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Shape of eigenvalue distribution for bosons in scalar space

V. K. B. Kota
Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

V. Potbhare
Department of Physics, Faculty of Science, M. S. University, Baroda 390 002, India
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Analytic expressions for obtaining first four moments of noninteracting as well as interacting bosons have been given in terms of matrix elements. These expressions have been utilized to obtain general conditions under which the eigenvalue distributions for bosons tend towards the Gaussian distribution. Results for pairing and $Q\cdot Q$ operators in an s-d boson shell have also been given.

NUCLEAR STRUCTURE Analytic expressions, first four moments, noninteracting and interacting bosons, eigenvalue distribution, conditions leading to Gaussian distribution, pairing and Q-Q operators, s-d boson space.

The assumption of applicability of a strong central limit theorem (CLT) to two-body interactions has brought forth a tremendous simplification in the nuclear shell model spectroscopy.[1] The conditions on traces of matrix elements, which give rise to normal eigenvalue distributions (EVD) for fermion systems (noninteracting and interacting), have been clearly formulated.[2,3] The question of normality of EVD for boson systems was recently raised by Bortignon *et al.*[4,5] They have given analytical expressions for the first four moments of noninteracting boson EVD. These expressions, however, do not reveal the conditions for normality of EVD; also, the more interesting and more realistic case of interacting bosons was not treated. In this paper, we present simple analytic expressions for the first four moments of EVD for noninteracting and interacting bosons. We shall also discuss the conditions under which EVD for bosons tend to Gaussian distribution.

The m-particle scalar average $\langle H(k)\rangle^m$ of a k-body operator $H(k)$ in the space defined by N single-particle states, corresponds to the average of expectation values of $H(k)$ over all the states of m particles, and can be expressed in terms of $\langle H(k)\rangle^k$:

$$\langle H(k)\rangle^m = \binom{m}{k}\langle H(k)\rangle^k\,, \tag{1}$$

where $\binom{m}{k}$ is the usual binomial factor. Equation 1 is valid for both the fermions and the bosons. The number of states of m bosons in the space of N single-particle states is given by $\binom{N+m-1}{m}$; hence, in terms of the trace of $H(k)$ in k-particle space $\langle\langle H(k)\rangle\rangle^k$, we obtain

$$\langle H(k)\rangle^m = \binom{m}{k}\binom{N+k-1}{k}^{-1}\langle\langle H(k)\rangle\rangle^k\,, \tag{2}$$

where the double bracket $\langle\langle\ \rangle\rangle$ indicates the trace. For noninteracting bosons, the Hamiltonian H is a one-body operator $H=\sum_i \epsilon_i n_i$; here ϵ_i is single-particle energy and n_i is the number operator for the ith single-particle state. Except for the centroid which defines the location of a distribution, the shape of the EVD is determined by its central moments. Therefore, we shall deal only with traceless single-particle energies $\tilde{\epsilon}_i=\epsilon_i - N^{-1}\sum\epsilon_i$, and shall evaluate moments of $\tilde{H} = H - nN^{-1}\sum\epsilon_i$, where $n=\sum_i n_i$, for noninteracting bosons. We evaluate the second moment of H by separating $(\tilde{H})^2$ into its one-body and two-body parts:

$$(\tilde{H})^2=\sum_i \tilde{\epsilon}_i^{\,2} n_i+\sum_i \tilde{\epsilon}_i^{\,2} n_i(n_i-1) + \sum_{i\neq j}\tilde{\epsilon}_i\tilde{\epsilon}_j n_i n_j\,. \tag{3}$$

The first term above is the one-body part of $(\tilde{H})^2$ and its scalar average in m-particle space is given by $m\sum\epsilon_i^{\,2}/N$. The remaining terms form the two-body part of $(\tilde{H})^2$ and their scalar average equals $[m(m-1)/N(N+1)]\sum_i\tilde{\epsilon}_i^{\,2}$. Hence the average value of $(\tilde{H})^2$ in m-particle space can be expressed as

$$\langle\tilde{H}^2\rangle^m=\frac{m(N+m)}{N(N+1)}\sum\tilde{\epsilon}_i^{\,2}\,. \tag{4}$$

