the foregoing results into (3.5),

$$\mathcal{V}_{SS} = V_C^{SS} + V_\sigma^{SS}\vec{\sigma}_a\cdot\vec{\sigma}_b + V_\tau^{SS}\vec{\tau}_a\cdot\vec{\tau}_b + V_{\sigma\tau}^{SS}(\vec{\sigma}_a\cdot\vec{\sigma}_b)(\vec{\tau}_a\cdot\vec{\tau}_b) \,, \qquad (3.15)$$

and taking the trace of both sides of (3.15), one obtains

$$\begin{aligned}
V_C^{SS} &= -\tfrac{1}{4}[3m_S(36) + \tfrac{1}{3}m_S(14) + 2m_S(16)] \,,\\
V_\sigma^{SS} &= -\tfrac{1}{36}[-m_S(36) + m_S(14) - 2m_S(16)] \,,\\
V_\tau^{SS} &= -\tfrac{1}{108}[9m_S(36) + m_S(14) - 6m_S(16)] \,,\\
V_{\sigma\tau}^{SS} &= -\tfrac{1}{324}[-25m_S(36) + m_S(14) + 10m_S(16)] \,,
\end{aligned} \qquad (3.16)$$

where the $m_s(ij)$ are the radial EV's given by

$$m_s(ij) = -\frac{8\pi\hbar^2\alpha_s}{3m^2c^2}\langle\phi_a\phi_b|\delta^3(\vec{r}_{ij})\hat{P}^x_{36}|\phi_a\phi_b\rangle\,. \quad (3.17)$$

2. *The spin-orbit interaction*

The starting point is again (3.5) with the qq spin-orbit potential being given by

$$V_{ls}(ij) = \frac{-\hbar\alpha_s}{2m^2c^2} \times\left(\frac{\vec{r}_{ij}\times(\vec{p}_i - 2\vec{p}_j)\cdot\vec{S}_i - \vec{r}_{ij}\times(\vec{p}_j - 2\vec{p}_i)\cdot\vec{S}_j}{r_{ij}^{\ 3}}\right). \quad (3.18)$$

It is shown starting from this interaction that it implies a spin-orbit interaction between the two nucleons. As has been the practice up to now, the 16 calculation is carried out in some detail, and subsequently the corresponding results for the remaining two cases are presented. The general EV for the 16 case has the structure

$$\mathcal{V}_{ls}(16) = 8\left(\frac{-\hbar\alpha_s}{2m^2c^2}\right) \cdot \times\langle\phi_a\phi_b|(\vec{A}\cdot\vec{S}_1 + \vec{B}\cdot\vec{S}_6)\hat{P}^{\sigma\tau}_{36}\hat{P}^x_{36}|\phi_a\phi_b\rangle\,. \quad (3.19)$$

From (3.18) it is seen that the vectors $\vec{A}$, $\vec{B}$ act only upon the 0(3) states so that, as far as the SI space is concerned, they may be treated as constant vectors.

The factorization property embodied in (A11) of Appendix A leads to the following structure of the SI EV:

$$\langle\phi_a\phi_b|\vec{A}\cdot\vec{S}_1\hat{P}^{\sigma\tau}_{36}|\phi_a\phi_b\rangle = \frac{1}{4}\sum_{ll'}4\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & l' & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}\mu_{\vec{A}\cdot\vec{S}_1}(16), \quad (3.20)$$

with

$$\mu_{\vec{A}\cdot\vec{S}_1}(16) = \langle((l\tfrac{1}{2})\tfrac{1}{2},(l\tfrac{1}{2})\tfrac{1}{2};S,M|\vec{A}\cdot\vec{S}_1\hat{P}^{\sigma}_{36}| \times(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M\rangle\,,$$

$$\mu_{\vec{A}\cdot\vec{S}_1}(16) = \vec{A}\cdot\vec{S}\left[l(l+1)\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & l' & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1\end{Bmatrix} - \frac{l(l+1)}{6}\begin{Bmatrix} \frac{1}{2} & l' & \frac{1}{2} \\ \frac{1}{2} & l & \frac{1}{2}\end{Bmatrix}\right], \quad (3.21)$$

and

$$\mu_{\vec{B}\cdot\vec{S}_6}(16) = \vec{B}\cdot\vec{S}\left[\frac{[3-2l'(l'+1)]}{3}\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & l' & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1\end{Bmatrix} - \tfrac{1}{2}[3-2l(l+1)]\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & l \\ \frac{1}{2} & \frac{1}{2} & l'\end{Bmatrix}\right], \quad (3.22)$$

and the EV of $\vec{B}\cdot\vec{S}_6$ has the structure similar to that of $\vec{A}\cdot\vec{S}_1$ [see Eq. (3.20)]. The results corresponding to (3.21) and (3.22) for the 14 and 36 cases are

$$\mu_{\vec{C}\cdot\vec{S}_4}(14) = \vec{C}\cdot\vec{S}\left[l'(l'+1)\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & l' & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1\end{Bmatrix} - \frac{l'(l'+1)}{6}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & l \\ \frac{1}{2} & \frac{1}{2} & l'\end{Bmatrix}\right], \quad (3.23)$$

$$\mu_{\vec{D}\cdot\vec{S}_3}(36) = \vec{D}\cdot\vec{S}\left[\frac{[3-2l(l+1)]}{3}\begin{Bmatrix} l' & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & l & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1\end{Bmatrix} - \frac{[3-2l'(l'+1)]}{6}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & l \\ \frac{1}{2} & \frac{1}{2} & l'\end{Bmatrix}\right]. \quad (3.24)$$

These could have been obtained from (3.21) and (3.22)—in so far as their ll' dependence is concerned—by interchanging l and l'. This symmetry stems from the $(123)\longleftrightarrow(456)$ symmetry which is possessed by all these EV's. Performing the ll' summation in (3.20), after substituting for (3.21) and (3.22), one has

$$\mathcal{V}_{ls}(16) = -\frac{8\hbar\alpha_s}{2m^2c^2}\langle\phi_a\phi_b|\hat{O}_{16}\hat{P}^x_{36}|\phi_a\phi_b\rangle\,, \quad (3.25)$$

where

$$\hat{O}_{16} = \vec{A}(16)\cdot\vec{S}f_1(T) + \vec{B}(16)\cdot\vec{S}f_2(T)\,,$$

and

$$\vec{A}(16)[\vec{B}(16)] = \frac{\vec{r}_{16}\times\vec{p}_1 - 2\vec{r}_{16}\times\vec{p}_6}{r_{16}^{\ 3}}\left[\frac{2\vec{r}_{16}\times\vec{p}_1 - \vec{r}_{16}\times\vec{p}_6}{r_{16}^{\ 3}}\right], \quad (3.26)$$

$$f_1(T)=\frac{8(3+2T)}{36\times 36},$$

$$f_2(T)=\frac{(2T-1)}{8}-\frac{(2T+3)}{54\times 36}-\frac{(3-2T)}{12\times 9}.$$

The spin-orbit NN interaction arising from the 14 and 36 exchanges are, using (3.23) and (3.24),

$$\mathcal{V}_{ls}(14)=\frac{4\hbar\alpha_s}{2m^2c^2}\langle\phi_a\phi_b|\hat{O}_{14}\hat{P}^x_{36}|\phi_a\phi_b\rangle, \tag{3.27}$$

with

$$\hat{O}_{14}=[\vec{A}(14)+\vec{B}(14)]\cdot\vec{S}f_1(T),$$

and

$$\mathcal{V}_{ls}(36)=\frac{4\hbar\alpha_s}{2m^2c^2}\langle\phi_a\phi_b|\hat{O}_{36}\hat{P}^x_{36}|\phi_a\phi_b\rangle, \tag{3.28}$$

with

$$\hat{O}_{36}=[\vec{A}(36)+\vec{B}(36)]\cdot\vec{S}f_2(T).$$

The total spin-orbit potential is the sum of (3.25), (3.27), and (3.28).

The total spin-orbit interaction between nucleons can be cast in the canonical form. Firstly,

$$\begin{aligned}\mathcal{V}_{ls}&=\mathcal{V}_{ls}(16)+\mathcal{V}_{ls}(14)+\mathcal{V}_{ls}(36)\\&=\mathcal{V}^{even}_{ls}+\mathcal{V}^{odd}_{ls}\\&=\langle\phi_a\phi_b|\vec{\alpha}^{even}\cdot(\vec{S}_a+\vec{S}_b)\hat{P}^x_{36}|\phi_a\phi_b\rangle\\&\quad+\langle\phi_a\phi_b|\vec{\alpha}^{odd}\cdot(\vec{S}_a+\vec{S}_b)\hat{P}^x_{36}|\phi_a\phi_b\rangle,\end{aligned}$$

where the even and odd parts are given by

$$\vec{\alpha}^{even\,(odd)}=\frac{2\hbar\alpha_s}{m^2c^2}\left[\left(\frac{\vec{A}(14)+\vec{B}(14)}{r_{14}^{\;3}}-\frac{2\vec{A}(16)}{r_{16}^{\;3}}\right)\tfrac{5}{162}\left(\tfrac{1}{54}\right)+\left(\frac{\vec{A}(36)+\vec{B}(36)}{r_{36}^{\;3}}-\frac{2\vec{B}(16)}{r_{16}^{\;3}}\right)\tfrac{55}{486}\left(-\tfrac{100}{648}\right)\right]. \tag{3.29}$$

To bring out the NN spin-orbit potential in the canonical form $\propto\vec{L}_R\cdot\vec{S}$, it is necessary to carry out the radial EV's in the foregoing equations. This will be taken up in the section on radial EV's (see Sec. IV C).

