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# Parametrization of $N/D$ input

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A modified version of the usual single channel $N/D$ equations is presented, which makes possible a mathematically sound parametrization of the effect of those distant parts of the left hand cut and right hand cut for which the discontinuities are not known. A conformal mapping technique is used to develop a generally valid parametrization involving only real parameters. An application to the Phillips plot of triton energy versus doublet scattering length is given so as to demonstrate the superiority of the present scheme over ones used earlier by us.

[NUCLEAR REACTIONS Parametrization of unknown discontinuities in $N/D$ equations, Phillips plot.]

## I. INTRODUCTION

The $N/D$ method was devised by Chew and Mandelstam[1] in order to overcome the difficult problem of solving the nonlinear singular integral equations that describe the analytic structure of partial wave scattering amplitudes. The method immediately found widespread application.[2] In the last decade the method has been used extensively in the few-body problems of nuclear physics.[3-17]

In its simplest form the $N/D$ method allows one to construct single-channel partial wave amplitudes that are analytic in the complex energy plane except for a right hand cut (RHC) associated with two-particle unitarity, separated from a left hand cut (LHC) whose discontinuity is obtained from the underlying dynamics, as well as bound state poles. One of the earliest extensions[18] of the method made it possible to account for deviations from single-channel, two-particle unitarity by using inelasticities taken from experiment or a model as part of the input. Schemes for handling multichannel, two-particle unitarity have been in existence for some time now.[19,20] The first attempt toward the inclusion of three-particle unitarity within the $N/D$ framework was made by Blankenbecler,[21] and worked out in detail shortly thereafter.[22,23]

The presence of three-particle channels complicates the unitarity relations significantly. The $2\to 3$ and $3\to 3$ amplitudes depend not only on the total energy, but on the energies of the two-particle subsystems as well. For this reason various approximate methods[3,6,7,13,15] have been proposed for treating the three-particle contribution to the unitarity equations. In the $N/D$ approach scattering amplitudes are determined by their singularities in the whole energy plane, which implies that in principle contributions to the unitarity equations from channels with arbitarily large numbers of particles must be considered. This is impractical, therefore one must truncate the unitarity relations, which means that the RHC is treated exactly only over a limited energy range.

A similar problem exists for the LHC. In general as one moves off to the left, calculation of the discontinuity across the cut involves processes in which more and more particles are exchanged. In the field theory approach the various particles exchanged are described explicitly.[4,9] In many of the nuclear physics applications[6,7,10-17] only the nucleon exchanges are put in explicitly, and the pionic degrees of freedom are absorbed in form factors or vertex functions. In any case it is only possible to treat the low energy end of the LHC carefully. Model calculations[8,11] indicate that it is often necessary to treat a significant part of the LHC correctly in order to obtain satisfactory results.

The purpose of the present paper is to present a method for parametrizing in a systematic and mathematically satisfying way the effect of the distant parts of the RHC and LHC. In recent calculations on the three-nucleon system, we have presented methods for parametrizing the effect of the distant part of the LHC by means of power series in the square root of the energy,[14] and in a variable obtained by a conformal mapping technique.[16,17] However, no systematic treatment of the RHC was given. We assume the partial wave amplitude is analytic in the complex energy plane except for an RHC separated from an LHC, and possible bound state poles. We also assume the discontinuity is known across the low energy end of the LHC, and allow for input of the inelasticity parameter over a limited part of the RHC. Everything else is parametrized by means of a power series in a variable obtained from a conformal mapping technique. The method presented here is an improvement over our earlier work[14,16,17] in that it makes it possible to parametrize the effect of the neglected parts of both the RHC and LHC by means of one mathematical form.

The outline of the paper is as follows. In Sec. II we present a modification of the usual $N/D$ equations which is convenient for our purposes. In a sense these modified equations are a mixture of those that have been in the literature for some time.[18,19] Here we also develop the conformal transformation which is the basis for our parametrization scheme. In Sec. III we present an application of our method which demonstrates the improvement over our earlier work.[14,16,17] Our application is to the well known Phillips plot[24,25] which summarizes the relation between the triton binding energy and the doublet scattering length. Section IV gives a brief summary as well as suggestions for future work.

