PHYSICAL REVIEW C VOLUME 21, NUMBER 6 JUNE 1980

Phenomenological relativistic quantum mechanics of the $NN\pi$ system

M. Betz and F. Coester
Argonne National Laboratory, Argonne, Illinois 60439
(Received 7 December 1979)

The theoretical basis is given for a class of mathematically well defined $NN\pi$ models. They are designed to describe nucleon-nucleon scattering, pion-nucleon scattering, pion-deuteron scattering, pion production, and absorption. The elementary degrees of freedom of the models are the nucleon, the Δ isobar, and the pion. The models are relativistic and do not require renormalization.

[NUCLEAR SCATTERING Relativistic models, N-N scattering, π-N scattering, π-d scattering.]

I. INTRODUCTION

Traditional nuclear theory assumes that nuclei can be treated as a collection of nucleons with the effects of the other degrees of freedom absorbed in the phenomenological Hamiltonian. For a treatment of pion-nucleus reactions it is desirable to extend this scheme so that pions and Δ isobars can play an explicit role.[1,2] A relativistic treatment is indicated because of the small pion mass.

Relativistic quantum mechanics requires a unitary representation of the Poincaré group on the Hilbert space of states. Relativistic field theories satisfy the requirement; they realize the generators of the infinitesimal transformations by space integrals over the energy-momentum tensor.[3] Inevitably there are then infinitely many degrees of freedom and the one-nucleon problem is intractable. Truncation of a relativistic field theory to states with a finite number of particles necessarily destroys the relativistic invariance. This lack of invariance may imply the lack of "cluster separability"[4,5]; specifically, the pion-nucleon scattering in the $NN\pi$ channel with a distant nucleon spectator is not the same as the pion-nucleon scattering in the πN channel, unless special precautions are taken. It is reasonable in most cases to assume that real baryons move with nonrelativistic velocities, but renormalization problems and problems of internal consistency persist in this approximation. There are many models of the $NN\pi$ system based on truncated field theories.[6]

The purpose of this paper is to construct relativistic models of the $NN\pi$ system in which π, N, and Δ are the elementary degrees of freedom; the pion can be absorbed and no renormalization is required. The underlying theory is not new.[5] Relativistic quantum mechanics need not be a field theory. It is possible to construct representations of the Poincaré group for interacting particles by modifying the mass operator M of a system of free particles.[7,4]

The formal structure, which is not unlike that of nonrelativistic quantum mechanics, is based on the following remark. The ten generators of the Poincaré group ($\vec{P}, H$ for space and time translations, $\vec{J}$ for rotations, and $\vec{K}$ for Lorentz boosts) can be constructed as functions of operators $\vec{P}$, $\vec{X}$, $\vec{j}$, and M defined such that the total momentum $\vec{P}$ and the c.m. position $\vec{X}$ satisfy canonical commutation relations and commute with the spin operator $\vec{j}$ and the mass M. The mass operator must also commute with $\vec{j}$.

Conversely, if the ten Poincaré generators are known, then $\vec{P}$, $\vec{X}$, $\vec{j}$, and M are defined as functions of these generators and satisfy the commutation relations specified above as a consequence of the Lie algebra of the Poincaré group. For noninteracting particles the Poincaré generators are well determined. Phenomenological interactions can be introduced by retaining the operators $\vec{P}$, $\vec{X}$, and $\vec{j}$ of the noninteracting system and modifying only the mass operator M:

$$M = M^0 + V \,, \tag{1}$$

where M^0 is the noninteracting mass operator and V commutes with $\vec{P}$, $\vec{X}$, and $\vec{j}$. For two particles, the construction of V is simple.[8,9] For three particles, it is an essential requirement that the right two-body scattering is recovered if one of the particles is far away.[5]

In Sec. II we construct a model of the $NN\pi$ system along the following lines. Let $\mathcal{H}_N$, $\mathcal{H}_\pi$, and $\mathcal{H}_\Delta$ be one-particle Hilbert spaces of a nucleon, a pion, and a Δ isobar. We begin with a description of nucleon-nucleon scattering in the space

$$\mathcal{H}_{NN} = \mathcal{H}_N \otimes \mathcal{H}_N \,, \tag{2}$$

with the mass operator

$$M_{NN} = M^0_{NN} + v_{NN} \,. \tag{3}$$

The interaction v_{NN} is fitted to N-N scattering data and to the properties of the deuteron.