3. *The tensor interaction*

The NN interaction arising from the qq tensor potential also turns out to be tensorial. The calculation of the SI parts starts from (3.5) with the qq interactions given by

$$V_t(ij)=\frac{\hbar^2\alpha_s}{m^2c^2}\left(\frac{\vec{S}_i\cdot\vec{S}_j-3(\vec{S}_i\cdot\hat{n})(\vec{S}_j\cdot\hat{n})}{r_{ij}^{\;3}}\right),$$

and the NN tensor potential (anticipating the result) is

$$\mathcal{V}_t=-4\langle\phi_a\phi_b|[V_t(14)+V_t(36)-2V_t(16)]\hat{P}^{\sigma\tau x}_{36}|\phi_a\phi_b\rangle. \tag{3.30}$$

The proof is first given—for the above statement—for the case of the 16 term, and then the corresponding results for the other two cases are presented. The tensor potential (3.30) is actually a scalar obtained by contracting two second rank tensors.[23] One of these is compounded from the vector operator $\hat{n}_{ij}$ ($\hat{n}_{ij}$ is the unit vector in the direction $\vec{r}_{ij}$), while the other is a second rank tensor compounded from two vector spin operators of the two particles $\vec{S}_i$ and $\vec{S}_j$. Therefore (3.30) can be written as

$$V_t(ij)=\frac{\hbar^2\alpha_s}{m^2c^2}\frac{1}{r_{ij}^{\;3}}T_2(\vec{S}_i,\vec{S}_j)\cdot T_2(\hat{n}_{ij}). \tag{3.31}$$

The NN potential $\mathcal{V}_t$ arising from this, if it is truly tensorial, will have the same general structure as in (3.31), i.e., $\propto T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n})$, where $\hat{n}$ is a unit vector in the direction $\vec{R}$. In attempting to derive the "macroscopic" tensor potential from the "microscopic" one, it will be useful to bear in mind that this derivation must necessarily proceed in two stages. The SI calculations will lead to the replacement

$$T_2(\vec{S}_i,\vec{S}_j)\to T_2(\vec{S}_a,\vec{S}_b),$$

and the radial calculations to the replacement $T_2(\hat{n}_{ij})\to T_2(\hat{n})$ for all the (ij)'s in (3.31). Therefore the proof will be completed only towards the end of the section on the evaluations of the radial EV's (see Sec. IV D). For the present, the proof is carried out only for the SI parts. For illustration, the 16 case is considered in some detail and the corresponding results for the remaining two cases are then presented.

$$\mathcal{V}_t(16)=\frac{8\hbar^2\alpha_s}{m^2c^2}\left\langle\phi_a\phi_b\left|\frac{1}{r_{16}^{\;3}}T_2(\vec{S}_1,\vec{S}_6)\cdot T_2(\hat{n}_{16})\times\hat{P}^{\sigma\tau}_{36}\hat{P}^x_{36}\right|\phi_a\phi_b\right\rangle. \tag{3.32}$$

Using the factorization property of the SI EV, embodied in (A11) of Appendix A,

$$\langle\phi_a\phi_b|T_2(\vec{S}_1,\vec{S}_6)\cdot T_2(\hat{n}_{16})\hat{P}^{\sigma\tau}_{36}|\phi_a\phi_b\rangle=\tfrac{1}{4}\sum_{ll'}4\begin{Bmatrix}l&\tfrac{1}{2}&\tfrac{1}{2}\\\tfrac{1}{2}&l'&\tfrac{1}{2}\\\tfrac{1}{2}&\tfrac{1}{2}&T\end{Bmatrix}\mu_t(16), \tag{3.33}$$

where

$$\mu_t(16)=\langle(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M|T_2(\vec{S}_1,\vec{S}_6)\cdot T_2(\hat{n}_{16})\hat{P}^{\sigma}_{36}|(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M\rangle, \tag{3.34}$$

and after some manipulation, (3.34) can be reduced to the form

$$\left\langle S,M\left|\tfrac{1}{2}\sum_q(-1)^q T_{2-q}(\hat{n}_{16})T_{2q}(\vec{S}_1,\vec{S}_6)\right|S,M\right\rangle .$$

By successive application of Wigner-Eckart theorem and more powerful ones derived thereof,[24] one has, for this matrix element,

$$\tfrac{1}{2}\left(\frac{T_{20}(\hat{n}_{16})2[3M^2-S(S+1)]}{[(2S+3)(2S+2)(2S+1)(2S)(2S-1)]^{1/2}}\right.$$

$$\left.\times[S]\sqrt{[2]}\,\delta_{S,1}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & 1\\ 1 & 1 & 2\end{Bmatrix}\right)$$

$$\times\tfrac{1}{2}\langle(l)\tfrac{1}{2}\|\vec{S}_{12}\|(l\tfrac{1}{2})\tfrac{1}{2}\rangle$$

$$\times\langle(l'\tfrac{1}{2})\tfrac{1}{2}\|\vec{S}_6\|(l'\tfrac{1}{2})\tfrac{1}{2}\rangle . \qquad (3.35)$$

By considering now the EV $\langle S,M|\hat{T}_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{16}|S,M\rangle$, it can be shown to be $=\frac{3}{2}\times$the quantity in the curly brackets in (3.35). Finally evaluating the two reduced matrix elements in (3.35), the EV $\mu_t(16)$ can be written as a tensor interaction between the two nucleons a and b, in so far as the spin structure is concerned,

$$\mu_t(16)=T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{16})\times\{\tfrac{1}{9}l(l+1)[\tfrac{3}{2}-l'(l'+1)]\} . \qquad (3.36)$$

The remainder of the proof will be carried out in the section on the radial EV's (see Sec. IV D). The results analogous to (3.36) for the 14 and 36 cases are

$$\mu_t(14)=T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{14})\left(\frac{l(l+1)l'(l'+1)}{9}\right), \qquad (3.37)$$

$$\mu_t(36)=T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{36})\{\tfrac{2}{3}[\tfrac{3}{2}-l(l+1)]\times\tfrac{2}{3}[\tfrac{3}{2}-l'(l'+1)]\} . \qquad (3.38)$$

Inserting (3.36) in (3.33) and carrying out the l, l' summation, one has

$$\mathcal{V}_t(16)=\frac{8\hbar^2\alpha_s}{m^2c^2}\frac{1}{36}\left((3-2T)-\frac{(3+2T)}{9}\right)\times\langle\phi_a\phi_b|\frac{T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{16})}{r_{16}^3}\hat{P}^x_{36}|\phi_a\phi_b\rangle , \qquad (3.39)$$

$$\mathcal{V}_t(14)=-\frac{4\hbar^2\alpha_s}{m^2c^2}\left(\frac{3+2T}{81}\right)\times\langle\phi_a\phi_b|\frac{T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{14})}{r_{14}^3}\hat{P}^x_{36}|\phi_a\phi_b\rangle , \qquad (3.40)$$

and

$$\mathcal{V}_t(36)=-\frac{4\hbar^2\alpha_s}{m^2c^2}\left(\frac{(3+2T)}{9\times 36}+\frac{(2T-1)}{4}+\frac{(2T-3)}{18}\right)\times\langle\phi_a\phi_b|\frac{T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{n}_{36})}{r_{36}^3}\hat{P}^x_{36}|\phi_a\phi_b\rangle . \qquad (3.41)$$

The tensor interaction is generally decomposed into even ($T=1$) and odd ($T=0$) parts,

$$\mathcal{V}_t=\mathcal{V}_t^{even}+\mathcal{V}_t^{odd},$$

$$\mathcal{V}_t^{even(odd)}=\frac{4\hbar^2\alpha_s}{m^2c^2}\langle\phi_a\phi_b|T_2(\vec{S}_a,\vec{S}_b)\cdot\mathcal{T}^{even(odd)}\times\hat{P}^x_{36}|\phi_a\phi_b\rangle ,$$

where

$$\mathcal{T}^{even(odd)}=\tfrac{2}{81}(\tfrac{12}{81})\frac{T_2(\hat{n}_{16})}{r_{16}^3}-\tfrac{5}{81}(-\tfrac{3}{81})\frac{T_2(\hat{n}_{14})}{r_{14}^3}-\tfrac{17}{81}(+\tfrac{33}{81})\frac{T_2(\hat{n}_{36})}{r_{36}^3} . \qquad (3.42)$$

When the evaluation of these radial EV's is complete, it will be found that $T_2(\hat{n}_{ij})\to T_2(\hat{n})$ where $\hat{n}$ = unit vector in the direction of $\vec{R}$.