## II. THE METHOD

We shall use a dimensionless energy variable $z$, defined so that $z=0$ is the elastic threshold and $z=1$ is the first inelastic threshold. We write the elastic scattering amplitude in the form

$$f(z)=(\eta e^{2i\delta}-1)/(2iz^{1/2}), \tag{1}$$

where $\delta$ is a real phase shift, $\eta$ is the inelasticity and has the value $\eta=1$ for $0\leq z\leq 1$ and satisfies $\eta<1$ for $z>1$. We assume $\eta$ is known on the interval $1<z<c$. We also assume that $f(z)$ is a real, analytic function of $z$ with an RHC beginning at $z=0$, an LHC beginning at $z=-a$ $(a>0)$, and possible bound state poles on the real, negative $z$ axis. We introduce an effective amplitude $F(z)$ through the relation

$$F(z)=\frac{f(z)}{R(z)}+\frac{1}{2iz^{1/2}}\left[\frac{1}{R(z)}-1\right], \tag{2}$$

where

$$R(z)=\exp\left[-\frac{iz^{1/2}}{\pi}\int_1^c dy\,\frac{\ln\eta(y)}{y^{1/2}(y-z)}\right]. \tag{3}$$

The function $R(z)$ has one RHC for $z\geq 0$ and another one for $1\leq z\leq c$, but no LHC's. It has been defined so that $F(z)$ satisfies a two-body like unitarity relation for $0<z<c$, i.e.,

$$\operatorname{Im}F^{-1}(z)=-z^{1/2},\quad 0<z<c\,. \tag{4}$$

Whenever we write the imaginary part of a function on one of its cuts, we shall mean the value just above the cut. On the LHC we have

$$\operatorname{Im}F(z)=\operatorname{Im}f(z)/R(z),\quad z<-a\,. \tag{5}$$

We assume that $\operatorname{Im}f(z)$ is known on the interval $-b<z<-a$. We write

$$F(z)=N(z)/D(z)\,, \tag{6}$$

with $N(z)$ carrying the entire LHC and the RHC for $z>c$, while $D(z)$ carries the RHC for $0<z<c$. From Cauchy's theorem it follows that we can write

$$N(z)=\frac{1}{\pi}\left(\int_{-\infty}^{-a}+\int_c^{\infty}\right)dy\,\frac{\operatorname{Im}N(y)}{y-z} \tag{7}$$

and

$$D(z)=1+\frac{z}{\pi}\int_0^c dy\,\frac{\operatorname{Im}D(y)}{y(y-z)}\,. \tag{8}$$

In writing these equations we have assumed that $N(z)\to 0$ and $D(z)\to$ constant when $|z|\to\infty$. Also we have chosen $D(0)=1$. From (4) and (6) we have

$$\operatorname{Im}N(z)=D(z)\operatorname{Im}F(z),\; z<-a,\quad z>c \tag{9}$$

and

$$\operatorname{Im}D(z)=-z^{1/2}N(z),\quad 0<z<c\,. \tag{10}$$

Using the above equations, putting (7) into (8), and interchanging the order of integration we find the following equation for $D(z)$:

$$D(z)=1-\frac{z}{\pi}\left(\int_{-\infty}^{-a}+\int_c^{\infty}\right)dy\,K(z^{1/2},y^{1/2})D(y)\operatorname{Im}F(y)\,, \tag{11}$$

where

$$K(z^{1/2},y^{1/2})=\frac{I(z^{1/2})-I(y^{1/2})}{y-z}, \tag{12}$$

with

$$I(z^{1/2})=\frac{1}{\pi}\int_0^c\frac{dy}{y^{1/2}(y-z)}\,. \tag{13}$$

Carrying out the integration in (13) we have

$$I(z^{1/2})=\frac{i}{z^{1/2}}-\frac{1}{\pi z^{1/2}}\ln\left(\frac{c^{1/2}+z^{1/2}}{c^{1/2}-z^{1/2}}\right). \tag{14}$$

