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Scattering amplitudes and integral equations for the collision of two charged composite particles

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Transition operators for the collision of two clusters composed of an arbitrary number of *charged* and neutral particles are represented as a sum of pure Coulomb and Coulomb-modified short-range operators. Sandwiching this relation between the corresponding channel states, correct two-fragment scattering amplitudes are obtained by adapting the conventional two-body screening and renormalization procedure. Furthermore, integral equations are derived for off-shell extensions of the full screened amplitudes and of the unscreened Coulomb-modified short-range amplitudes. For three particles, the final results coincide with those derived previously in a different approach. The proposed theory is valid for pure Coulomb scattering as well as for systems containing, in addition, two-body interactions of short range.

[NUCLEAR REACTIONS N-body scattering theory for charged particles. Scattering amplitudes for two-fragment collisions defined via screening and renormalization procedure. Derived effective two-body integral equations. Formalism also applicable to atomic problems.]

I. INTRODUCTION

In recent publications three-body scattering processes with two[1-3] or three[4] *charged* particles have been investigated by employing the quasiparticle method[5] which consists in replacing the original three-body integral equations by fully equivalent Lippmann-Schwinger (LS)-type matrix equations for effective two-body operators. By making use of the close analogy between this effective two-body formalism and the well-established genuine two-body Coulomb theory, scattering amplitudes for (in)elastic and rearrangement collisions of two fragments could be defined, and at the same time manageable integral equations for their calculation have been obtained.

In the present paper we develop an alternative approach which is based on the fundamental operator relations between the relevant quantities of the problem. Hereby, the previously derived expressions for the on-shell scattering amplitudes are reproduced in a direct and transparent manner. Consequently, their generalization to collisions of two fragments consisting of an arbitrary number of charged particles[6-14] becomes straightforward. We, moreover, show that these amplitudes may again be calculated from manageable integral equations.

The general idea of our procedure can be stated in the following way. As suggested by the results of Refs. 1–4, which are very much in accord with physical intuition, a *center-of-mass Coulomb potential,* acting between the total charges of the two fragments concentrated in their respective centers of mass, is introduced besides the original Coulomb interaction. We then derive a representation of the full transition amplitude as a pure center-of-mass two-body Coulomb amplitude, plus a Coulomb-modified short-range transition operator sandwiched between the bound state wave functions of the fragments and between the two-body Coulomb scattering states associated with their relative movement.

As in Refs. 1 to 4 the derivation of this representation is first performed for screened Coulomb potentials. Applying then the usual two-body renormalization and limiting procedure,[15-19] the occurring two-body Coulomb scattering amplitudes and states go over into the corresponding, explicitly known, unscreened quantities. However, no renormalization is needed when performing the zero-screening limit in the Coulomb-modified short-range transition operator sandwiched between bound states. For, in this expression there occurs, besides any short-ranged potentials, the difference between the original Coulomb interaction and the center-of-mass Coulomb potential. Expanding this difference into a multipole series with respect to the relative distance of the two fragments, the first nonvanishing term is the *dipole* term. Since it decreases sufficiently fast for large spatial separations, the transition to infinite screening radius, indeed, poses no problems in this amplitude. Of course, the justification of the multipole expansion rests on the localization property of the cluster wave functions.

The screening technique used to *define* the relevant transition amplitudes represents also an appropriate *practical* method. For, it allows us to apply *any* method of short-range collision theory to calculate first the screened scattering amplitudes, and then to perform after renormalization the zero-screening limit, e.g., numerically.[20] One of these short-range methods is the quasiparticle approach of Ref. 5 which has the advantage of providing us with exact integral equations for off-shell extensions of the two-fragment amplitudes considered. Hence, we describe its application to the present problem, reproducing in this way the integral equations given for three particles in Refs. 1–4, and deriving their generalization to arbitrary particle numbers. In addition, the quasiparticle method leads to a second approach which is based on integral equations directly for the *unscreened* Coulomb-modified short-range transition amplitudes.

The paper is organized as follows. The simple example of the scattering of two particles interacting via short-ranged plus Coulomb potentials is recalled in Sec. II. Two-fragment collisions of three particles are dealt with in Sec. III, utilizing the close structural analogy of the three-body formalism of Ref. 5 with the two-body theory. The generalization to the scattering of two fragments built up of an arbitrary number of charged particles is contained in Sec. IV. Quasiparticle equations for the full screened amplitudes and for the unscreened Coulomb-modified short-range amplitudes are described in Sec. V for three, and in Sec. VI for N particles.

II. TWO-BODY COLLISIONS

We recapitulate in this section some aspects of the scattering theory of two charged particles[19] in a way which suggests straightforward generalization to two-fragment arrangement collisions with an arbitrary number of (charged) particles.