Similarly, we have a description of the πN system in the space

$$\mathcal{H}_{N\pi}=\mathcal{H}_N\otimes\mathcal{H}_\pi\oplus\mathcal{H}_\Delta \ , \tag{4}$$

with the mass operator

$$M_{N\pi}=M^0_{N\pi}+v_{N\pi} \ . \tag{5}$$

The operator $v_{N\pi}$ includes two-body interactions in $\mathcal{H}_N\otimes\mathcal{H}_\pi$ as well as a vertex $N\pi\rightleftarrows\Delta$. For a large class of models the properties of v_{NN} and $v_{N\pi}$ assure the existence and completeness of Møller wave operators.[10]

As an intermediate step we proceed to an equivalent description of the NN system in the presence of a spectator pion, and a description of the $N\pi$ system in the presence of a spectator nucleon. The Hilbert space of states is then

$$\mathcal{H}'=\mathcal{H}_N\otimes\mathcal{H}_N\otimes\mathcal{H}_\pi\oplus\mathcal{H}_N\otimes\mathcal{H}_\Delta \ , \tag{6}$$

and the interacting mass operator is a function of v_{NN} and $v_{N\pi}$. If both NN and $N\pi$ interactions are present it is possible to construct the interacting mass operator M^0+V' in such a manner that, if one of the three particles is moved away, the remaining particles scatter correctly.

For the full model, the Hilbert space of states is $\mathcal{H}=\mathcal{H}'\oplus\mathcal{H}_N\otimes\mathcal{H}_N$, with a mass operator of the form (1) with

$$V=V_0+V'+V'' \ , \tag{7}$$

where V' vanishes on states in $\mathcal{H}_N\otimes\mathcal{H}_N$. The operator V_0 is a phenomenological baryon-baryon interaction defined on the states in $\mathcal{H}_N\otimes\mathcal{H}_N\oplus\mathcal{H}_N\otimes\mathcal{H}_\Delta$. By definition V_0 vanishes on states in $\mathcal{H}_N\otimes\mathcal{H}_N\otimes\mathcal{H}_\pi$. The operator V'' is defined as a three-body interaction on states in $\mathcal{H}_N\otimes\mathcal{H}_N\otimes\mathcal{H}_\pi$ with possible off-diagonal elements to states in $\mathcal{H}_N\otimes\mathcal{H}_N$. It vanishes if any one of the particles is moved away.

The model is designed to allow pion production and absorption and the parameters of V_0 are to be determined by fits to N-N scattering data both above and below the pion threshold. Since the model does not allow inelastic N-N scattering when a pion is present, the cluster separability can be satisfied only to the extent that inelastic N-N scattering is negligible in the $NN\pi$ channel.

The scheme outlined above meets our specifications. (i) It is relativistically invariant. (ii) If all stable particles are distant from each other, then $M\to M^0$ and thus no renormalization is called for. (iii) True pion absorption is included by the mechanism $N\pi\to\Delta$, $N\Delta\to NN$, as well as the off-diagonal part of V''. To the extent that pion production is dominated by the Δ resonance, it is reasonable to consider models in which $V''=0$.

In Sec. III, the multichannel scattering theory for the $NN\pi$ system is summarized and the expressions for the scattering and reaction amplitudes are brought into a form which reflects the strategy for numerical computation that will be used in a subsequent paper.[11] The relevant properties of the Poincaré generators are listed in the Appendix.

II. CONSTRUCTION OF THE $NN\pi$ MODEL

We begin with a representation for two interacting nucleons on $\mathcal{H}_{NN}$. States $|\psi\rangle\in\mathcal{H}_{NN}$ can be represented by functions $\psi(\vec{p}_a,\vec{p}_b)$, where $\vec{p}_a$ and $\vec{p}_b$ are the nucleon momenta. We suppress the spin variables to simplify the notation. They are not important for the general structure of the model. Alternatively we may represent $|\psi\rangle$ by functions $\psi(\vec{P}_{NN},\vec{k}_a)$, where

$$\vec{P}_{NN}=\vec{p}_a+\vec{p}_b \, . \tag{8}$$

The vector $\vec{k}_a$ is defined by

$$k_a=L(\vec{Q}_{NN})\, p_a \, , \tag{9}$$

where k_a and p_a are four-vectors and $L(\vec{Q})$ is the Lorentz transformation that transforms $\{\vec{Q},(1+\vec{Q}^2)^{1/2}\}$ into $\{0,0,0,1\}$. The vector $\vec{Q}_{NN}$ is a center-of-mass velocity defined by

$$\vec{Q}_{NN}=\vec{P}_{NN}/M^0_{NN} \ . \tag{10}$$

In this representation the mass operator (3) has the kernel

$$(\vec{P}'_{NN},\vec{k}'_a|M_{NN}|\vec{k}_a,\vec{P}_{NN})=(\vec{k}'_a|\hat{M}_{NN}|\vec{k}_a)\delta(\vec{P}'_{NN}-\vec{P}_{NN}) \, , \tag{11}$$