Similarly the scalar averages of $(\tilde{H})^3$ and $(\tilde{H})^4$

can be obtained by separating $\tilde{H}^3$ into (1+2+3)-body parts and $\tilde{H}^4$ into (1+2+3+4)-body parts. Adding the contributions from different k-body parts gives us the following expressions:

$$\langle\tilde{H}^3\rangle^m = \frac{m(N+m)(N+2m)}{N(N+1)(N+2)}\sum_i \tilde{\epsilon}_i^{\,3},$$
$$\langle\tilde{H}^4\rangle^m = \frac{m(N+m)}{N(N+1)}\sum_i \tilde{\epsilon}_i^{\,4} + \frac{m(m-1)(N+m)(N+m+1)}{N(N+1)(N+2)(N+3)} \times\left[3\left(\sum_i \tilde{\epsilon}_i^{\,2}\right)^2 + 6\sum_i \tilde{\epsilon}_i^{\,4}\right]. \tag{5}$$

It is quite interesting to note that, by substituting $-N$ for N in the corresponding expressions for fermions, and then by taking the absolute value of each term, one obtains the above expressions for the boson systems. For example,

$$\langle\tilde{H}^3\rangle^m_{\text{fermions}} = \frac{m(N-m)(N-2m)}{N(N-1)(N-2)}\sum_i \tilde{\epsilon}_i^{\,3} \tag{6}$$

goes to its boson counterpart by substituting $-N$ for N and taking modulo of the final result (however, one cannot obtain moments of EVD for fermions from those of bosons as we will be left with phase ambiguities). This was previously conjectured by Nomura[6]; using this principle one can obtain several new results. For example, a class of Wigner coefficients of the O(N) group can be obtained from those of the Sp(N) group via moments in quasispin subspaces.[7]

Another interesting one-body Hamiltonian is $H=\sum \epsilon_{ij}A_iB_j$, where ϵ_{ij} is an off diagonal single-particle matrix element when $i\neq j$, ϵ_{ii} is the usual single-particle energy, and A_i, B_j are single-particle creation and destruction operators, respectively. Such a Hamiltonian is widely used in solid-state physics, where ϵ_{ij} $(i\neq j)$ can be identified with the exciton hopping integral between sites i and j. The traceless single-particle matrix elements $\tilde{\epsilon}_{ij}$, in this case, are given by $\tilde{\epsilon}_{ij}=\epsilon_{ij}-\delta_{ij}N^{-1}\sum_k \epsilon_{kk}$. The expressions for second, third, and fourth central moments of this Hamiltonian can be obtained by substituting $\sum\tilde{\epsilon}_i^{\,2}$, $\sum\tilde{\epsilon}_i^{\,3}$, and $\sum\tilde{\epsilon}_i^{\,4}$ by $\sum\tilde{\epsilon}_{ij}\tilde{\epsilon}_{ji}$, $\sum\tilde{\epsilon}_{ij}\tilde{\epsilon}_{jk}\tilde{\epsilon}_{ki}$, and $\sum\tilde{\epsilon}_{ij}\tilde{\epsilon}_{jk}\tilde{\epsilon}_{kl}\tilde{\epsilon}_{li}$, respectively, in Eqs. 4 and 5 given above.

The preceding treatment of a one-body Hamiltonian was simple and elementary. We shall start with some general results before going to the case of moments of EVD for bosons interacting via two-body interactions. A number conserving, k-body operator for bosons in the space defined by N single-particle states can be decomposed according to the irreducible (IR) representations of the U(N) group; $F(k)=\sum_{\nu=0}^{k}F^\nu(k)$, where $F^\nu(k)$ is the rank-ν part of $F(k)$ and transforms as a $[2\nu,\nu^{N-2}]$ partition under a U(N) transformation. $F^\nu(k)$ can be written as a product of a unitary scalar operator and a completely IR ν-body and rank-ν operator $\mathfrak{F}(\nu)$:

$$F^\nu(k) = \binom{n-\nu}{k-\nu}\mathfrak{F}(\nu). \tag{7}$$

In order to find out the different unitary IR parts systematically, we introduce a contraction operator D_-, defined by the following double commutator:

$$D_-F = \sum_{i=1}^{N}\Big[[B_i, F]_-, A_i\Big]_-. \tag{8}$$

The operator D_- reduces the particle rank (or the body rank) of F by one, without altering its unitary rank. Since $\mathfrak{F}(\nu)$ is completely irreducible, any reduction of its particle rank by D_- destroys it; $D_-\mathfrak{F}(\nu)=0$. It should be noted that this contraction operator is different from similar contraction operators for fermions.[1,8] Contracting $F(k)$ r times by D_- gives us

$$D_-^rF(k) = \sum_\nu r!\binom{N+k+\nu-1}{r}F^\nu(k-r). \tag{9}$$

By inverting this expression we obtain

$$\mathfrak{F}(\nu) = \frac{1}{(k-\nu)!}\binom{N+\nu+k-1}{k-\nu}^{-1} \times\sum_{t=k-\nu}^{k}\frac{(-1)^{t-k+\nu}}{(t-k+\nu)}\binom{N+2\nu-2}{t-k+\nu}^{-1} \times\binom{n-k+t}{t-k+\nu}D_-^tF(k). \tag{10}$$

Application of this to a one-body operator $H=\sum\epsilon_in_i$ gives us the IR $\nu=0$ part as $n\sum\epsilon_i/N$, and the IR $\nu=1$ part as $\sum\tilde{\epsilon}_in_i$ in terms of the traceless single-particle energies $\tilde{\epsilon}_i$. Similar application of Eq. (10) to the two-body interaction given by $\frac{1}{4}\sum V_{ijkl}A_iA_jB_kB_l$, expressed in terms of the two-body matrix elements V_{ijkl}, gives us the following three parts: (i) the IR $\nu=0$ part

$$n(n-1)\sum V_{ijij}/[2N(N+1)],$$

(ii) the IR $\nu=1$ part

$$\sum_{i,k}(n-1)(N+2)^{-1}\left[\sum_j\left(V_{ijkj}-\delta_{ik}\sum_{ij}V_{ijij}/N\right)A_iB_k\right],$$

and (iii) the IR $\nu=2$ part

$$\tfrac{1}{4}\sum\tilde{V}_{ijkl}A_iA_jB_kB_l.$$

Here $\tilde{V}_{ijkl}$ is the IR $\nu=2$ matrix element, obtained by subtracting $\nu=0$ and $\nu=1$ contributions from V_{ijkl}.

We have already given expressions for the first four central moments of EVD generated by the one-body (which is essentially IR $\nu=1$) interaction. Now we give the final expressions for second, third, and fourth central moments of EVD generated by a two-body, IR rank $\nu=2$ Hamiltonian H. The procedure leading to these final expressions is straightforward, and consists of (a) decomposing H^n into parts according to the body (particle) rank, and (b) using Eq. (2) separately for each part. In order to take care of the bookkeeping of various terms, which is the only difficult part in the whole procedure, we use the Feynman diagram type approach.[8,9] The following is the expression for the ith central moment μ_i [$\mu_i^m = \langle(\tilde{H})^i\rangle^m$, $\tilde{H}$ is the traceless part of H in m-particle scalar space] of EVD for m bosons interacting via two-body, IR rank $\nu=2$ interaction $\tilde{V}$:

$$\langle(\tilde{V})^i\rangle^m = \sum_{s=2}^{i} \binom{m}{s} \binom{N+s+m-1}{s} \times \binom{2s}{s}^{-1} \binom{N+2s-1}{2s} C_i^s . \tag{11}$$

Here i goes from 2 to 4. All the necessary information about the two-body, $\nu=2$ interaction goes in the evaluation of coefficients C_i^s. The expressions for these in terms of the two-body matrix elements are given below:

$$\begin{aligned}
&C_2{}^2 = \tfrac{1}{4}\sum \tilde{V}_{ijkl}\tilde{V}_{klij}, \quad C_3{}^2 = \tfrac{1}{8}\sum \tilde{V}_{ijkl}\tilde{V}_{klmn}\tilde{V}_{mnij},\\
&C_3{}^3 = 2C_3{}^2 + \sum \tilde{V}_{iljk}\tilde{V}_{knlm}\tilde{V}_{mjni},\\
&C_4{}^2 = \tfrac{1}{16}(AA1),\\
&C_4{}^3 = \tfrac{1}{4}(AA1) + CC1 + \tfrac{1}{2}BA1 + 2(CA1),\\
&C_4{}^4 = \tfrac{3}{8}(AA1) + 6(CC1) + 3(BA1) + 6(CA1)\\
&\qquad + 3(AB1) + 3(C_2{}^2)^2,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
&AA1 = \sum \tilde{V}_{ijkl}\tilde{V}_{klmn}\tilde{V}_{mnop}\tilde{V}_{opij},\\
&AB1 = \sum \tilde{V}_{ijkl}\tilde{V}_{lmjn}\tilde{V}_{nomp}\tilde{V}_{pkoi},\\
&BA1 = \sum \tilde{V}_{ijkl}\tilde{V}_{klin}\tilde{V}_{mnop}\tilde{V}_{opmj},\\
&CA1 = \sum \tilde{V}_{ijkl}\tilde{V}_{kmno}\tilde{V}_{olmp}\tilde{V}_{npij},\\
&CC1 = \sum \tilde{V}_{ijkl}\tilde{V}_{mnjo}\tilde{V}_{olnp}\tilde{V}_{pkmi}.
\end{aligned} \tag{13}$$

Here the summation is over all labels. These results can be very easily extended to evaluation of the moments of EVD generated by a general (1+2)-body interaction H; the corresponding expressions are given in the Appendix.

The necessary information C_i^s for the evaluation of the ith moment of EVD in the m-boson space, generated by IR $\nu=2$ interaction, can also be obtained from the ith moment in 2 to the i-boson spaces only. One, however, would expect, due to the two-body character of the interaction, that the ith moments in 2 to $2i$ particle spaces would have been necessary[10] for evaluation of the ith moment of EVD in m-boson space. The validity of this fact in fermion spaces[3,10] can be attributed to the particle-hole symmetry of the IR $\nu=2$ interaction. There is no concept of hole for bosons, and hence one can see that the IR $\nu=2$ nature of interaction is responsible for this simplicity, and in fermion spaces, the particle-hole symmetry is a consequence of the IR $\nu=2$ nature of the interaction.

We shall now discuss the conditions for the normality of the EVD. The first two moments of the EVD tell us about the location (centroid) and the spread (width) of the EVD. The information about the shape of the spectrum is given by the third and higher central moments of the EVD. In principle, all the higher moments are necessary for determining the shape of EVD; but here we shall deal only with the moments up to fourth order. The corresponding shape parameters are the skewness $\gamma_1 = \mu_3/(\mu_2)^{3/2}$, and the excess $\gamma_2 = \mu_4/(\mu_2)^2 - 3$. For the Gaussian or the normal distribution, both γ_1 and γ_2 are equal to zero.

Let us start with a pure one-body interaction; the central moments for such an interaction are given by Eqs. (4) and (5). In the dilute limit, defined by $m\to\infty$, $N\to\infty$, but $m/N \ll 1$, we find that the shape parameters $\gamma_1(m)$ and $\gamma_2(m)$ for m-boson EVD approach asymptotically to values given by the expressions

$$\begin{aligned}
&\gamma_1(m) = m^{-1/2}\gamma_1(1),\\
&\gamma_2(m) = [\gamma_2(1)+3](m^{-1}+6N^{-1}).
\end{aligned} \tag{14}$$

Thus, in the dilute limit, both $\gamma_1(m)$ and $\gamma_2(m)$ approach zero if the shape parameters $\gamma_1(1)$ and $\gamma_2(1)$ for one-boson EVD are finite. The same result was obtained for the fermion EVD in the dilute limit.[2] Thus in the dilute system, one gets the same result irrespective of the fermion or the boson nature of particles, since the Pauli's exclusion principle has negligibly small effect when $N \gg m$. However, for bosons, since there is no limit to the number of particles in any single-particle state, unlike fermions, we can consider the limit $m \to \infty$ with finite N. This is called the dense limit; the second central moment and the shape parameters in this limit are given by

$$\begin{aligned} \langle(\tilde{H})^2\rangle^m &= \sigma^2(m) = m^2\sigma^2(1)/(N+1)\,, \\ \gamma_1(m) &= 2\gamma_1(1)\frac{(N+1)^{1/2}}{(N+2)}\,, \\ \gamma_2(m) &= 6\frac{\{[\gamma_2(1)+3](N+1)-(2N+3)\}}{(N+2)(N+3)}\,. \end{aligned} \tag{15}$$