A. Operator relations

Let the Hamiltonian $H^{(R)}$ be composed of the kinetic energy part H_0, a short-ranged potential V^S, and a screened Coulomb potential V^R,

$$H^{(R)} = H_0 + V^S + V^R \,, \tag{2.1}$$

with R denoting the screening radius. Then, the full resolvent, the free, and the "Coulomb-distorted free" resolvents, are given by

$$G^{(R)}(z) = (z - H_0 - V^S - V^R)^{-1} = (z - H^{(R)})^{-1} \,, \tag{2.2}$$

$$G_0(z) = (z - H_0)^{-1} \,, \tag{2.3}$$

$$G^R(z) = (z - H_0 - V^R)^{-1} \,. \tag{2.4}$$

Factoring out from $G^{(R)}$ the resolvents G_0 and G^R, respectively, and from G^R the resolvent G_0, by means of

$$G^{(R)} = G_0 + G_0 T^{(R)} G_0 \,, \tag{2.5}$$

$$G^{(R)} = G^R + G^R T^{SR} G^R \,, \tag{2.6}$$

$$G^R = G_0 + G_0 T^R G_0 \,, \tag{2.7}$$

the full transition operator $T^{(R)}$, the Coulomb-modified short-range operator T^{SR}, and the pure Coulomb transition operator T^R are introduced. These definitions emphasize the basic fact that appropriate singularities of the Green's functions have to be removed when going over to T operators. Moreover, the fundamental interrelationship

$$T^{(R)} = T^R + (1 + T^R G_0) T^{SR} (1 + G_0 T^R) \tag{2.8}$$

is obtained in a particularly simple manner by equating the right-hand sides of (2.5) and (2.6), and by replacing the occurring resolvent G^R by the representation (2.7). Let us recall, however, that the definitions of the various T operators via Eqs. (2.5)–(2.7) are equivalent to the more conventional ones

$$T^{(R)} = V^S + V^R + (V^S + V^R) G^{(R)} (V^S + V^R) \,, \tag{2.9}$$

$$T^{SR} = V^S + V^S G^{(R)} V^S \,, \tag{2.10}$$

$$T^R = V^R + V^R G^R V^R \,. \tag{2.11}$$

B. Zero-screening limit

We are now going to perform the zero-screening limits of the transition amplitudes obtained by sandwiching the operators (2.9)–(2.11) between plane waves $|\vec{p}\rangle$. This is unproblematic for T^{SR} since in its definition (2.6) the singularities of G^R corresponding to the *Coulomb-distorted free movement* of the particles are separated off from $G^{(R)}$. In fact, the explicit representation (2.10) shows that the full resolvent $G^{(R)}$ appears in $\langle \vec{p}' | T^{SR}(E + i0) | \vec{p} \rangle$ between states $V^S | \vec{p} \rangle$ which are normalizable and smooth under appropriate conditions on V^S. Thus, the existence of the limit $R \to \infty$, which is denoted by $\langle \vec{p}' | T^{SC}(E + i0) | \vec{p} \rangle$, is guaranteed for all values of the initial and final momenta.

In contrast, the amplitudes $T^{(R)}$ and T^R are obtained by extracting from the respective Green's functions only those singularities which correspond to the *undistorted free relative motion*. Since such an unperturbed movement does not exist in the presence of unscreened Coulomb potentials, the performance of the limit $R \to \infty$ requires the well-known renormalization procedure.[15-19] In fact, the momentum representation of the relation (2.8) takes on the energy shell [$p' = p = (2\mu E)^{1/2}$, with μ being the reduced mass] the form

$$\langle \vec{p}' | T^{(R)}(E+i0) | \vec{p} \rangle = \langle \vec{p}' | T^{R}(E+i0) | \vec{p} \rangle + \langle \vec{p}_R'^{(-)} | T^{SR}(E+i0) | \vec{p}_R^{(+)} \rangle , \tag{2.12}$$

with the screened Coulomb scattering states defined according to

$$| \vec{p}_R^{(\pm)} \rangle = [1 + G_0(E \pm i0) T^R(E \pm i0)] | \vec{p} \rangle . \tag{2.13}$$

When these wave functions are multiplied by the appropriate renormalization factors $Z_R^{\mp 1/2}(p)$, and the amplitude $\langle \vec{p}' | T^R(E+i0) | \vec{p} \rangle$ by $Z_R^{-1}(p)$, the unscreened Coulomb scattering states $| \vec{p}_C^{(\pm)} \rangle$ and the Coulomb amplitude $T^C(\vec{p}', \vec{p})$, respectively, are recovered in the zero screening limit. (For instance, in the case of exponential screening, $V^R(r) = e_1 e_2 \exp(-r/R)/r$, we have $Z_R(p) = \exp[-2ie_1e_2(\mu/p)(\ln 2pR - C)]$. Here, e_i is the charge of particle i, and $C = 0.5772\ldots$ the Euler number.) Hence, the limit $R \to \infty$ exists in both terms on the right-hand side of Eq. (2.12) *after renormalization* by $Z_R^{-1}(p)$, thus providing the full transition amplitude for a short-ranged plus an unscreened Coulomb potential according to

$$T(\vec{p}', \vec{p}) = T^C(\vec{p}', \vec{p}) + \langle \vec{p}_C'^{(-)} | T^{SC}(E+i0) | \vec{p}_C^{(+)} \rangle . \tag{2.14}$$