4. The central interaction

The SI structure of the central interactions will obviously be the same as that of the norm $\mathfrak{N}$ of the six-quark WF. There are essentially four different types of central interactions between quarks: Coulombian, oscillator, $\delta(\vec{r})$, and the momentum dependent potential [see (2.2)]. The last of these is the most difficult to evaluate, as will be seen in the forthcoming section on the radial EV's.

The central qq interactions give rise to central NN potentials with all the four types of exchanges. It is customary to cast this interaction in the standard form, analogous to (3.15). The direct part of the interaction is absent on account of nucleons being color singlets [see the text preceeding (3.1)]; only the exchange interaction contributes. From the starting point (3.5), one has

$$\mathcal{V}_c=-4\langle\phi_a\phi_b|[V_c(14)+V_c(36)-2V_c(16)]\hat{P}^x_{36}|\phi_a\phi_b\rangle\times\left(\frac{(2S-1)(2T-1)}{4}+\frac{2(3-2S)(3-2T)}{36}+\frac{(3+2S)(3+2T)}{9\times 36}\right), \qquad (3.43)$$

with the SI dependence being identical to that in the norm $\mathfrak{N}$ given by (3.7). Putting this equal to

$$V_c+V_\sigma\cdot\vec{\sigma}_a\cdot\vec{\sigma}_b+V_\tau\cdot\vec{\tau}_a\cdot\vec{\tau}_b+V_{\sigma\tau}(\vec{\sigma}_a\cdot\vec{\sigma}_b)(\vec{\tau}_a\cdot\vec{\tau}_b) . \qquad (3.44)$$

The component V's can be evaluated by trace methods already outlined [see sequel to (3.15)]. Finally, substituting these values of V's back into (3.44),

$$\mathcal{V}_c = -\rho - \tfrac{1}{9}\rho \circ \vec{\sigma}_a \circ \vec{\sigma}_b - \tfrac{1}{9}\rho \circ \vec{\tau}_a \circ \vec{\tau}_b - \tfrac{25}{81}\rho \circ (\vec{\sigma}_a \circ \vec{\sigma}_b)(\vec{\tau}_a \circ \vec{\tau}_b), \qquad (3.45)$$

with

$$\rho = \langle \phi_a \phi_b | [V_c(14) + V_c(36) - 2V_c(16)] \hat{P}^x_{36} | \phi_a \phi_b \rangle,$$

where V_c is the sum of the four central interactions mentioned earlier. This structure of ρ is very crucial, in making the *NN* interaction independent of the OGEC constant b in (2.2). The $[11-2]$ structure of the *NN* interaction [see (3.5)] is responsible for b being totally absent in the *NN* potential. Contrary to this, the role played by b in baryon spectroscopy was very decisive in obtaining a good fit to the baryonic levels.[15]

Having determined the SI structure of the *NN* QEP, attention is now turned to the determination of the radial dependence of the QEP.

IV. THE STRUCTURE OF THE QEP IN COORDINATE SPACE

Since the EV's are the averages in a six-quark state, the radial matrix elements are really 18-dimensional integrals. As in the treatment of the SI EV's, the various types of interactions can be examined sequentially. Before embarking on this program, it is proper that the nucleon radial WF ϕ_a be accurately defined. Up to this point, these WF's were kept quite general, but for calculational purposes these must be given specific forms. In the work on baryon spectroscopy, carried out by these authors,[15] two arguments R were used for ϕ_a, as were two forms for ϕ_a. The arguments were of a collective nature: $r_{12}+r_{23}+r_{31}$, $(r_{12}^2+r_{23}^2+r_{31}^2)^{1/2}$. For ϕ_a, both exponential and Gaussian forms were used. Without going into the reasons for this choice here, the following definition of ϕ_a suffices:

$$\phi_a(123) = \eta \exp[-\tfrac{1}{2}\beta^2(r_{12}^2 + r_{23}^2 + r_{31}^2)], \qquad (4.1)$$

where η, the normalization factor, $=(3\sqrt{3}/\pi^3)^{1/2}\beta^3$. In the present calculation only the normalized WF in (4.1) is used. Physicists have long since been familiar with Gaussian integrals, which can be evaluated in closed form; this is their main advantage. The value of β, which characterizes the WF and is compatible with s-wave baryonic data, is given in Table I.

A. The radial norm

A glance at (3.6) suggests the following definition for the radial norm:

$$\mathfrak{N}(R) = \langle \phi_a(123)\phi_b(456) | \hat{P}^x_{36} | \phi_a(123)\phi_b(456) \rangle. \qquad (4.2)$$

The structure of the radial EV's appears in (2.11). The value of such a matrix element will depend critically upon the definition of the exchange operator $\hat{P}^x_{36}$. This exchange operator not only exchanges $\vec{r}_3$ and $\vec{r}_6$ between $\phi_a(123)$ and $\phi_b(456)$, but also interchanges these two coordinates in the argument of the WF $\psi_{ST}(\vec{R})$, for the relative motion. Since $\vec{R} = \frac{1}{3}(\vec{r}_4 + \vec{r}_5 + \vec{r}_6 - \vec{r}_1 - \vec{r}_2 - \vec{r}_3)$, one has

$$\hat{P}^x_{36}\phi_a(123)\phi_b(456)\psi_{ST}(\vec{R}) = \phi_a(126)\phi_b(453)\psi_{ST}\left(\frac{\vec{r}_4 + \vec{r}_5 + \vec{r}_3 - \vec{r}_1 - \vec{r}_2 - \vec{r}_6}{3}\right).$$

Taylor expanding the WF of relative motion to first order,

$$\psi_{ST}(\hat{P}^x_{36}\vec{R}) = \psi_{ST}(\vec{R}) + \tfrac{2}{3}\vec{r}_{36} \circ \vec{\nabla}_R \psi_{ST}(\vec{R}), \qquad (4.3)$$

and the omission of the higher order terms is justified since $\psi_{ST}(\vec{R})$ is expected to vary slowly with $\vec{R}$. Now it turns out that *for all interactions excepting the spin-orbit one*, the contribution of the gradient term to the radial EV vanishes by virtue of the parity selection rule, whereas for the spin-orbit interaction, the matrix element corresponding to the first term vanishes while that corresponding to the second makes a nonzero contribution. Thus the action of $\hat{P}^x_{36}$ on $\vec{R}$ is included here contrary to the practice in molecular physics, and this action can be consistently defined for all the parts of the qq interactions in (2.2).

The radial norm (4.2) falls in the first category of EV's which do not receive any contribution from the second term in (4.3). This amounts to ignoring the action of $\hat{P}^x_{36}$ on $\psi_{ST}(\vec{R})$. All the calculations are performed in the center-of-mass frame of the dinucleon system. Thus (4.2) has the structure, if (4.1) is used,

$$\mathfrak{N}(R) = \eta^4 \int \prod_{i=1}^{6} d\vec{r}_i \exp(-\tfrac{1}{2}\beta^2\{\rho_a^2 + \rho_b^2 + \rho_a'^2 + \rho_b'^2\}) \times \delta\left(\frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3} + \frac{\vec{R}}{2}\right) \times \delta\left(\frac{\vec{r}_4 + \vec{r}_5 + \vec{r}_6}{3} - \frac{\vec{R}}{2}\right), \qquad (4.4)$$

with

$$\begin{aligned}\rho_a^2(\rho_a'^2) &= r_{12}^2 + r_{23}^2(r_{26}^2) + r_{13}^2(r_{16}^2),\\ \rho_b^2(\rho_b'^2) &= r_{45}^2 + r_{46}^2(r_{43}^2) + r_{56}^2(r_{53}^2).\end{aligned} \qquad (4.5)$$

The presence of the two delta functions needs some explanation. Going back to (2.11) and remembering that the QEP is defined at a fixed value of the argument of ψ_{ST}, namely $\vec{R}$, it can be seen that all these radial EV's must be carried out for a fixed value of $\vec{R}$. This is achieved by inserting a delta function $\delta[\vec{R} - \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3 - \vec{r}_4 - \vec{r}_5 - \vec{r}_6)]$ in the EV. The fact that all calculations are carried out in the center-of-mass frame of reference, when stated mathematically, is tantamount to the insertion of another delta function in the radial EV, $\delta[\frac{1}{3}(\vec{r}_1+\vec{r}_2+\vec{r}_3)-\frac{1}{3}(\vec{r}_4+\vec{r}_5+\vec{r}_6)]$. The pair of delta functions appearing in (4.4) are entirely equivalent to the ones mentioned above, as can be verified by inspection. These Gaussian three-dimensional integrals are easily evaluated in closed form using the results of Appendix B. The radial norm is thus found to be

$$\mathfrak{N}(R)=\frac{64}{27\times 5\times\sqrt{5}}\exp(-\tfrac{9}{5}\beta^2R^2). \tag{4.6}$$

B. The EV's of central potentials

The central potentials are four in number: Coulomb, oscillator (confinement), delta function, and the potential bilinear in the quark momenta. The Coulomb potential will be considered first. From (3.5), it is seen that three distinct EV's contribute to the *NN* potential. Calculations show that a Coulomb potential between quarks leads to a $\mathrm{erf}(CR)/R$ potential between the nucleons. Contrary to the practice of illustrating the calculation for a specific pair of quark indices, only the final results of the calculations are presented for the EV's of the central potentials.