Because of the logarithm, $I(z^{1/2})$ is a multivalued function in the $z^{1/2}$ plane. We choose the branch defined below,

$$\ln\left(\frac{c^{1/2}+z^{1/2}}{c^{1/2}-z^{1/2}}\right)=\ln(r_2/r_1)+i(\theta_2-\theta_1+\pi),\; 0<\theta_1<2\pi,\quad -\pi<\theta_2<\pi \tag{15}$$

where $r_1$, $r_2$, $\theta_1$ and $\theta_2$ are shown in Fig. 1. This gives an RHC and an LHC in the $z^{1/2}$ plane, beginning at $z^{1/2}=c^{1/2}$ and $z^{1/2}=-c^{1/2}$, respectively.

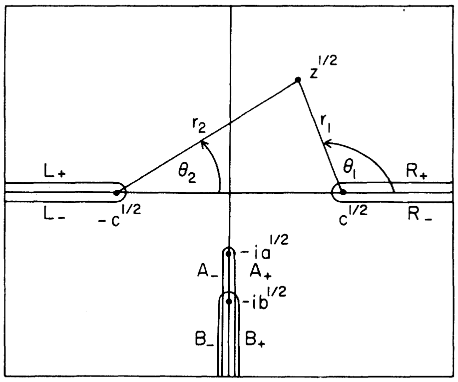


FIG. 1. $z^{1/2}$ plane.

We denote the location of the RHC and LHC by $R$ and $L$, and we distinguish the two sides of the cuts by plus and minus signs. The values of $I(z^{1/2})$ just above and below these cuts are given by

$$I(\pm z^{1/2}+i\epsilon)=-\frac{1}{\pi z^{1/2}}\ln\left(\frac{z^{1/2}+c^{1/2}}{z^{1/2}-c^{1/2}}\right),\quad z^{1/2}\text{ on }R \tag{16}$$

and

$$I(\pm z^{1/2}-i\epsilon)=\pm\frac{2i}{z^{1/2}}-\frac{1}{\pi z^{1/2}}\ln\left(\frac{z^{1/2}+c^{1/2}}{z^{1/2}-c^{1/2}}\right),\quad z^{1/2}\text{ on }R\,. \tag{17}$$

Throughout we assume $0<\epsilon\ll 1$. It should be kept in mind that if $z$ is just above (below) the real, positive $z$ axis, $z^{1/2}$ is just above the real, positive (negative) $z^{1/2}$ axis. Using this it is easy to verify that the function defined by (14) and (15) has the singularity structure in the $z$ plane implied by (13), namely, a cut on the real $z$ axis for $0<z<c$ and no other singularities.

We now discuss the singularities of $D(z)$ in the $z^{1/2}$ plane. From (11) we see that these are determined by the singularities of the kernel $K(z^{1/2},y^{1/2})$. Singularities can arise from the branch cuts at $L$ and $R$ in $I(z^{1/2})$, as well as from the vanishing of the denominator in (12), i.e., when $z^{1/2}=\pm y^{1/2}$. It appears as if we have singularities given by $z^{1/2}=\pm i|y|^{1/2}$, with $-\infty<y<-a$, but actually there is no singularity for $z^{1/2}=+i|y|^{1/2}$, since the numerator in (12) vanishes when this occurs. The singularity given by $z^{1/2}=-i|y|^{1/2}$, $-\infty<y<-a$, leads to a branch cut in $D(z)$ on the negative, imaginary axis as shown in Fig. 1. We denote its location by $A$. In determining whether or not there are singularities given by $z^{1/2}=\pm y^{1/2}$, with $c<y<\infty$, we have to be a little careful. In (12), $I(y^{1/2})$ is given by (16) with $z=y$. With this in mind and using (17), we see that as $z^{1/2}$ approaches $\pm y^{1/2}$, $c<y<\infty$, through negative, imaginary values, the numerator in (12) does not vanish, so we get singularities on both the positive and negative, real $z^{1/2}$ axis from the vanishing of the denominator in (12). These singularities contribute to the discontinuities of $D(z)$ across $L$ and $R$.