where

$$(\vec{k}'_a|\hat{M}_{NN}|\vec{k}_a)=2\omega(\vec{k}_a)\delta(\vec{k}'_a-\vec{k}_a)+(\vec{k}'_a|\hat{v}_{NN}|\vec{k}_a) \tag{12}$$

and

$$\omega(\vec{k})=(\vec{k}^2+m_N^2)^{1/2} \ . \tag{13}$$

It follows from Eq. (11) and the definition of the Møller wave operators $\Omega_\pm$,

$$\Omega_\pm(M,M^0)=s-\lim_{\tau\to\pm\infty} e^{iM\tau}\ e^{-iM^0\tau} \ , \tag{14}$$

that $\Omega_{NN\pm}=\Omega_\pm(M_{NN},M^0_{NN})$ has matrix elements of the form

$$(\vec{P}'_{NN},\vec{k}'_a|\Omega_{NN\pm}|\vec{k}_a,\vec{P}_{NN})=\delta(\vec{P}'_{NN}-\vec{P}_{NN})(\vec{k}'_a|\hat{\Omega}_{NN\pm}|\vec{k}_a). \tag{15}$$

From Eq. (14) and the definition

$$T=v\ \Omega_\pm \ , \tag{16}$$

it follows that the Lippmann-Schwinger equation for the nucleon-nucleon T matrix is

$$(\vec{k}'|\hat{T}_{NN}|\vec{k}) = (\vec{k}'|\hat{v}_{NN}|\vec{k}) + \int_{\omega(\vec{k})} d^3k'' \frac{(\vec{k}'|\hat{v}_{NN}|\vec{k}'')(\vec{k}''|\hat{T}_{NN}|\vec{k})}{2[\omega(\vec{k}) - \omega(\vec{k}'') + i\varepsilon]} , \tag{17}$$

where the limit $\varepsilon \to +0$ is implied.

If there is a pion present, the space of states is $\mathcal{H}_{NN} \otimes \mathcal{H}_\pi$ and the operator defined as $M_{NN} \otimes 1$ on this tensor product will be denoted again by M_{NN}. Its kernel in the representation (11) is

$$(\vec{p}_\pi', \vec{P}_{NN}, \vec{k}_a'|M_{NN}|\vec{k}_a, \vec{P}_{NN}, \vec{p}_\pi) = (\vec{k}_a'|\hat{M}_{NN}|\vec{k}_a)\delta(\vec{P}_{NN}' - \vec{P}_{NN})\delta(\vec{p}_\pi' - \vec{p}_\pi) . \tag{18}$$

Similarly,

$$(\vec{p}_\pi', \vec{P}_{NN}', \vec{k}_a'|\Omega_{NN\pm}|\vec{k}_a, \vec{P}_{NN}, \vec{p}_\pi) = (\vec{k}_a'|\hat{\Omega}_{NN\pm}|\vec{k}_a)\delta(\vec{P}_{NN}' - \vec{P}_{NN})\delta(\vec{p}_\pi' - \vec{p}_\pi) . \tag{19}$$

The three-body Hamiltonian and the three-body boost operator are additive in this representation if the pion does not interact with the nucleons;

$$H_{NN,\pi} = H_{NN} + H_\pi , \tag{20}$$

$$\vec{K}_{NN,\pi} = \vec{K}_{NN} + \vec{K}_\pi . \tag{21}$$

Neither the mass operator (18) nor the wave operator (19) commute with the center-of-mass position $\vec{X}$ defined by (A7) for the noninteracting system. To implement the construction outlined in the Introduction we need a scattering equivalent mass operator M_{NN} that commutes with the noninteracting $\vec{X}$.

Let $\vec{q}_\pi$ be defined by

$$q_\pi = L(\vec{P}/M_{NN}^0)p_\pi . \tag{22}$$

It is possible to represent states $|\psi\rangle \in \mathcal{H}_{NN} \otimes \mathcal{H}_\pi$ by functions $\psi(\vec{P}, \vec{k}_a, \vec{q}_\pi)$. We define the mass operator M_{NN} by its kernel in this representation

$$(\vec{q}_\pi', \vec{P}', \vec{k}_a'|\mathsf{M}_{NN}|\vec{k}_a, \vec{P}, \vec{q}_\pi) = (\vec{k}_a'|\hat{M}_{NN}|\vec{k}_a)\delta(\vec{P}' - \vec{P})\delta(\vec{q}_\pi' - \vec{q}_\pi) . \tag{23}$$