The Coulomb radial EV's, which enter into (3.45), are proportional to

$$M_{cb}(ij)=\alpha_s\left\langle\phi_a\phi_b\left|\frac{1}{|\vec{r}_{ij}|}\hat{P}^x_{36}\right|\phi_a\phi_b\right\rangle. \tag{4.7}$$

The general structure of these EV's is found to be

$$M_{cb}(ij)=\alpha_s\times\mathfrak{N}(R)\frac{C_{ij}}{R}\mathrm{erf}(A_{ij}\beta R), \tag{4.8}$$

where $\mathrm{erf}(x)$ is the error function (see Appendix B). The values of the coefficients in (4.8) for all the three cases appear in Table II.

The radial EV's of the next central potential, namely, that of the harmonic oscillator, are proportional to

$$M_{cn}(ij)=-\frac{\alpha_s}{a^3}\langle\phi_a\phi_b\,|\,r_{ij}{}^2\hat{P}^x_{36}\,|\,\phi_a\phi_b\rangle. \tag{4.9}$$

Calculations show that these EV's can be brought to the form

$$M_{cn}(ij)=-\frac{\alpha_s}{a^3}\frac{\mathfrak{N}(R)}{\beta^2}\left(C'_{ij}+\frac{\beta^2R^2}{C_{ij}{}^2}\right). \tag{4.10}$$

The coefficients C'_{ij} for the three cases are listed in Table II. It can be seen from this expression that the harmonic confinement potential between quarks leads to a harmonic potential between the nucleons.

The radial EV's of delta-function potentials are proportional to

$$M_d(ij)=-\frac{\alpha_s\pi\hbar^2}{m^2c^2}\langle\phi_a\phi_b\,|\,\delta(\vec{r}_{ij})\hat{P}^x_{36}\,|\,\phi_a\phi_b\rangle, \tag{4.11}$$

and the radial spin-spin EV's are simply related to these

$$M_s(ij)=\tfrac{8}{3}M_d(ij). \tag{4.12}$$

Calculations yield the structure

$$M_d(ij)=-\frac{\alpha_s\pi\hbar^2}{m^2c^2}\mathfrak{N}(R)\beta^3\times C^{3/2\,(1)}_{ij}\exp(\beta^2A_{ij}{}^2R^2). \tag{4.13}$$

The presence of the Gaussian factors, over and above that in the radial norm, indicates that the *qq* delta-function interaction leads to an extremely short range *NN* potential. And even the strength of this potential is quite small, in view of the c^2 factor in the denominator.

The last of the central potentials whose radial EV's are necessary is the potential between the quarks which is bilinear in the momenta of the quarks. From the mathematical point of view, the evaluation of these EV's is extremely laborious. The EV's are proportional to

$$M_{pp}(ij)=-\frac{\alpha_s}{2m^2c^2}\left\langle\phi_a\phi_b\left|\left(\frac{\vec{p}_i\cdot\vec{p}_j}{|\vec{r}_{ij}|}+\frac{\vec{r}_{ij}\cdot(\vec{r}_{ij}\cdot\vec{p}_i)\vec{p}_j}{|\vec{r}_{ij}|^3}\right)\times\hat{P}^x_{36}\right|\phi_a\phi_b\right\rangle. \tag{4.14}$$

The calculations (with suppressed details) show that these EV's have a structure given by

TABLE II. The entries in the table [see Eqs. (4.8), (4.10), and (4.13)] define completely the radial part of the *NN* potential arising from the indicated *qq* interactions.

ij	Coulomb: $1/r$ C_{ij}	A_{ij}	Confinement: r^2 C'_{ij}	$(1/C_{ij}{}^2)$	Delta function: $\delta(\vec{r})$ $C^{(1)}_{ij}$	A_{ij}
14	$\frac{10}{12}$	$3(\frac{3}{10})^{1/2}$	$\frac{4}{5}$	$(\frac{6}{5})^2$	$15/8\pi$	$3(\frac{3}{10})^{1/2}$
36	$\frac{10}{6}$	$3(\frac{1}{20})^{1/2}$	$\frac{6}{5}$	$(\frac{3}{5})^2$	$5/4\pi$	$3(\frac{1}{20})^{1/2}$
16	$\frac{10}{9}$	$3(\frac{27}{140})^{1/2}$	$\frac{7}{10}$	$(\frac{9}{10})^2$	$15/7\pi$	$3(\frac{27}{140})^{1/2}$

TABLE III. The entries in the table [see (4.15)] define completely the radial part of the NN potential arising from the qq potential bilinear in the quark momenta.

ij	A_{ij}	C_{ij}	D_{ij}	G_{ij}	F_{ij}
14	$3(\frac{3}{10})^{1/2}\beta R$	$5/9\beta^2$	$\frac{9}{32}$	$\frac{15\sqrt{15}}{90\sqrt{2\pi}}\frac{1}{\beta}$	0
36	$3\beta R/\sqrt{20}$	$44/3\beta^2$	$\frac{54}{5}$	$\frac{72}{\sqrt{20\pi}}\frac{1}{\beta}$	0
16	$9\sqrt{3}/2\sqrt{35}\,\beta R$	$145/27\beta^2$	1	$\frac{17\sqrt{15}}{21\sqrt{7\pi}}\frac{1}{\beta}$	$\frac{45\sqrt{15}}{98\sqrt{7\pi}}\beta$

$$M_{pp}(ij)=\frac{\alpha_s\hbar^2}{2m^2c^2}\beta^4\mathfrak{N}(R)\times\left(\frac{\mathrm{erf}(A_{ij})}{R}(C_{ij}-D_{ij}R^2)-\exp(-A_{ij}{}^2)\{G_{ij}+F_{ij}R^2\}\right). \quad (4.15)$$

The values of these coefficients are tabulated in Table III. It is seen that this relativistic correction term leads to a Coulomb-type interaction between the nucleons. There is an additional contribution which resembles a delta-like interaction [see (4.13)] between nucleons. And yet another contribution which amounts to a damped harmonic interaction between the two nucleons.

C. The radial EV's of the spin-orbit term

The proof for the qq spin-orbit interaction going over to the NN spin-orbit interaction was suspended halfway in Sec. III B 2. Here, in this section, this remainder of the proof is carried to its conclusion. In essence it must be shown now that the radial EV's of the vectors $\vec{\alpha}^{\,even(odd)}$ in (3.29) are proportional to the angular momentum $\vec{L}_R$ of the relative motion of the two nucleons.

Recapitulating the discussion in Sec. IV A on the definition of $\hat{P}^x_{36}$, this operator must now be made to act on $\psi_{ST}(\vec{R})$, the WF for the relative motion of the two nucleons. The radial spin-orbit EV's are proportional to

$$M_{ls}(ij)=-\frac{\hbar\alpha_s}{2m^2c^2}\langle\phi_a\phi_b|\left(\frac{\vec{A}(ij)+\vec{B}(ij)}{|\vec{r}_{ij}|^3}\right)\cdot\vec{S}\times\hat{P}^x_{36}|\phi_a\phi_b\psi_{ST}\rangle, \quad (4.16)$$

where the vectors $\vec{A}(ij)$, $\vec{B}(ij)$ are defined in (3.26), and $\vec{S}$ is the total spin of the two nucleons, i.e., $=\vec{S}_a+\vec{S}_b$. The result of the calculations of the radial EV's is

$$M_{ls}(ij)=\frac{\alpha_s\hbar^2}{2m^2c^2}\frac{\mathfrak{N}(R)}{R}C_{ij}\frac{d}{dR}\times\left\{\frac{D'_{ij}}{R}\mathrm{erf}(A_{ij}\beta R)\right\}\vec{L}_R\cdot\vec{S}\psi_{ST}. \quad (4.17)$$

TABLE IV. The entries in the table define completely the radial structure [see (4.17) and (4.23)] of the NN spin orbit and tensor potentials arising from the corresponding qq potentials.

ij	Spin-orbit potential C_{ij}	D'_{ij}	A_{ij}	Tensor potential C_{ij}	A_{ij}
14	$\frac{5}{6}$	$\frac{10}{6}$	$3(\frac{3}{10})^{1/2}$	$\frac{5}{6}$	$3(\frac{3}{10})^{1/2}$
36	$\frac{5}{3}$	$\frac{10}{3}$	$3(\frac{1}{20})^{1/2}$	$\frac{5}{3}$	$3(\frac{1}{20})^{1/2}$
16	$\frac{10}{9}$	$\frac{10}{9}$	$\frac{9}{2}(\frac{3}{35})^{1/2}$	$\frac{10}{9}$	$\frac{9}{2}(\frac{3}{35})^{1/2}$

The coefficients occurring in this result are all listed in Table IV. The constant $\hbar$ that usually goes with an angular momentum operator has been lumped, in this case, along with the multiplicative constants. Thus the spin-orbit NN interaction has now been finally put in the canonical form.

D. The radial EV's of the tensor interaction

A glance at the end of Sec. III B3 shows that when the radial EV's are evaluated. it would amount to the quantities $\mathcal{T}^{even(odd)}$ being replaced by $T_2(\hat{n})$. In this section the latter half of this proof is carried through.