It is not difficult to determine the discontinuity of $K(z^{1/2},y^{1/2})$ across the various cuts. The results are

$$K(z^{1/2}+\epsilon,y^{1/2})-K(z^{1/2}-\epsilon,y^{1/2})=4\pi\delta(y-z)/z^{1/2},\quad z^{1/2}\text{ on }A \tag{18}$$

and

$$K(\pm z^{1/2}+i\epsilon,y^{1/2})-K(\pm z^{1/2}-i\epsilon,y^{1/2})=\mp\frac{2i}{z^{1/2}}\frac{1}{y-z\pm i\epsilon},\quad z^{1/2}\text{ on }R\,. \tag{19}$$

The corresponding discontinuities in $D(z)\equiv d(z^{1/2})$ are obtained from (11) and are given by

$$d(z^{1/2}+\epsilon)-d(z^{1/2}-\epsilon)=-4z^{1/2}\,\mathrm{Im}N(z),\quad z^{1/2}\text{ on }A \tag{20}$$

and

$$d(\pm z^{1/2}+i\epsilon)-d(\pm z^{1/2}-i\epsilon)=\pm 2iz^{1/2}N(z\mp i\epsilon),\quad z^{1/2}\text{ on }R\,. \tag{21}$$

Here we have also used (7) and (9). It should be noted that the $z$ appearing in the argument of $N$ in (20) and (21) is on the physical sheet.

For $z$ real, we have

$$D(z\pm i\epsilon)=d(\pm z^{1/2}),\quad 0<z^{1/2}<c^{1/2} \tag{22}$$

and

$$D^*(z+i\epsilon)=D(z-i\epsilon)\,, \tag{23}$$

which when combined with (10) leads to

$$N(z)=-[d(z^{1/2})-d(-z^{1/2})]/(2iz^{1/2})\,. \tag{24}$$

Since the right hand side of this equation has a well defined analytic continuation, it can be used everywhere. It is easy to verify this relation by using (9), (11), (12), and (14) to reproduce (7). This relation can be used to check (20) and (21), by expressing the discontinuities of $N(z)$ across its LHC and RHC in terms of $d(z^{1/2})$.

By assumption $\mathrm{Im}F(z)$ is not known for $z>c$ and $z<-b$, so we write (11) in the form

$$D(z)=1+U(z^{1/2})-\frac{z}{\pi}\int_{-b}^{-a}dy\,K(z^{1/2},y^{1/2})D(y)\,\mathrm{Im}F(y)\,, \tag{25}$$

where $U(z^{1/2})$ is an unknown function. From the discussion above, we know that $U(z^{1/2})$ is an analytic function in the $z^{1/2}$ plane, except for the cuts $L$, $R$, and $B$ in Fig. 1, where $B$ starts at $z^{1/2} = -ib^{1/2}$. Also, we know that

$$U(0)=0\,. \tag{26}$$

This information can be used to obtain a mathematically meaningful parametrization of $U(z^{1/2})$. For example, if $b<c$, $U(z^{1/2})$ can be expressed as a power series in $z^{1/2}$ which will converge when $|z|<b$. A more sophisticated parametrization can be obtained by a conformal mapping technique. If $w(z^{1/2})$ maps a region $\Upsilon$ of the $z^{1/2}$ plane, in which $U(z^{1/2})$ is analytic, conformally onto the interior of the unit circle in the $w$ plane, centered on the origin, then the series

$$U(z^{1/2})=\sum_{n=0}^{\infty}c_n w^n(z^{1/2}) \tag{27}$$

converges for $z^{1/2}$ in $\Upsilon$. The $c_n$'s which are constrained by (26) can be determined by fitting the amplitude obtained from (25) to experimental information. In general, it is not necessary to completely solve (25) every time a new choice for the $c_n$'s is made. If a quadrature rule is used to replace (25) with a matrix equation, only one matrix inversion need be carried out, since the $c_n$'s appear in the inhomogeneous term. In some situations it is possible to eliminate the $c_n$'s explicitly by using the techniques of Ref. 16.