Manifestly, the wave operators

$$\tilde{\Omega}_{NN\pm} = \Omega_\pm(\mathsf{M}_{NN}, M^0) , \tag{24}$$

exist and are complete. From Eqs. (23) and (24) it follows that the matrix elements of $\tilde{\Omega}_{NN\pm}$ are

$$(\vec{q}_\pi', \vec{P}', \vec{k}_a'|\tilde{\Omega}_{NN\pm}|\vec{k}_a, \vec{P}, \vec{q}_\pi) = (\vec{k}_a'|\hat{\Omega}_{NN\pm}|\vec{k}_a)\delta(\vec{P}' - \vec{P})\delta(\vec{q}_\pi' - \vec{q}_\pi) . \tag{25}$$

It is clear from (19) and (25) that $\Omega_{NN\pm}$ and $\tilde{\Omega}_{NN\pm}$ differ. But the S matrices are identical,

$$S_{NN} = \Omega_{NN+}^\dagger \Omega_{NN-} = \tilde{\Omega}_{NN+}^\dagger \tilde{\Omega}_{NN-} , \tag{26}$$

because S commutes with M^0. (Note that $\vec{q}_\pi' = \vec{q}_\pi$ follows from $\vec{p}_\pi' = \vec{p}_\pi$ and $\vec{P}_{NN}' = \vec{P}_{NN}$ if and only if $|\vec{k}_a'| = |\vec{k}_a|$.) Since

$$M^0 = (\vec{q}_\pi^2 + M_{NN}^{0\,2})^{1/2} + (\vec{q}_\pi^2 + m_\pi^2)^{1/2} , \tag{27}$$

it follows from

$$\mathsf{M}_{NN}\tilde{\Omega}_{NN\pm} = \tilde{\Omega}_{NN\pm} M_{NN}^0 \tag{28}$$

that

$$\tilde{\Omega}_{NN\pm} M^0 = \mathsf{M}_{NN,\pi} \tilde{\Omega}_{NN\pm} , \tag{29}$$

where

$$\mathsf{M}_{NN,\pi} = (\vec{q}_\pi^2 + \mathsf{M}_{NN}^2)^{1/2} + (\vec{q}_\pi^2 + m_\pi^2)^{1/2} . \tag{30}$$

We may therefore expect the desirable result

$$\Omega_\pm(\mathsf{M}_{NN,\pi}, M^0) = \Omega_\pm(\mathsf{M}_{NN}, M_{NN}^0) . \tag{31}$$

A proof of (31) uses the Birman-Kato invariance principle[12,13] and Lebesgue's bounded convergence theorem.[14]

The Hamiltonian

$$\tilde{H}_{NN,\pi} = [\vec{P}^2 + \mathsf{M}_{NN,\pi}^2]^{1/2} \tag{32}$$

is not additive as in (20). The two-body interaction term

$$V_{NN,\pi} = \mathsf{M}_{NN,\pi} - M^0 = (\vec{q}_\pi^2 + \mathsf{M}_{NN}^2)^{1/2} - (\vec{q}_\pi^2 + M_{NN}^{0\,2})^{1/2} \tag{33}$$

depends parametrically on the spectator momentum $\vec{q}_\pi$.

The treatment of the two-body πN problem is similar to that of N-N scattering described above except that the masses are not equal and πN is coupled to a virtual Δ channel. Any state $|\psi\rangle$ in $\mathcal{H}_{N\pi}$ has a component in $\mathcal{H}_N \otimes \mathcal{H}_\pi$ represented by a function $\psi_{N\pi}(\vec{P}_{N\pi}, \vec{k}_\pi)$ and a component in $\mathcal{H}_\Delta$ represented by a function $\psi_\Delta(\vec{P}_{N\pi})$, where $\vec{P}_{N\pi}$ denotes the total momentum and $\vec{k}_\pi$ is defined by

$$k_\pi = L(\vec{P}_{N\pi}/M_{N\pi}^0)p_\pi . \tag{34}$$

The mass operator $M_{N\pi}$ has matrix elements proportional to $\delta(\vec{P}_{N\pi}' - \vec{P}_{N\pi})$ as in (11) and the restricted matrix $\hat{M}_{N\pi}$ is of the form

$$(\vec{k}_\pi'|\hat{M}_{N\pi}|\vec{k}_\pi) = W(\vec{k})\,\delta(\vec{k}_\pi' - \vec{k}_\pi) + (\vec{k}_\pi'|v_{N\pi}|\vec{k}_\pi) , \tag{35}$$

where

$$W(\vec{k}) = (\vec{k}^2 + m_N^2)^{1/2} + (\vec{k}^2 + m_\pi^2)^{1/2} , \tag{36}$$

$$(\Delta|\hat{M}_{N\pi}|\Delta) = m_\Delta , \tag{37}$$

$$(\Delta|\hat{M}_{N\pi}|\vec{k}_\pi) = (\Delta|\hat{v}_{N\pi}|\vec{k}_\pi) . \quad (38)$$