With the aid of the identity

$$\frac{\vec{S}_a\cdot\vec{S}_b-3(\vec{S}_a\cdot\hat{n})(\vec{S}_b\cdot\hat{n})}{r^3}=-(\vec{S}_a\cdot\vec{\nabla})(\vec{S}_b\cdot\vec{\nabla})\left(\frac{1}{r}\right)-\frac{4\pi}{3}\vec{S}_a\cdot\vec{S}_b\delta(\vec{r}), \quad (4.18)$$

the EV of (3.31) can be put in the form

$$t_{ij}=\langle\phi_a\phi_b|\tfrac{1}{3}T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\vec{\nabla}_{ij})\frac{1}{r_{ij}}\hat{P}^x_{36}|\phi_a\phi_b\rangle\ . \quad (4.19)$$

Using Fourier integral representations and completeness expansions, the EV (4.19) can be calculated, and the results are

$$t_{36}=\frac{1}{2\pi^2}\left(\frac{4\pi}{3}\right)\mathfrak{N}(R)T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{R})\times\int K^2dK\,j_2\left(\frac{3KR}{5}\right)\exp\left(\frac{-K^2}{5\beta^2}\right). \quad (4.20)$$

For the remaining two cases, one has

$$t_{14}=\frac{1}{2\pi^2}\left(\frac{4\pi}{3}\right)\mathfrak{N}(R)T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{R})\times\int K^2dK\,j_2\left(\frac{6KR}{5}\right)\exp\left(\frac{-2K^2}{15\beta^2}\right), \quad (4.21)$$

$$t_{16}=\frac{1}{2\pi^2}\left(\frac{4\pi}{3}\right)\mathfrak{N}(R)T_2(\vec{S}_a,\vec{S}_b)\cdot T_2(\hat{R})\times\int K^2dK\,j_2\left(\frac{9KR}{10}\right)\exp\left(\frac{-7K^2}{60\beta^2}\right). \quad (4.22)$$

Defining $m_t(ij) = -(\alpha_s \hbar^2/m^2c^2)t_{ij}$ and carrying out the $\vec{K}$ integrations, one obtains

$$m_t(ij) = -\frac{\alpha_s \hbar^2}{m^2c^2}\mathfrak{N}(R)\left(\frac{C_{ij}}{R}\right)^3 n\gamma(\tfrac{5}{2}, A_{ij}{}^2\beta^2R^2) \times T_2(\vec{S}_a, \vec{S}_b)\cdot T_2(\hat{R}), \qquad (4.23)$$

where $n\gamma(\frac{5}{2}, x)$ is the normalized incomplete Gamma function[25] of order $\frac{5}{2} = \gamma(\frac{5}{2}, x)/\Gamma(\frac{5}{2})$, and the coefficients C_{ij}, A_{ij} are the ones listed in Tabel IV. At this stage it can be appreciated that the occurrence of the incomplete Gamma function of order $\frac{5}{2}$ can be traced to the use of the Gaussian quark WF's. If some other choice had been made for these quark WF's then there would be, correspondingly, a different function in place of $n\gamma(\frac{5}{2}, x)$. In any case, whichever quark radial WF's one might use, the tensorial character of the NN interaction will not be perturbed and will emerge finally, as in (4.23). From (4.23) it is possible to evaluate the even and the odd parts of the NN tensor potentials $\mathcal{V}^{even(odd)}$ defined towards the end of Sec. III B3.

Up to this point all the effort has gone into the derivation of the NN potential—starting with the qq interaction—and its casting in the canonical form. Now it remains to be seen whether this NN potential, arising from the exchange of colored quarks, has any measure of truth in it. For this purpose the 14, 16, and 36 parts which have been separately calculated must be brought together according to (3.5), the general expression for the NN QEP. Once this is done, it will be possible to numerically calculate the strengths of these NN potentials in the various channels.

V. COMPARISON OF THE QEP WITH PHENOMENOLOGICAL POTENTIALS

The three main parts of the QEP, appearing in (1.1), are now assembled with a view to compare their radial profiles with the corresponding ones of phenomenological potentials.

A. The central part of the QEP

Both the central potential (i.e., all the four types of qq potentials discussed in Sec. III B4) and the spin-spin potential contribute to the total NN central potential $V_c^{\rm tot}(R)$ in (1.1). The detailed structure is given by (3.5),

$$V_c^{\rm tot}(R) = V_c(R) + V_\sigma(R)\vec{\sigma}_a\cdot\vec{\sigma}_b + V_\tau(R)\vec{\tau}_a\cdot\vec{\tau}_b + V_{\sigma\tau}(R)(\vec{\sigma}_a\cdot\vec{\sigma}_b)(\vec{\tau}_a\cdot\vec{\tau}_b), \qquad (5.1)$$

where

$$V_c(R) = -\tfrac{2}{3}[3m_d(36) + \tfrac{1}{3}m_d(14) + 2m_d(16)] + \phi(R), \qquad (5.2)$$

$$V_\sigma(R) = -\tfrac{2}{27}[-m_d(36) + m_d(14) - 2m_d(16)] + \tfrac{1}{9}\phi(R), \qquad (5.3)$$

$$V_\tau(R) = -\tfrac{2}{81}[9m_d(36) + m_d(14) - 6m_d(16)] + \tfrac{1}{9}\phi(R), \qquad (5.4)$$

$$V_{\sigma\tau}(R) = -\tfrac{2}{243}[-25m_d(36) + m_d(14) + 10m_d(16)] + \tfrac{25}{81}\phi(R). \qquad (5.5)$$

In the above four equations the quantities occurring in the square brackets represent the spin-spin-delta-function interaction, while $\phi(R)$ represents the contribution arising from the four types of central potentials. This latter quantity is given by

$$\phi(R) = 2[m_d(16) + m_{cn}(16) + m_{cb}(16) + m_{pp}(16)] - [m_d(14) + m_{cn}(14) + m_{cb}(14) + m_{pp}(14)] - [m_d(36) + m_{cn}(36) + m_{cb}(36) + m_{pp}(36)], \qquad (5.6)$$

and these matrix elements $m_x(ij)$ have already been calculated in Secs. III and IV. The phenomenological NN potentials[1,2,8] are known in each of

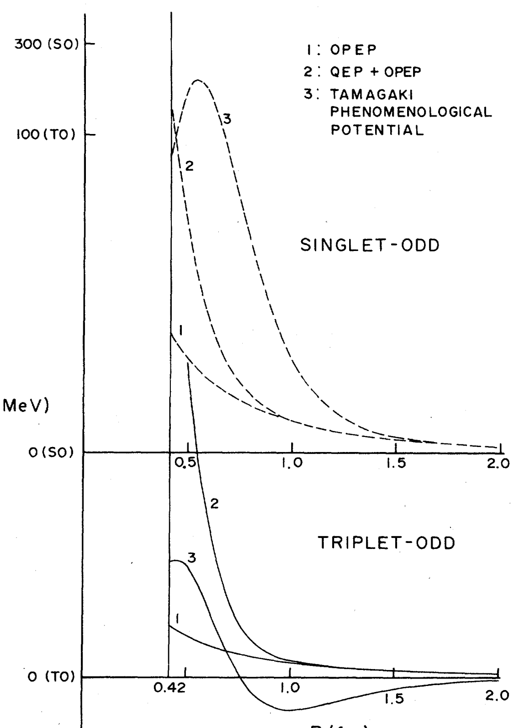

FIG. 1. A comparison of the central components of the QEP+OPEP, OPEP (Ref. 1) and the Tamagaki phenomenological potentials (Ref. 8) in the singlet-odd (SO) and the triplet-odd (TO) channels. The Tamagaki hard-core radius is at 0.42 fm.

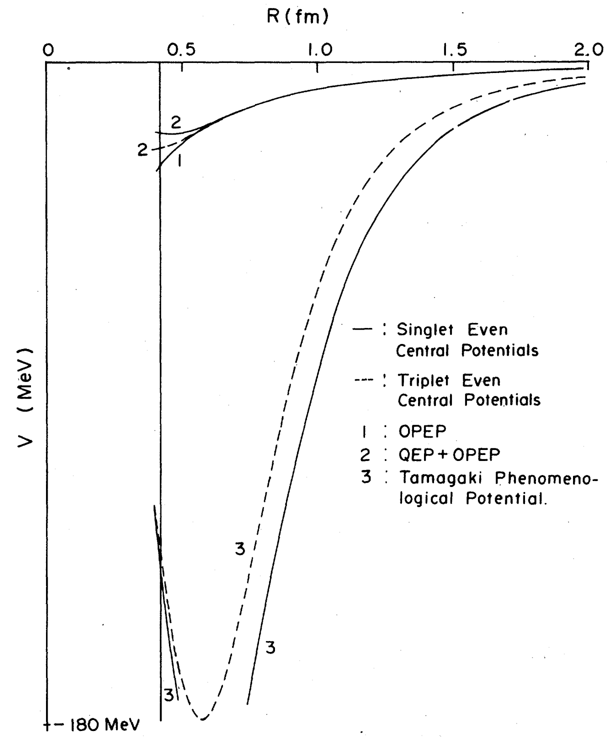

FIG. 2. A comparison of the central components of the QEP+OPEP, OPEP (Ref. 1) and the Tamagaki phenomenological potentials in the even channels. The OPEP has the same radial variation for both the even channels. The Tamagaki hard-core radius is at 0.42 fm.