Here we shall develop a transformation that maps the entire $z^{1/2}$ plane cut along $L$, $R$, and $B$ onto the interior of the unit circle in the $w$ plane. This leads to a series for $U(z^{1/2})$, which converges everywhere in the cut $z^{1/2}$ plane. We carry out the mapping in three stages. First, we map the cut $z^{1/2}$ plane onto the upper half of the $u$ plane, cut along part of the imaginary axis, with the transformation

$$\begin{aligned} u(z^{1/2}) &= z^{1/2}+(z-c)^{1/2}\,,\\ &= z^{1/2}+(r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2},\quad 0<\theta_1<2\pi,\\ &\qquad -\pi<\theta_2<\pi \end{aligned} \tag{28}$$

where $r_1$, $r_2$, $\theta_1$, and $\theta_2$ are shown in Fig. 1. As indicated in Fig. 2, the cuts along $L$ and $R$ have been opened up along the real $u$ axis, and the cut along $B$ has been mapped onto the imaginary $u$ axis between 0 and $u(-ib^{1/2})$. Now we map the cut, upper half of the $u$ plane onto the upper half of the $v$ plane by means of the transformation

$$\begin{aligned} v(z^{1/2}) &= [u^2(z^{1/2})-u^2(-ib^{1/2})]^{1/2}\,,\\ &= (\rho_1\rho_2)^{1/2}e^{i(\phi_1+\phi_2)/2},\\ &\qquad -\pi/2<\phi_1,\ \ \phi_2<3\pi/2 \end{aligned} \tag{29}$$

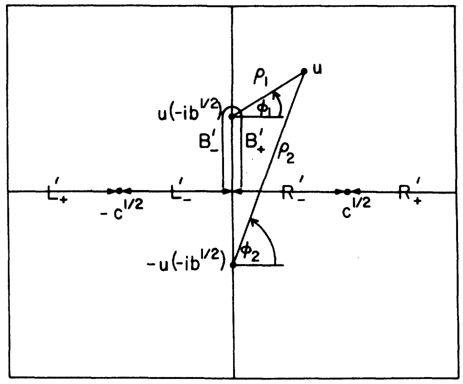


FIG. 2. $u$ plane. $L'_+$, $L'_-$, ... are the images of $L_+$, $L_-$, ....

where $\rho_1$, $\rho_2$, $\phi_1$, and $\phi_2$ are shown in Fig. 2. The cuts along $L$, $R$, and $B$ are now spread out along the real $v$ axis, as shown in Fig. 3. Finally, by means of the linear fractional transformation, we map the upper half of the $v$ plane onto the interior of the unit circle in the $w$ plane ($|w|<1$). We have

$$w(z^{1/2})=\frac{\zeta+iv(z^{1/2})}{\zeta-iv(z^{1/2})}\,,\quad \zeta>0\,. \tag{30}$$

We have arranged things so that the line $\mathrm{Re}(z^{1/2})=0$, $-b^{1/2}<\mathrm{Im}(z^{1/2})<\infty$, is mapped onto the line $\mathrm{Im}(w)=0$, $-1<\mathrm{Re}(w)<1$, with $z^{1/2}=-ib^{1/2}$ going into $w=1$ and $z^{1/2}=i\infty$ going into $w=-1$. The cuts are completely wrapped around $|w|=1$.

It is straightforward to show that for real positive $z$, we have

$$w(-z^{1/2})=w^*(z^{1/2}),\quad -c^{1/2}<z^{1/2}<c^{1/2}\,, \tag{31}$$

which according to (22) and (23) is also a property of $d(z^{1/2})$. From this it follows that the coefficients $c_n$ in (27) are real.

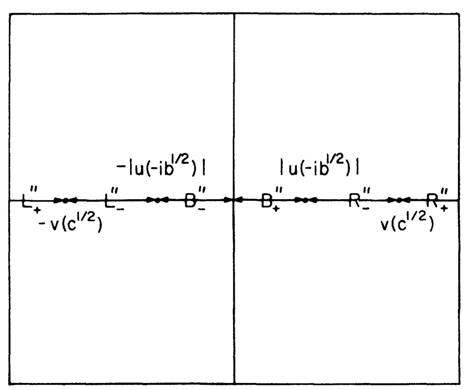


FIG. 3. $v$ plane. $L''_+$, $L''_-$, ... are the images of $L'_+$, $L'_-$ ....