The parameters are such that $\hat{M}_{N\pi}$ has no point spectrum. The T matrix for πN scattering satisfies the Lippmann-Schwinger equation

$$(\vec{k}'|\hat{T}_{N\pi}|\vec{k}) = (\vec{k}'|\hat{U}(\vec{k})|\vec{k}) + \int d^3k'' \frac{(\vec{k}'|\hat{U}(\vec{k})|\vec{k})(\vec{k}''|T_{N\pi}|\vec{k})}{W(\vec{k}) - W(\vec{k}'') + i\varepsilon} , \quad (39)$$

where the effective potential $\hat{U}(\vec{k})$ is defined by

$$(\vec{k}')|\hat{U}(\vec{k})|\vec{k}'') = (\vec{k}'|\hat{v}_{N\pi}|\vec{k}'') + \frac{(\vec{k}'|\hat{v}_{N\pi}|\Delta)(\Delta|\hat{v}_{N\pi}|\vec{k}'')}{W(\vec{k}) - m_\Delta + i\varepsilon} . \quad (40)$$

The wave operators $\Omega_{N\pi}$ are known to exist and to be complete.[10]

The pion-nucleon interaction in the three-body space can be defined by the same procedure that led to Eq. (33). The result is

$$V_{a\pi,b} = \mathrm{M}_{a\pi,b} - M^0 = (\vec{q}_b^2 + \mathrm{M}_{a\pi}^2)^{1/2} - (\vec{q}_b^2 + M_{a\pi}^{02})^{1/2} , \quad (41)$$

where a labels the nucleon that interacts with the pion and b labels the spectator.

The fully interacting mass operator is then of the form (1) with (7) and

$$V' = V_{NN,\pi} + V_{a\pi,b} + V_{b\pi,a} . \quad (42)$$

The Hamiltonian $H = (\vec{P}^2 + M^2)^{1/2}$ does not become additive when a particle is moved away, but the wave operators

$$\Omega_\pm = s\text{-}\lim_{\tau\to\pm\infty} e^{iM\tau} e^{-iM^0\tau} \quad (43)$$

satisfy the cluster relations[5]

$$\lim_{|\vec{d}|\to\infty} \| (\Omega_\pm - \tilde{\Omega}_{NN\pm}) e^{i\vec{q}_\pi\cdot\vec{d}} |\psi\rangle \| = 0 , \quad (44)$$

$$\lim_{|\vec{d}|\to\infty} \| (\Omega_\pm - \tilde{\Omega}_{a\pi\pm}) e^{i\vec{q}_b\cdot\vec{d}} |\psi\rangle \| = 0 , \quad (45)$$

where $|\psi\rangle \in \mathcal{H}_{NN} \otimes \mathcal{H}_\pi$. The S operator has therefore the required cluster properties,

$$\lim_{|\vec{d}|\to\infty} \| (S - S_{NN} \otimes 1) e^{i\vec{p}_\pi\cdot\vec{d}} |\psi\rangle \| = 0 \quad (46)$$

and

$$\lim_{|\vec{d}|\to\infty} \| (S - S_{a\pi} \otimes 1) e^{i\vec{p}_b\cdot\vec{d}} |\psi\rangle \| = 0 . \quad (47)$$

This completes the proof that the models constructed above satisfy the conditions stated in the Introduction. In any concrete realization, a convenient form is assumed for the two-body interactions v_{NN}, $v_{N\pi}$, and V_0. The mass operator is then of the form (1) and (7) with V' given by (42). This mass operator together with the operators $\vec{P}$, $\vec{X}$, and $\vec{j}$ of the noninteracting system can be used to obtain the Poincaré generators using (A13)–(A15).

Contact with experiment requires equations for the relevant scattering amplitudes. The derivation of these equations from the general theory is sketched in the Sec. III.