the four dinucleon states: singlet even, singlet odd, triplet even, and triplet odd. To compare the QEP with such potentials, it is necessary to take the EV of Eq. (5.1) in each of these four states $|S,T\rangle$. Doing this, one has

$$\begin{aligned}
\langle V_c^{\rm tot}(R)\rangle_{\rm SE} &= V_c(R) - 3V_\sigma(R) + V_\tau(R) - 3V_{\sigma\tau}(R)\,,\\
\langle V_c^{\rm tot}(R)\rangle_{\rm SO} &= V_c(R) - 3V_\sigma(R) - 3V_\tau(R) + 9V_{\sigma\tau}(R)\,,\\
\langle V_c^{\rm tot}(R)\rangle_{\rm TE} &= V_c(R) + V_\sigma(R) - 3V_\tau(R) - 3V_{\sigma\tau}(R)\,,\\
\langle V_c^{\rm tot}(R)\rangle_{\rm TO} &= V_c(R) + V_\sigma(R) + V_\tau(R) + V_{\sigma\tau}(R)\,.
\end{aligned} \tag{5.7}$$

Here, S and T stand for the singlet and triplet, and E and O stand for even and odd, respectively. These four potentials are plotted as function of R in Figs. 1 and 2. In these same figures one can contrast the QEP with the one-pion exchange potential (OPEP) and the phenomenological potentials. The contributions to the central part of the QEP from only the spin-spin interactions are also exhibited in Figs. 3 and 4.

1. The hard core

The phenomenological $^1S_0$ phase-shift data show that beyond about 250 MeV the phase shift changes

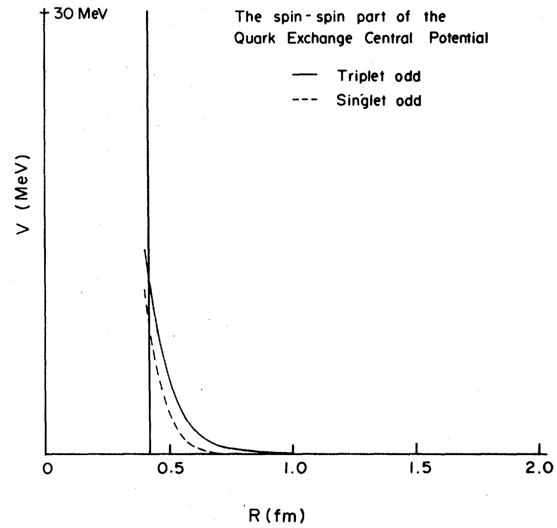

FIG. 3. The spin-spin qq interaction (Eq. 3.12) leads to the above repulsive central NN interaction in the odd channels. The potentials continue to be more repulsive within the Tamagaki hard-core radius at 0.42 fm.

from a positive to a negative value. With the phase shift being roughly the difference between the perturbed action and the unperturbed one, it is easy to see that its sign will be opposite to that of the scattering potential.[26] What this means is that at higher energies the NN interaction acquires a repulsive component. The occurrence of such a repulsion is welcome since it is found adequate to explain the saturation property of nuclei. The onset of such a repulsion only at higher energies can be simulated by a potential which is very strongly repulsive at short distances (the hard

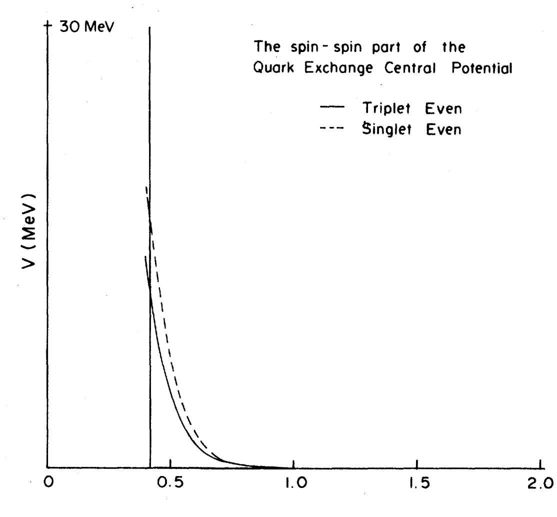

FIG. 4. Same as Fig. 3, but for the even channels.

core). This particular representation of the energy dependence of the $^1S_0$ phase shift is by no means unique, even within the potential framework. Finite repulsive potentials (i.e., soft core) at short distances, as also momentum dependent potentials, are capable of reproducing this phase-shift data. In the following it will be seen that the QEP gives the short-range strong repulsion and hence the property of saturation of nuclear forces.

The central part of the QEP in the odd channels is strongly repulsive at short distances. This is shown in Fig. 1. Also, there is reasonable agreement with the Tamagaki phenomenological potential.[8] However, the central part is only weakly attractive in the even channels (see Fig. 2), contrary to the behavior of the pehnomenological potentials. The insignificant attraction in the triplet even state suggests that the colored quark exchange mechanism is not adequate to keep the deuteron bound. The spin-spin interaction is found to contribute repulsion in all the four channels of the central potential (see Figs. 3 and 4). It is also found that in the even channels the QEP is repulsive inside the hard core, a feature which also appears in the investigation of Ribeiro.[27] Whereas, in Ribeiro's analysis this short-range repulsion is ascribed to the qq confinement interaction; in the present model it arises from the short-range spin-spin qq intreaction. Similar conclusions are also reached in the work of Liberman[9] and in a calculation of the NN potential as given by the MIT[28] bag model of QCD.

As regards the origin of this repulsion, Machida and Namiki[29] had ascribed the hard core of the NN interaction to the exclusion principle. However, it turns out that when color is introduced one can construct[7] a six-quark wave function with the spatial structure $(1s)^6$ so that the exclusion principle in itself does not rule out six quarks being localized—as in a NN collision. The repulsion observed in the QEP at short distances originates from the gluon interaction rather than from the exclusion principle. Lastly, if one were to define the radius of the short-range repulsion in the QEP to be nearabout the point of steepest rise, it is seen from Fig. 1 that this $\simeq 0.55$ fm. The consistency of this value with the hard-core radius expected on the grounds that the nucleon core size is $\simeq 0.3$ fm (Ref. 15) is indeed noteworthy.

2. *The intermediate range*

As pointed out above, the QEP's central part falls short of providing sufficient attraction in the triplet even and singlet even states in the intermediate range. The implication is that the deuteron cannot be bound by this mechanism of exchange of colored quarks. As it is impossible to alter the shape of the QEP curves—since the theory is truly microscopic and all the OGEC constants have been reliably fixed once for all—this conclusion of insufficient binding is irrevocable.

B. Spin-orbit part

Putting together the 14, 16, and 36 terms of the spin-orbit potentials into (3.5), one has in the various dinucleon states

$$\langle V_{ls}^{\text{tot}}(R)\rangle_{\text{TO}} = \tfrac{2}{243}[140 m_{ls}(16) - 15 m_{ls}(14) - 55 m_{ls}(36)], \qquad (5.8)$$

$$\langle V_{ls}^{\text{tot}}(R)\rangle_{\text{TE}} = \tfrac{1}{81}[-88 m_{ls}(16) - 6 m_{ls}(14) + 50 m_{ls}(36)], \qquad (5.9)$$

and the spin-orbit potential obviously vanishes in both the singlet states. It is seen from Figs. 5 and 6 that the spin-orbit potential arising from the exchange of colored quarks is very weak, contrary to that observed in phenomenological potentials. However, the inverted signature of this interaction, as demanded by experiment, emerges correctly from the present theory. Within the framework of meson theory, a close connection exists between the hard-core repulsion (produced by the exchange of vector mesons) and the inverted signature of the spin-orbit NN potential.[30] Even though the QEP yields both the hard-core and the inverted spin-orbit interaction, the link between these two features is not transparent.