## III. AN APPLICATION

Here we shall apply the $N/D$ approach to the three-nucleon system in an attempt to calculate the triton energy from a knowledge of the doublet scattering length and the discontinuity across the low energy end of the LHC which for this system begins at $z=-\frac{1}{3}$. For the discontinuity we shall use the *reference potential* model of Refs. 16 and 17. This model for the discontinuity includes contributions from the one- and two-nucleon exchange cuts and from the $\Delta'$ cut. In Ref. 16 the discontinuities are given by Eqs. (3.5), (3.6), and (3.13), and are displayed in Fig. 2. The reference potential model gives the discontinuity out to $b=12.134$. This potential is a spin-dependent, central, $s$-wave separable interaction which has been fitted to the low-energy two-nucleon data.[16] It reproduces the experimental triton energy of $E_t=8.48$ MeV when used in the Amado-Lovelace[26,27] equations, and leads to a value for the doublet scattering length of $a_2=0.986$ fm.

We consider three $N/D$ calculations in order of increasing sophistication. In each of the calculations there is one free parameter which is adjusted to the above value for $a_2$, and then a value of $E_t$ is obtained by locating the zero of the $D$ function. In our first two calculations we set the inelasticity parameter $\eta=1$ and the high-energy cutoff parameter $c=\infty$. This means we have completely ignored the effects of three-particle unitarity, and are essentially treating the three-body problem as a two-body problem. In our first calculation we follow the treatment of Ref. 14 and expand $U(z^{1/2})$ in (25) in powers of $z^{1/2}$ and retain only the first nonvanishing term, i.e., we write

$$U(z^{1/2})=-ic_1 z^{1/2}, \tag{32}$$

where $c_1$ is real.[14] Since we are completely ignoring three-particle unitarity, such a series converges for[14] $|z|<b$. Adjusting $c_1$ to $a_2$, we find $E_t=7.74$ MeV which is to be compared with the exact result of $E_t=8.48$ MeV. The range of $z$ values we are dealing with is $-12.134\leq z\leq 0$, so there is no reason to expect the approximation of keeping the leading term of the series to be a good one.

For our next calculation we use the conformal transformation of Ref. 16, which maps the $z^{1/2}$ plane, cut along the negative imaginary axis from $-i\infty$ to $-ib^{1/2}$, onto the interior of the unit circle in the $w$ plane, centered on the origin. The mapping is given by Eqs. (3.22) and (3.23) of Ref. 16 with $d=(2b)^{1/4}$, and maps $-b\leq z\leq 0$ onto $-0.0864\leq w\leq 0.0864$. We expand $U(z^{1/2})$ in powers of $w(z^{1/2})$ and keep the two leading terms, i.e., we write

$$U(z^{1/2})=c_1[w(z^{1/2})-w(0)], \tag{33}$$

where we have used the constraint (26). Adjusting $c_1$ to $a_2$, we find $E_t=7.95$ MeV, which is an improvement over the previous result.

For our last calculation we choose $c=1$ which is the breakup threshold, so we do not need any inelasticities in (3), and, of course, $R(z)=1$. From (12) and (14), it follows that the kernel in this calculation is different from the one in the previous two calculations where $c=\infty$. Here we write $U(z^{1/2})$ in the form (33), and use the mapping described in Sec. II for $w(z^{1/2})$. We choose the parameter $\xi$ in (30) so that the interval $-b\leq z\leq 0$ maps onto an interval of the real $w$ axis for which the end points are as close to the origin as possible. This leads to the condition