The width of the m-boson EVD $\sigma(m)$ increases linearly with m. The shape parameters, however, approach values which are independent of m, but which very much depend on the shape parameters for one-boson EVD and the size of the single-boson space N. For large values of N again, the EVD can be considered Gaussian if $\gamma_1(1)$ and $\gamma_2(1)$ are reasonably small. For example, in the case of noninteracting particles, with N equidistant single-particle states, we have $\gamma_1(1)=0$ and $\gamma_2(1)=-1.2\ [(N^2+1)/(N^2-1)]$. We can see that for $N=5$, the absolute value of excess in the dense limit is less than 0.24; for $N=10$ it is less than 0.12. Thus, the EVD in this case approaches Gaussian in the dense limit if $N>5$.

We now consider the dilute and the dense limits of the shape of EVD generated by a purely two-body, IR rank-2 interaction V, by introducing dimensionless quantities $P(m,n)$, defined by

$$P(m,n)=\frac{C_m^n}{(C_2{}^2)^{m/2}}\,. \tag{16}$$

These are the ratios of traces of various combinations of the matrix elements, and are characteristic of the interaction used to generate the EVD. In the dilute limit, $\gamma_1(m)$ approaches $NP(3,2)/m+P(3,3)$ and $[\gamma_2(m)+3]$ tends to $N^2P(4,2)/m^2+NP(4,3)/m+P(4,4)$. Thus the EVD approach Gaussian distribution if $P(3,2)\ll m/N$, $P(3,3)\ll 1$, $P(4,2)\ll m^2/N^2$, $P(4,3)\ll m/N$, and $(P(4,4)-3)\ll 1$. One can translate these conditions to conditions on the shape parameters of EVD for 2, 3, and 4 bosons. The skewness and excess in dilute limit approach constant values $-\sqrt{2}\,\gamma_1(2)+\sqrt{6}\,\gamma_1(3)$, and $\gamma_2(2)-6\gamma_2(3)+6\gamma_2(4)$, respectively. Thus the approach to Gaussian distribution in the dilute limit is decided by these shape parameters of EVD for 2, 3, and 4 bosons. Since in the dilute limit, bosons and fermions behave similarly, the same conditions will apply for fermions. This clearly shows that for purely two-body, IR rank-2 interaction, the assumption of applicability of a strong central limit theorem is not valid. The shape of the EVD in the dilute limit does not necessarily approach the Gaussian distribution with increasing m, as the limiting values of $\gamma_1(m)$ and $\gamma_2(m)$ reach constant values depending on the shape parameters of the EVD for $m=2$, 3, and 4. Of course, the addition of the one-body part of the interaction and the IR rank-1 part of the two-body interaction to the two-body IR rank-2 interaction is likely to alter the whole picture, and the conditions for normality of the EVD will have complicated structures. It is known that the IR rank-1 part of the interaction is quite important for the nuclear structure studies.[11]

In the dense limit ($m\to\infty$, N finite) valid only for bosons, the values of $P(3,3)$ and $P(4,4)$ completely determine the shape of the EVD. For,

$$\begin{aligned} \gamma_1(m) &\to \frac{P(3,3)[N(N+1)(N+2)(N+3)]^{1/2}}{(N+4)(N+5)}\,, \\ \gamma_2(m)+3 &\to \frac{P(4,4)N(N+1)(N+2)(N+3)}{(N+4)(N+5)(N+6)(N+7)}\,. \end{aligned} \tag{17}$$

For large N, we get $\gamma_1(m)=P(3,3)$ and $\gamma_2(m)+3=P(4,4)$ in the dense limit. Again one can express these conditions in terms of the shape parameters of EVD for 2, 3, and 4 bosons, using the following expressions for $P(3,3)$ and $P(4,4)$:

$$\begin{aligned} P(3,3) &= (N+4)\left(\frac{2}{N(N+1)}\right)^{1/2} \times\left[\left(\frac{3(N+4)}{(N+2)}\right)^{1/2}\gamma_1(3)-\gamma_1(2)\right]\,, \\ P(4,4) &= \frac{(N+4)}{N(N+1)}\left(6[\gamma_2(4)+3]\,\frac{(N+4)(N+5)^2}{(N+2)(N+3)} -6[\gamma_2(3)+3]\,\frac{(N+6)(N+4)}{(N+2)} +[\gamma_2(2)+3](N+7)\right)\,. \end{aligned} \tag{18}$$

So in the dense limit also, the skewness and the excess of EVD for m bosons approach constant values depending on the values of $\gamma_1(i)$ for $i=2,3$ and $\gamma_2(j)$ for $j=2,3,4$. The approach to Gaussian limit depends only on these numbers.