At this stage one might suppose that this deficiency of the QEP might be overcome by introducing a spin-dependent confinement potential.[16,31] Already, to have a OGEC theory of the states of charmonium—that explains all the observed splittings of levels—it has been found necessary to postulate a spin-dependent confinement potential. One therefore wonders whether a similar correction is necessary here, too. Moreover, the inclusion of this correction to the spin-orbit potential does not vitiate the values of the OGEC constants, since, for s-wave baryons, there is no contribution to their mass from the spin-orbit potential. Thus, replacing the quantity $-(1/r_{ij})^3$ by $(1/r_{ij})(d/dr_{ij})V(r_{ij})$ in the spin-orbit terms, one can see that if $V(r_{ij}) = (1/r_{ij}) - (r_{ij}/a)^2(1/a)$, then $-(1/r_{ij})^3 \to -(1/r_{ij})^3 - (2/a^3)$. This correction does not perturb the already right signatures obtained for the spin-orbit potential. Carrying out a little algebra, it can be seen that the correction amounts to the replacement

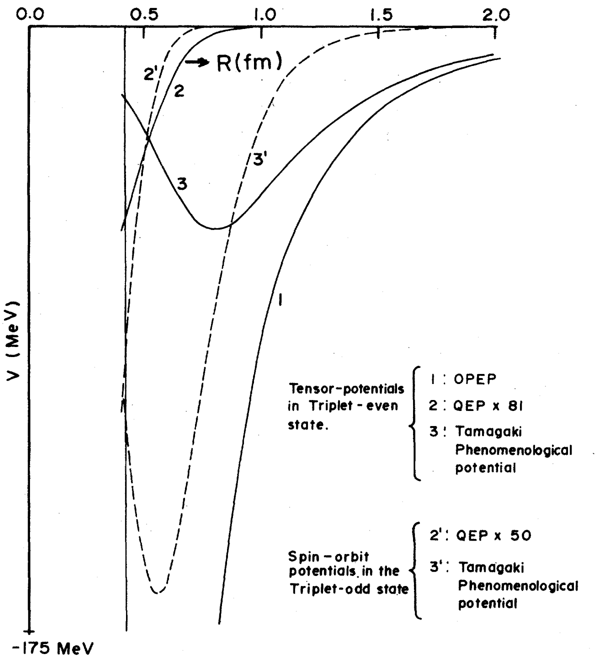

FIG. 5. A comparison of the respective tensor components and the respective spin-orbit components of the QEP, OPEP, and the Tamagaki phenomenological potential. The large magnification of the QEP indicates that its strength is quite weak. The signatures of the QEP are nevertheless correct.

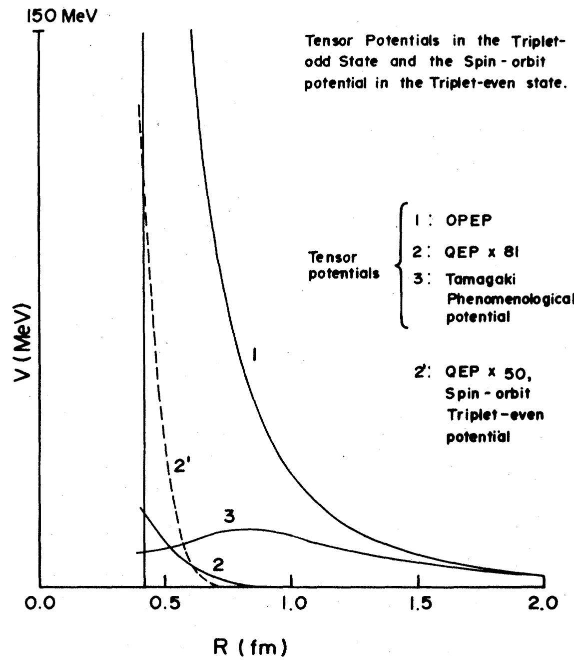

FIG. 6. See caption to Fig. 5.

$$m_{ls}(ij) \to m_{ls}(ij) - \frac{\alpha_s \hbar^2}{2m^2c^2}\left(\frac{2}{a^3}\right)\mathfrak{N}(R)\eta(ij), \qquad (5.10)$$

where $\eta(14)=\frac{12}{5}$, $\eta(16)=\frac{9}{10}$, and $\eta(36)=\frac{6}{5}$. It has been found that this correction is too insignificant to alter the conclusions already reached regarding the spin-orbit potential.

It is not out of place here to record also the futility of applying a similar correction to the central part of the QEP, due to the additional spin-dependent confinement potential. This correction is tantamount to replacing $\delta(\vec{r})$—multiplying $\vec{S}_i \cdot \vec{S}_j$ in the qq interaction—by $\delta(\vec{r})+3/2\pi a^3$. In terms of the radial EV's, the replacement is $m_s(ij) \to m_s(ij) + (4/a^3)(-\alpha\hbar^2/m^2c^2)\mathfrak{N}(R)$. That is to say, the correction is independent of (ij) and consequently, in view of the structure in (3.5), the total correction averages to zero. As regards the tensor potential, it can be seen that an oscillator confinement potential cannot possibly lead to a tensor potential.

C. The tensor part

The total tensor potential is found to be

$$\langle V_t^{\text{tot}}(R)\rangle_{\text{TO}} = \tfrac{1}{81}[2m_t(16) - 17m_t(36) - 5m_t(14)], \qquad (5.11)$$

$$\langle V_t^{\text{tot}}(R)\rangle_{\text{TE}} = \tfrac{1}{27}[4m_t(16) + 11m_t(36) - m_t(14)]. \qquad (5.12)$$

Figures 5 and 6 exhibit these plots, in comparison with those of OPEP and the phenomenological potential.[8] The QEP is seen to suffer from the drawback of insufficient strengths, even though the signatures in both the channels come out right. The weak nature of the quark-exchange tensor potentials is, however, a happy circumstance, since the OPEP supplies most of what is called for by experiment. Neither of the theoretical potentials by themselves are close to reality, but their sum, however, provides a better approximation to what is observed.

D. The L^2 and the quadratic spin-orbit terms within the QEP

In defining the action of the exchange operator on coordinate space WF's, the higher order gradient operators acting on the WF of relative motion $\psi_{ST}(R)$ were neglected on the grounds that $\psi_{ST}(\vec{R})$ varies slowly with R. If the higher order gradient operators are retained then one will arrive at the $\hat{L}^2$ term which is often included in the analysis of the phase-shift data.[1,2] The way to include the quadratic spin-orbit interaction at the NN level is straightforward. In the derivation of the qq interaction (the Fermi-Breit interaction), if the relativistic corrections $O(1/c^4)$

are also included, this term will naturally come about. In the interests of simplicity, many of such finer refinements have not been included in the present analysis.

VI. COMPARISON WITH CONTEMPORARY CALCULATIONS BASED ON QUARKS

Except for the work of Barry,[10] all the approaches to the NN interaction via the quark model have used colored quarks. In the work of Barry, the assumption is made that a diquark is a fermion, and the problem is thus reduced to that of two "hydrogenic atoms" interacting with each other. This treatment, moreover, lacks the merit of using reliable coupling constants and quark masses. The defect in the work of Libermann[9] is that the constants have not been well determined, and for the qq interaction, too restricted a choice is used. (The qq potential is taken to be just the sum of an oscillator potential and a "confinement" spin-spin potential.) These are probably the reasons which lead to a strong repulsion in the even channels as well of the central NN potential, calculated by Libermann. Kislinger[11] has calculated the NN force arising from the exchange of a massive dressed gluon. By arguing that interquark distances (when the quarks are in different nucleons) are nearly the same as the distance between the nucleonic clusters, he replaces the qq interaction by the NN interaction. While this assumption is justified for the radial coordinates, the same cannot be said of the spin-isospin coordinates. Moreover, the constants used have no authenticity, and quark dynamics is totally ignored. The basic observations made by Robson[12] and Ribeiro[27] are borne out in the present study—regarding the signs of the central part of the QEP in the various dinucleon channels.

VII. CONCLUSION

These labors having drawn to a close, one must examine with hindsight what ground one has covered and learn the lessons one can. What measure of reality is there in the colored quark exchange mechanism as a possible origin of the nuclear force? Certainly OGEC is not the whole story in so far as all the aspects of the NN interaction are concerned. Its chief merit lies in the strong short-range repulsion that it gives in the odd channels of the central part, i.e., quark exchange yields the property of saturation of nuclear forces. Its weak tensor part ties in well with the close agreement between the OPEP's tensor potential and the tesnor potential of the phenomenological potential. Considering the lack of sufficient attraction in the intermediate range, one can say that the QEP is somewhat complimentary to the OPEP.

If QCD is to be taken as the path leading to reality, then one must show on its basis how meson exchanges can exist. A preliminary step in this direction has been taken by Lovas,[3] who has shown by a simple generalization of the Okubo-Zweig-Iizuka (OZI) rule that the mesons δ, S^* will have the weak coupling strengths they have. The next step ought to be in the direction of deriving the OPEP from a possible generalization of the present OGEC model so as to include the sea quarks as well in baryons.

APPENDIX A

In evaluating the SI EV's, it is found that all of them share a common factorization property which can be brought to light by studying the structure of a typical EV,

$$M(\hat{O};S,T)=\langle\phi_a\phi_b|\hat{O}(\text{spin})\hat{P}^{\sigma}_{36}\hat{P}^{\tau}_{36}|\phi_a\phi_b\rangle\,. \qquad \text{(A1)}$$

The state $|\phi_a\phi_b\rangle$ is a six-quark state of total spin (isospin) $=S(T)$. ϕ_a, ϕ_b are three quark states of spin (isospin) $=\frac{1}{2}(\frac{1}{2})$. The structure of this cluster state (being totally symmetric in the spin, isospin coordinates of three quarks) is

$$\phi_a=\frac{1}{\sqrt{2}}[\chi^{\sigma}_S(\tfrac{1}{2})\chi^{\tau}_S(\tfrac{1}{2})+\chi^{\sigma}_A(\tfrac{1}{2})\chi^{\tau}_A(\tfrac{1}{2})]\,, \qquad \text{(A2)}$$

where $\chi_{S(A)}$ is a three-quark state which is symmetric (antisymmetric) in the labels of the first and second quarks and has total spin or isospin $=\frac{1}{2}$, i.e.,

$$\chi_{S(A)}=|[1(0)\tfrac{1}{2}]\tfrac{1}{2}\rangle\,. \qquad \text{(A3)}$$

If the state is symmetric (antisymmetric) in the first two-quark coordinates, then the total spin or isospin of the first two quarks will be 1(0).