III. SCATTERING THEORY

The complete scattering theory of the $NN\pi$ system must consider the channels NN, $NN\pi$, and πd. Since a bound state—the deuteron—is involved, it is most convenient to use a two-Hilbert-space formulation.[5,15-17] The initial and final state are described in the Hilbert space $\mathcal{H}^f$,

$$\mathcal{H}^f = \mathcal{H}^f_{NN} \oplus \mathcal{H}^f_{NN\pi} \oplus \mathcal{H}^f_{\pi d} . \quad (48)$$

The Poincaré generators in $\mathcal{H}^f$ are those for free particles in the three channels NN, $NN\pi$, and πd. A time dependent scattering state $\psi(t) \in \mathcal{H}$ satisfies the initial condition

$$\lim_{t\to-\infty} \|\psi(t) - \Phi\chi(t)\| = 0 , \quad (49)$$

where $\chi(t)$ is a state of noninteracting particles in $\mathcal{H}^f$ and the operator Φ from $\mathcal{H}^f$ into $\mathcal{H}$ maps $\mathcal{H}^f_{NN}$ and $\mathcal{H}^f_{NN\pi}$ identically into $\mathcal{H}_{NN}$ and $\mathcal{H}_{NN\pi}$. It maps $\mathcal{H}^f_{\pi d}$ into $\mathcal{H}_{NN\pi}$ according to

$$(\vec{p}_\pi', \vec{P}_{NN}, \vec{k}|\Phi|\vec{P}_d, \vec{p}_\pi) = \delta(\vec{p}_\pi' - \vec{p}_\pi)\delta(\vec{P}_d - \vec{P}_{NN})\phi_d(\vec{k}), \quad (50)$$

where $\phi_d(\vec{k})$ is the momentum space deuteron wave function. The wave operators are then defined by

$$\Omega_\pm = s\text{-}\lim_{\tau\to\pm\infty} e^{iM\tau} \Phi e^{-iM^f\tau} . \quad (51)$$

From the definition of Φ it follows that

$$\Phi M^f|\alpha\rangle = M^0\Phi|\alpha\rangle \quad (52)$$

if $|\alpha\rangle \in \mathcal{H}^f_{NN}$ or $|\alpha\rangle \in \mathcal{H}^f_{NN\pi}$, and that

$$\Phi M^f|\delta\rangle = (M^0 + V_{NN,\pi})\Phi|\delta\rangle \quad (53)$$

if $|\delta\rangle \in \mathcal{H}^f_{\pi d}$.

An off-shell T operator in $\mathcal{H}^f$ can be defined by[18]

$$T(z) = \Phi^\dagger\mathcal{V} + \mathcal{V}^\dagger \frac{1}{z-M} \mathcal{V} , \quad (54)$$

where $\mathcal{V}$ is by definition

$$\mathcal{V}=M\Phi-\Phi M^f \ . \tag{55}$$

It follows from Eqs. (51) and (54) and the definition of S,

$$S=\Omega_+^\dagger\Omega_- \ , \tag{56}$$

that

$$\langle w'|S|w\rangle=\delta(w'-w)[1-2\pi i\langle w'|T(w)|w\rangle] \ , \tag{57}$$

where w is an eigenvalue of M^f. From the definition of $\mathcal{V}$ together with (52) and (53) it follows that

$$\mathcal{V}|\alpha\rangle=V\Phi|\alpha\rangle \tag{58}$$

and

$$\mathcal{V}|\delta\rangle=V_d\Phi|\delta\rangle \ , \tag{59}$$

where

$$V_d=V-V_{NN,\pi} \ . \tag{60}$$

It is therefore possibe to express the channel projections of the operator $T(z)$ in the form $\Phi^\dagger\mathcal{T}(z)\Phi$ where $\mathcal{T}(z)$ is a different operator for different channel projections. We have

$$\langle\beta|T(z)|\alpha\rangle=\langle\beta|\Phi^\dagger\mathcal{T}(z)\Phi|\alpha\rangle \ , \tag{61}$$

with

$$\mathcal{T}(z)=V+V\frac{1}{z-M}V \ , \tag{62}$$

if $|\alpha>$ and $|\beta>$ are in the NN channel or in the $NN\pi$ channel. If $|\delta>$ and $|\delta'>$ are in the πd channel we have

$$\langle\alpha|T(z)|\delta\rangle=\langle\alpha|\Phi^\dagger\mathcal{T}_d(z)\Phi|\delta\rangle \ , \tag{63}$$

$$\langle\delta'|T(z)|\delta\rangle=\langle\delta'|\Phi^\dagger\mathcal{T}_{dd}(z)\Phi|\delta\rangle \ , \tag{64}$$

where

$$\mathcal{T}_d(z)=\left(1+V\frac{1}{z-M}\right)V_d \tag{65}$$

and

$$\mathcal{T}_{dd}(z)=\left(1+V_d\frac{1}{z-M}\right)V_d \ . \tag{66}$$

For purposes of calculations it will be convenient to project out the three-body space $\mathcal{H}_{NN}\otimes\mathcal{H}_\pi$. Let $\mathcal{P}$ be the projection operator that projects onto $\mathcal{H}_{NN}\oplus\mathcal{H}_{N\Delta}$ and $\overline{\mathcal{P}}=1-\mathcal{P}$. We use the following definitions:

$$R(z)=(z-M)^{-1} \ , \tag{67}$$

$$\overline{R}(z)=(z-\overline{\mathcal{P}}M\overline{\mathcal{P}})^{-1}\overline{\mathcal{P}} \ , \tag{68}$$

$$\tilde{M}(z)=\mathcal{P}M\mathcal{P}+\mathcal{P}V\overline{R}(z)V\mathcal{P} \ , \tag{69}$$

$$\tilde{R}(z)=[z-\tilde{M}(z)]^{-1} \ . \tag{70}$$

Because of the $\Delta\to N\pi$ vertex, $\tilde{M}(z)$ does not go to M^0 if the nucleon in the space $\mathcal{H}_N\otimes\mathcal{H}_\Delta$ is moved to infinity. Let $V_D(z)$ be the surviving Δ self-interaction,

$$V_D(z)=\lim_{|\vec{d}|\to\infty}e^{i\vec{q}_N\cdot\vec{d}}\,\mathcal{P}_{N\Delta}[\tilde{M}(z)-M^0]\,\mathcal{P}_{N\Delta}\,e^{-i\vec{q}_N\cdot\vec{d}}$$

$$=\lim_{|\vec{d}|\to\infty}e^{i\vec{q}_N\cdot\vec{d}}\,\mathcal{P}V\overline{R}(z)V\mathcal{P}e^{-i\vec{q}_N\cdot\vec{d}} \ , \tag{71}$$

and define

$$\tilde{V}(z)=\tilde{M}(z)-\tilde{M}^0(z)=V_0+\mathcal{P}V\overline{R}V\mathcal{P}-V_D(z) \ , \tag{72}$$

where

$$\tilde{M}^0(z)=\mathcal{P}M^0\mathcal{P}+V_D(z) \ . \tag{73}$$

The following relations are simple consequences of these definitions and the identities

$$\frac{1}{z-A-B}=\frac{1}{z-A}+\frac{1}{z-A}B\frac{1}{z-A-B}$$
$$=\frac{1}{z-A}+\frac{1}{z-A-B}B\frac{1}{z-A} \tag{74}$$

which hold for any z, A, and B:

$$\tilde{R}(z)=\mathcal{P}R(z)\mathcal{P} \ , \tag{75}$$

$$R(z)=[1+\overline{R}(z)V]\tilde{R}(z)[V\overline{R}(z)+1]+\overline{R}(z) \ . \tag{76}$$

From (65) and (74) it follows that

$$\mathcal{T}(z)=U(z)+U(z)\tilde{R}(z)U(z) \ , \tag{77}$$

$$\mathcal{T}_d(z)=[1+U(z)\tilde{R}(z)]U_d(z) \ , \tag{78}$$

and

$$\mathcal{T}_{dd}(z)=U_{dd}(z)+U_d^\dagger(z^*)\tilde{R}(z)U_d(z) \ , \tag{79}$$

where

$$U(z)=V+V\overline{R}(z)V-V_D(z) \ , \tag{80}$$

$$U_d(z)=[1+V\overline{R}(z)]V_d \ , \tag{81}$$

and

$$U_{dd}(z)=V_d+V_d\overline{R}(z)V_d \ . \tag{82}$$

Further reductions involve the identities

$$\tilde{R}(z)=\tilde{R}_0(z)+\tilde{R}_0(z)\tilde{T}(z)\tilde{R}_0(z) \ , \tag{83}$$

where

$$\tilde{R}_0(z)=[z-\tilde{M}_0(z)]^{-1} \tag{84}$$

and

$$\tilde{T}(z)=\tilde{V}(z)+\tilde{V}(z)\tilde{R}(z)\tilde{V}(z)$$
$$=\tilde{V}(z)+\tilde{V}(z)\tilde{R}_0(z)\tilde{T}(z) \ . \tag{85}$$

Since

$$\mathcal{P}U(z)\mathcal{P}=\tilde{V}(z)\ , \tag{86}$$

we have

$$\mathcal{P}T(z)\mathcal{P}=\tilde{T}(z) \tag{87}$$

and

$$\langle\beta\,|\,T(z)\,|\,\alpha\rangle=\langle\beta\,|\,\Phi^{\dagger}\tilde{T}(z)\Phi\,|\,\alpha\rangle \tag{88}$$

if $|\alpha>$ and $|\beta>$ are in the NN channel.