Two sets of matrix elements have been given by Bortignon *et al.*[4] for the d bosons of the interacting boson approximation model.[12] Using these two sets of matrix elements, we obtain $P(3,3)=1.78$ and $P(4,4)=23.12$ for one interaction, while the second one gives $P(3,3)=2.09$ and $P(4,4)=25.55$. The size of the single-particle space is given by $N=5$. So, in the dense limit, the limiting values of the shape parameters are

$\gamma_1(m)=0.81, 0.95$ and $\gamma_2(m)=0.27, 0.61$, for the two interactions, respectively; hence the shape of the EVD is nowhere near Gaussian. A pairing interaction in the space of s-d bosons with $N=6$ gives $P(3,3)=1.46$ and $P(4,4)=15.64$; hence the limiting values of γ_1 and γ_2 in the dense limit are 0.73 and -0.24, respectively. Similarly, a $Q\cdot Q$ interaction in the s-d boson space (having both IR rank-1 and IR rank-2 parts) gives the following limiting values for $\gamma_1(m)=-0.76$ and $\gamma_2(m)=0.86$ in the dense limit. Incidentally, for fermions and the $Q\cdot Q$ interaction in a large j orbit, the values of γ_1 and γ_2 in the middle of the shell have been obtained by Nomura,[13] and these are 1.26 and -0.6, respectively.

One interesting feature of the boson systems is the complete domination of the two-body part of the interaction, over the one-body part for large m. This can be seen from the expressions given in the Appendix for moments of the (1+2)-body interaction; each factor containing the external single-particle energy term (ϵ) gets multiplied by m, while each factor containing a two-body matrix element (V) has a coefficient proportional to m^2 in the large m limit. For example, the one-body contribution to μ_4 has a coefficient which goes as m^4, and the corresponding coefficient for the two-body contribution to μ_4 goes as m^8 in the dense limit. One also sees that in this limit the shape parameters approach constant values, which are not necessarily those of a Gaussian distribution. For the dilute limit, one can get the result of noninteracting particles only if the one-body part (external single-particle energies) is dominant over the two-body part.

Finally, let us consider the shape of ensemble-averaged EVD, using an ensemble of random interactions. The ensemble averaged skewness will be zero for any ensemble (with matrix element mean = 0) of random interactions, because each necessary matrix element trace for the evaluation of the ensemble averaged third moment will have an odd number of matrix elements. We therefore need to consider only the ensemble averaged excess in this case. For the dilute limit one can show that the ensemble averaged eigenvalue distribution is Gaussian. For the dense limit, one gets the Gaussian distribution if N is sufficiently large; the shape parameters for finite N in the dense limit have complicated structure and shall be discussed elsewhere.

In conclusion, we have obtained analytic expressions for the first four moments of the EVD of interacting (via two-body interactions) and noninteracting boson systems. The conditions leading to normality of the EVD have been discussed. Extension of the present work for configuration subspaces is straightforward.

APPENDIX

Here we give expressions for the first four moments of the EVD of a general (1+2)-body Hamiltonian (H):

$$\begin{aligned} H &= \sum \epsilon_{ij} A_i B_j + \tfrac{1}{4}\sum V_{ijkl} A_i A_j B_k B_l \\ &= H^{\nu=0} + H^{\nu=1} + H^{\nu=2} \\ &= H_0 + \tilde{H}, \end{aligned} \tag{A1}$$

$$H^{\nu=0} = H_0 = n\sum \epsilon_{ii}/N + \frac{n(n-1)}{N(N+1)} \sum_{i\le j} V_{ijij},$$

$$H^{\nu=1} = h = \sum \xi_{ij} A_i B_j,$$

$$\xi_{ij} = \epsilon_{ij} - \frac{1}{N}\sum \epsilon_{ii}\delta_{ij} + \frac{n-1}{N+2} \times \left[\sum_k \left(V_{ikjk} - \frac{\delta_{ij}}{N}\sum V_{mnmn}\right)\right],$$