The structure of $|\phi_a\phi_b\rangle$ is now seen to be

$$|\phi_a\phi_b\rangle=\tfrac{1}{2}\sum_{ll'}|(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M\rangle \times|(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};T,M_T\rangle\,. \qquad \text{(A4)}$$

The EV (A1) is now seen to possess the factorization property

$$M(\hat{O};S,T)=\tfrac{1}{4}\sum_{l,l',l_1,l_1'}\langle(l_1\tfrac{1}{2})\tfrac{1}{2},(l_1'\tfrac{1}{2})\tfrac{1}{2};S,M|\hat{O}(\text{spin})\hat{P}^{\sigma}_{36}|(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M\rangle \times\langle(l_1\tfrac{1}{2})\tfrac{1}{2},(l_1'\tfrac{1}{2})\tfrac{1}{2};T,M_T|\hat{P}^{\tau}_{36}|(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};T,M_T\rangle\,. \qquad \text{(A5)}$$

Considering the simpler of these matrix elements, i.e., the one in isospin space, it can be seen that it will be diagonal in $l_1, l; l_1'l'$. The reason is that the operator $\hat{P}^{\tau}_{36}$ does not act on the coordinates 1, 2; 4, 5, which are characterized by $l_1, l; l_1', l'$. Thus, even though the spin operator $\hat{O}$ (spin) may act on the quarks 1, 2; 4, 5, the presence of the isospin-matrix element makes the spin-matrix element also diagonal in the ls. Therefore

$$M(\hat{O};S,T)=\tfrac{1}{4}\sum_{ll'}\langle(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M\,|\,\hat{O}(\text{spin})\hat{P}^{\sigma}_{36}\,|\,(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};S,M\rangle$$
$$\times\langle(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};T,M_T\,|\,\hat{P}^{\tau}_{36}\,|\,(l\tfrac{1}{2})\tfrac{1}{2},(l'\tfrac{1}{2})\tfrac{1}{2};T,M_T\rangle\,. \tag{A6}$$

Again considering the isospin-matrix element, denoting the states by $|l,l';T,M_T\rangle$ and changing the basis to one in which $\hat{P}^{\tau}_{36}$ diagonal, one has

$$\eta_T(l,l')=\langle l,l';T,M_T\,|\,\hat{P}^{\tau}_{36}\,|\,l,l';T,M_T\rangle$$
$$=\sum_{\substack{j_{1245},j'_{1245}\\ j_{36},j'_{36}}} 4\{[j_{1245}][j_{36}][j'_{1245}][j'_{36}]\}^{1/2}\begin{Bmatrix} l & l' & j_{1245}\\ \frac{1}{2} & \frac{1}{2} & j_{36}\\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}\begin{Bmatrix} l & l' & j'_{36}\\ \frac{1}{2} & \frac{1}{2} & j'_{36}\\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}$$
$$\times\langle(ll')j'_{1245},(\tfrac{1}{2}\tfrac{1}{2})j'_{36};T,M_T\,|\,\hat{P}^{\tau}_{36}\,|\,(ll')j_{1245},(\tfrac{1}{2}\tfrac{1}{2})j_{36};T,M_T\rangle\,,$$

where the last matrix-element is equal to

$$(-1)^{1+j_{36}}\delta(j_{1245},j'_{1245})\delta(j_{36},j'_{36})\,.$$

Therefore,

$$\eta_T(l,l')=\sum_{j_{36},j_{1245}}4[j_{1245}][j_{36}]\begin{Bmatrix} l & l' & j_{1245}\\ \frac{1}{2} & \frac{1}{2} & j_{36}\\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}^2(-1)^{1+j_{36}}\,, \tag{A7}$$

and $[l]=2l+1$. Using the theorem[32]

$$\sum_{j_{13},j_{24}}\begin{Bmatrix} j_1 & j_4 & j_{14}\\ j_3 & j_2 & j_{23}\\ j_{13} & j_{24} & j\end{Bmatrix}^2[j_{13}][j_{24}](-1)^{2j_2+j_{24}+j_{23}-j_{34}}=\begin{Bmatrix} j_1 & j_4 & j_{14}\\ j_2 & j_3 & j_{23}\\ j_{14} & j_{23} & j\end{Bmatrix}. \tag{A8}$$

Equation (A7) reduces to

$$\eta_T(l,l')=4\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & l' & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}. \tag{A9}$$

If $\hat{O}(\text{spin})=1$, then one has perfect symmetry between spin and isospin,

$$\eta_S(l,l')=4\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & l' & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & S\end{Bmatrix}. \tag{A10}$$

Thus (A6) becomes

$$M(\hat{O};S,T)=\tfrac{1}{4}\sum_{ll'}4\begin{Bmatrix} l & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & l' & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & T\end{Bmatrix}\times\text{spin matrix element}\,. \tag{A11}$$

Equation (A11) embodies the factorization property for the SI EV's. Repeated use will be made of this result in the text.

APPENDIX B

In evaluating the radial EV's, the following three-dimensional definite integrals are found to be useful:

$$\int d\vec{K}\exp(-B\vec{K}^2+iA\vec{K}\cdot\vec{R})=\left(\frac{\pi}{B}\right)^{3/2}\exp\left(-\frac{A^2R^2}{4B}\right), \tag{B1}$$

$$\int\frac{d\vec{K}}{\vec{K}^2}\exp(-B\vec{K}^2+iA\vec{K}\cdot\vec{R})=\pi^{3/2}\left(\frac{AR}{2}\right)^{-1}\gamma\left(\frac{1}{2},\frac{A^2R^2}{4B}\right), \tag{B2}$$

$$\int\vec{K}^2d\vec{K}\exp(-B\vec{K}^2+iA\vec{K}\cdot\vec{R})=\frac{\pi^{3/2}}{2B^{5/2}}\left(3-\frac{A^2R^2}{2B}\right)\exp\left(-\frac{A^2R^2}{4B}\right), \tag{B3}$$

$$\int\frac{d\vec{K}}{\vec{K}^2}(\vec{K}\cdot\vec{R})\exp(-B\vec{K}^2+iA\vec{K}\cdot\vec{R})=\frac{i\pi^{3/2}}{2}\times\frac{8}{A^2R}\gamma\left(\frac{3}{2},\frac{A^2R^2}{4B}\right), \tag{B4}$$

$$\int \vec{K}\cdot\vec{R}\, d\vec{K} \exp(-B\vec{K}^2 + iA\vec{K}\cdot\vec{R}) = \frac{i\pi^{3/2}}{2}\frac{AR^2}{B^{5/2}}\exp\left(-\frac{A^2R^2}{4B}\right), \quad \text{(B5)}$$

$$\int \frac{(\vec{K}\cdot\vec{R})^2}{\vec{K}^2} d\vec{K}\cdot \exp(-B\vec{K}^2 + iA\vec{K}\cdot\vec{R}) = \frac{8\pi^{3/2}}{A^3R}\left[-\frac{1}{2}\gamma\left(\frac{1}{2}, \frac{A^2R^2}{4B}\right) + \exp\left(-\frac{A^2R^2}{4B}\right) \times \left\{\left(\frac{A^2R^2}{4B}\right)^{1/2} + \left(\frac{A^2R^2}{4B}\right)^{3/2}\right\}\right], \quad \text{(B6)}$$

$$\int \vec{x}\exp(-B\vec{x}^2 - i\vec{K}\cdot\vec{x})d\vec{x} = \left(\frac{-i\vec{K}}{2B}\right)\left(\frac{\pi}{B}\right)^{3/2}\exp\left(-\frac{\vec{K}^2}{4B}\right), \quad \text{(B7)}$$

$$\int \vec{x}^2\exp(-B\vec{x}^2 - A\vec{K}\cdot\vec{x})d\vec{x} = \frac{\pi^{3/2}}{2B^{5/2}}\left(3 + \frac{A^2K^2}{2B}\right)\exp\left(\frac{A^2\vec{K}^2}{4B}\right), \quad \text{(B8)}$$

$$\int x_i x_j \exp(-B\vec{x}^2 - i\vec{K}\cdot\vec{x})d\vec{x} = \left(\frac{-1}{4B^2}\right)(K_iK_j - 2B\delta_{ij})\left(\frac{\pi}{B}\right)^{3/2}\exp\left(-\frac{\vec{K}^2}{4B}\right), \quad \text{(B9)}$$

where

$$\gamma(\tfrac{1}{2}, x^2) = \sqrt{\pi}\,\mathrm{erf}(x) = 2\int_0^x \exp(-t^2)dt, \quad \text{(B10)}$$

and

$$\gamma(n, x^2) = \int_0^{x^2} \exp(-t)t^{n-1}dt \quad \text{(B11)}$$

is the incomplete Gamma function of order n.

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