In practice, the parameters of $v_{N\pi}$ are determined from π-N elastic scattering data using (39) and (40). The parameters of v_{NN} are obtained from (17) by fits to N-N elastic scattering data in the energy region where inelasticity is negligible. The parameters of V_0 are then determined from N-N elastic scattering data both above and below the pion production threshold. This involves using (88) and (85). The study of other processes then provides tests of the validity of the model as well as means of removing any remaining ambiguity in the interactions. The numerical implementation of this scheme will be reported elsewhere.[11]

We wish to thank T.-S. H. Lee for helpful discussions. This work was performed under the auspices of the U.S. Dept. of Energy.

APPENDIX

The Poincaré generators $\vec{P}$, H, $\vec{J}$, $\vec{K}$ satisfy the commutation relations

$$[P_i,P_j]=[P_i,H]=0\ , \tag{A1}$$

$$[J_p,J_q]=i\sum_r \varepsilon_{pqr}J_r\ , \tag{A2}$$

$$[J_p,P_q]=i\sum_r \varepsilon_{pqr}P_r,\quad [J_i,H]=0\ , \tag{A3}$$

$$[J_p,K_q]=i\sum_r \varepsilon_{pqr}K_r\ , \tag{A4}$$

$$[K_p,K_q]=-i\sum_r \varepsilon_{pqr}J_r\ , \tag{A5}$$

$$[K_p,P_q]=i\,\delta_{pq}H,\quad [K_r,H]=i\,P_r\ . \tag{A6}$$

The Newton Wigner position operator[19] $\vec{X}$ is defined by

$$\vec{X}=\tfrac{1}{2}\,(H^{-1}\,\vec{K}+\vec{K}H^{-1})-\frac{\vec{P}\times(H\vec{J}+\vec{P}\times\vec{K})}{MH(M+H)} \tag{A7}$$

and the canonical spin $\vec{j}$ is

$$\vec{j}=\vec{J}-\vec{X}\times\vec{P}\ . \tag{A8}$$

It follows from (A1)–(A8) that

$$[X_r,X_s]=0\ , \tag{A9}$$

$$[X_r,P_s]=i\delta_{rs}\ , \tag{A10}$$

and

$$[X_r,j_s]=[P_r,j_s]=0\ . \tag{A11}$$

Since M commutes with all the generators, it follows that

$$[M,\vec{j}]=[M,\vec{X}]=[M,\vec{P}]=0\ . \tag{A12}$$

Conversely if $\vec{X}$, $\vec{P}$, $\vec{j}$, and M satisfy (A1)–(A12), then the generators H, $\vec{J}$, $\vec{K}$ defined by

$$H=(\vec{P}^2+M^2)^{1/2}\ , \tag{A13}$$

$$\vec{J}=\vec{X}\times\vec{P}+\vec{j}\ , \tag{A14}$$

and

$$\vec{K}=\tfrac{1}{2}\,(H\vec{X}+\vec{X}H)-\vec{j}\times\vec{P}\,(M+H)^{-1} \tag{A15}$$

satisfy the commutation relations (A1)–(A6).

[1] G. E. Brown and W. Weise, Phys. Rep. 22C, 279 (1975).

[2] L. S. Kisslinger and W. L. Wang, Ann. Phys. (N.Y.) 99, 374 (1976).

[3] J. Schwinger, Phys. Rev. 127, 324 (1962).

[4] L. L. Foldy, Phys. Rev. 122, 275 (1961).

[5] F. Coester, Helv. Phys. Acta 38, 7 (1965).

[6] See A. W. Thomas and A. S. Rinat, Phys. Rev. C 20, 216 (1979) and references therein.

[7] B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).

[8] R. Fong and J. Sucher, J. Math. Phys. 5, 956 (1964).

[9] F. Coester, S. C. Pieper, and F. J. D. Serduke, Phys. Rev. C 11, 1 (1975), Secs. II and III.

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1979), Vol. III, Theorem XI. 9, p. 27.

[11] M. Betz and T.-S. H. Lee (unpublished).

[12] J. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin-Heidelberg, 1966), Chap. X.4.

[13] Reference 10, Vol. I, Theorem XI.4, p. 30 and Appendix 3 to Chap. XI.3, p. 49ff.

[14] Reference 10, Vol. I, Theorem I.11.

[15] T. Kato, J. Funct. Anal. 1, 342 (1967).

[16] Reference 10, Vol. III, Chap. XI.3.

[17] F. Coester, Proceedings of the Conference on Mathematical Methods and Applications of Scattering Theory, May, 1979, Catholic University, Washington, D. C.

[18] F. Coester and L. Schlessinger, Ann. Phys. (N.Y.) 78, 90 (1973), see Appendix, p. 128.

[19] M. H. L. Pryce, Proc. R. Soc. London A195, 62 (1949), definition (e); T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949).