$$H^{\nu=2} = \tilde{V} = \tfrac{1}{4}\sum \tilde{V}_{ijkl} A_i A_j B_k B_l, \tag{A2}$$

$$\langle H\rangle^m = \frac{m}{N}\sum \epsilon_{ii} + \frac{m(m-1)}{N(N+1)} \sum_{i\le j} V_{ijij},$$

$$\langle\langle H\rangle^2\rangle^m = \frac{m(N+m)}{N(N+1)} \sum \xi_{ij}\xi_{ji} + \langle\langle \tilde{V}\rangle^2\rangle^m,$$

$$\langle\langle H\rangle^3\rangle^m = \frac{m(N+m)(N+2m)}{N(N+1)(N+2)} \sum \xi_{ij}\xi_{jk}\xi_{ki} + \frac{\binom{m}{2}\binom{N+m+1}{2}}{\binom{4}{2}\binom{N+3}{4}} \left(3D_1 + \tfrac{3}{2}E_1\right) + \frac{\binom{m}{3}\binom{N+m+1}{2}}{\binom{5}{2}\binom{N+4}{5}} (3E_1) + \langle\langle \tilde{V}\rangle^3\rangle^m,$$

where

$$D_1 = \sum \tilde{V}_{ijkl}\xi_{ik}\xi_{jl},$$

$$E_1 = \sum \tilde{V}_{ijkl}\tilde{V}_{kmij}\xi_{lm}, \tag{A3}$$

$$\langle\langle (H)^4\rangle\rangle^m = \frac{m(N+m)}{N(N+1)} M_2 + \frac{\binom{m}{2}\binom{N+m+1}{2}}{\binom{4}{2}\binom{N+3}{4}} [12K_1 + 2G_1 + 2G_2 + 2G_3 + F_1 + 3(M_1)^2 + 6M_2]$$

$$+ \frac{\binom{m}{3}\binom{N+m+1}{2}}{\binom{5}{2}\binom{N+4}{5}} [24K_1 + 2F_1] + \frac{\binom{m}{3}\binom{N+m+2}{3}}{\binom{6}{3}\binom{N+5}{6}} [12G_1 + 6G_2 + 12G_3 + 12G_4 + \tfrac{3}{2} G_5 + 2F_1 + 12F_2 + 6F_3]$$

$$+ \frac{\binom{m}{4}\binom{N+m+2}{3}}{\binom{7}{3}\binom{N+6}{7}} [4F_1 + 24F_2 + 12F_3] + \langle\langle (\tilde{V})^4\rangle\rangle^m ,$$

where

$$M_1 = \sum \xi_{ij}\xi_{ji}, \quad M_2 = \sum \xi_{ij}\xi_{jk}\xi_{kl}\xi_{li} ,$$

$$K_1 = \sum \tilde{V}_{ijkl}\xi_{ik}\xi_{jm}\xi_{ml} ,$$

$$G_1 = \sum \tilde{V}_{ijkl}\tilde{V}_{knij}\xi_{lm}\xi_{mn} ,$$

$$G_2 = \sum \tilde{V}_{ijkl}\tilde{V}_{mnij}\xi_{km}\xi_{ln} ,$$

$$G_3 = \sum \tilde{V}_{ijkl}\tilde{V}_{kmin}\xi_{lm}\xi_{nj} ,$$

$$G_4 = \sum \tilde{V}_{ijkl}\tilde{V}_{kmin}\xi_{jl}\xi_{mn} ,$$

$$G_5 = \sum \xi_{ij}\xi_{ji} \sum \tilde{V}_{klmn}\tilde{V}_{mnkl} ,$$

$$F_1 = \sum \tilde{V}_{ijkl}\tilde{V}_{kpmn}\tilde{V}_{mnij}\xi_{lp} , \tag{A4}$$

$$F_2 = \sum \tilde{V}_{ijkl}\tilde{V}_{lmjn}\tilde{V}_{npmi}\xi_{kp} ,$$

$$F_3 = \sum \tilde{V}_{ijkl}\tilde{V}_{klmj}\tilde{V}_{mpin}\xi_{np} .$$

Note that the expressions for $\langle\langle (\tilde{V})^i\rangle\rangle^m$ are given in Eq. (11) of the text.

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