$$w(ib^{1/2})=-w(0), \tag{34}$$

which when solved gives

$$\xi=[4(1-d^2)^2 b(1+b)]^{1/8}/d \tag{35}$$

with

$$d=(1+b)^{1/2}-b^{1/2}. \tag{36}$$

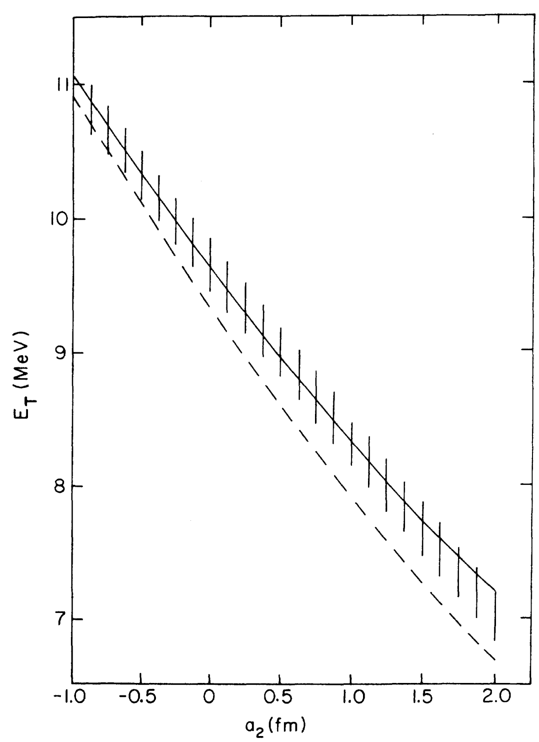


FIG. 4. Triton energy as function of doublet scattering length. Cross hatching from Ref. 25. Dashed line and solid line use mappings of Ref. 16 and this work, respectively.

Adjusting $c_1$ to $a_2$, we find $E_t = 8.39$ MeV, which agrees to about 1% with the exact value.

We have carried out[28] calculations using other choices for the parameters in the reference potential, thereby producing different "exact" values for $E_t$ and $a_2$. When we fit to these values of $a_2$ and compare the approximate $N/D$ results for the $E_t$'s with the exact results we find the same trend as reported above. Thus the method described in Sec. II is an improvement over the ones used previously,[14,16,17] at least insofar as the type of calculation reported here is concerned.

Figure 4 shows two sets of $N/D$ calculations in comparison with the Phillips plot data as summarized by Kim and Tubis.[25] The dashed and solid lines were obtained by following the methods used above for the second and third calculations, respectively. It is seen that the mapping of Sec. II, which takes into account the presence of the three-particle unitarity cut, gives consistently better results than the mapping of Ref. 16 which ignores this cut.

## IV. SUMMARY AND DISCUSSION

We have developed a set of single channel $N/D$ equations which can be used to calculate partial wave scattering amplitudes when the LHC and RHC discontinuities are known only over a limited energy range. The unknown information is characterized by the function $U(z^{1/2})$ appearing in Eq. (25). The location of the singularities of $U(z^{1/2})$ is known, which makes it possible to parametrize it in a mathematically sound way. We have presented one parametrization based on a conformal transformation. Our representation of $U(z^{1/2})$, which is by no means unique, is valid everywhere in the cut $z^{1/2}$ plane, and involves only real parameters. Of course one is usually not interested in calculating everywhere in the complex plane, so it is not necessary to map the entire cut $z^{1/2}$ plane into the unit circle. This possibility allows for a wide variety of representations for $U(z^{1/2})$.

As it stands our method is of use in carrying out uncoupled phase shift analyses. The constraints implied by knowledge of the low-energy LHC structure can be useful in eliminating the ambiguities that can arise in such analyses. Also, since the $N/D$ method gives the amplitude over a range of energies, it is of value in checking the consistency of phase shifts obtained at different energies.

The application presented in Sec. III indicates that our scheme is useful in checking the consistency of binding energies and phase shifts. Our earlier work[16] suggests that the type of parametrization considered here can lead to reliable extractions of asymptotic normalization parameters from experimental data.

In our scheme it is straightforward to analytically continue the partial wave amplitude into the lower half of the $z^{1/2}$ plane or, equivalently, onto the unphysical energy sheet. This makes it quite easy to determine the positions and strengths of virtual state and resonance poles. We have already done a successful virtual state determination along these lines with our earlier mapping.[17]

It should be possible to generalize our method so as to allow for multichannel, two-particle unitarity, and Coulomb effects. This would give the method an even larger range of applicability.

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