C. Integral equations

These considerations immediately suggest two different approaches of how to proceed in practice. One of them consists in solving the partial-wave LS equations for $T^{(R)}$ and T^R for finite values of R, and subtracting the result of the latter from that of the former. Summing up these partial-wave contributions yields the Coulomb-modified short-range amplitude in the (screened) Coulomb representation, $\langle \vec{p}_R'^{(-)} | T^{SR}(E+i0) | \vec{p}_R^{(+)} \rangle$. Renormalizing then by $Z_R^{-1}(p)$ the corresponding on-shell elements, and repeating the calculation for successively increased values of R, the transition to large ("infinite") screening radii is performed numerically.[20]

Alternatively, we can calculate directly the unscreened Coulomb-modified short-range amplitude occurring in (2.14) by making use of the LS equation

$$T^{SC} = V^S + V^S G^C T^{SC} \tag{2.15}$$

and of the analytically known Coulomb wave functions. Equation (2.15) follows from (2.6) or (2.10) in the zero-screening limit which exists according to the above discussion.

In both methods we then have to add the explicitly known Coulomb amplitude $T^C(\vec{p}', \vec{p})$ in order to get the full amplitude via (2.14).

III. SCATTERING AMPLITUDES FOR TWO-FRAGMENT COLLISIONS OF THREE CHARGED PARTICLES

For short-ranged potentials the three-body theory was formulated in Ref. 5 in structural analogy to the two-body case. The same concept is now being used to extend the considerations of Sec. II to three particles interacting via potentials with Coulomb tails. We remark that the *pure* Coulomb scattering is contained as a special case in our treatment.

A. Operator relations

The interaction in the total Hamiltonian

$$H^{(R)} = H_0 + \sum_\gamma V_\gamma^S + \sum_\gamma V_\gamma^R = H_0 + V^S + V^R \tag{3.1}$$

is assumed to be a superposition of short-ranged pair potentials V_γ^S and screened Coulomb potentials V_γ^R acting between particles $\alpha, \beta \neq \gamma$ (here and in the following, the familiar cyclic notation is used).

Let us introduce, in addition, the screened Coulomb interaction v_α^R between particle α and the *center of mass* of particles β and γ. In coordinate space this potential reads

$$v_\alpha^R(\vec{\rho}_\alpha) = e_\alpha(e_\beta + e_\gamma) f_\alpha^R(\rho_\alpha)/\rho_\alpha , \tag{3.2}$$

with $\vec{\rho}_\alpha$ denoting the corresponding relative coordinate, and $e_\alpha, e_\beta, e_\gamma$ the charges of the three particles.[21-23] The screening function $f_\alpha^R(\rho_\alpha)$ has to be suitably chosen, e.g., exponentially decreasing. Now, the full resolvent, the channel, and the Coulomb-distorted channel resolvents, are introduced as

$$G^{(R)}(z) = (z - H_0 - \sum_\gamma V_\gamma^S - \sum_\gamma V_\gamma^R)^{-1} = (z - H^{(R)})^{-1} , \tag{3.3}$$

$$G_\alpha^{(R)}(z) = (z - H_0 - V_\alpha^S - V_\alpha^R)^{-1} , \tag{3.4}$$

$$g_\alpha^R(z) = (z - H_0 - V_\alpha^S - V_\alpha^R - v_\alpha^R)^{-1} . \tag{3.5}$$

We emphasize that in two-fragment reactions of the three particles the channel resolvents (3.4) and (3.5) play the same role as the resolvents (2.3) and (2.4) in the two-elementary particle case. This fact suggests defining the full transition operators $U_{\beta\alpha}^{(R)}$, the Coulomb-modified short-range operators $U_{\beta\alpha}^{SR}$, and the pure center-of-mass Coulomb operators t_α^R, in analogy to Eqs. (2.5)–(2.7), via

$$G^{(R)} = \delta_{\beta\alpha} G_\alpha^{(R)} + G_\beta^{(R)} U_{\beta\alpha}^{(R)} G_\alpha^{(R)} , \tag{3.6}$$

$$G^{(R)} = \delta_{\beta\alpha} g_\alpha^R + g_\beta^R U_{\beta\alpha}^{SR} g_\alpha^R , \tag{3.7}$$

$$g_\alpha^R = G_\alpha^{(R)} + G_\alpha^{(R)} t_\alpha^R G_\alpha^{(R)} . \tag{3.8}$$

Equating the right-hand sides of (3.6) and (3.7) and inserting the representation (3.8) of g^R_α we end up with the fundamental relation

$$U^{(R)}_{\beta\alpha} = \delta_{\beta\alpha} t^R_\alpha + (1 + t^R_\beta G^{(R)}_\beta) U^{SR}_{\beta\alpha} (1 + G^{(R)}_\alpha t^R_\alpha) \tag{3.9}$$

which generalizes Eq. (2.8).

We mention that in writing the definitions (3.6) and (3.7) of $U^{(R)}_{\beta\alpha}$ and $U^{SR}_{\beta\alpha}$, respectively, we have followed the idea of Ref. 5, namely to introduce transition operators by extracting subsystem singularities from the full resolvent in a most symmetric way. Similarly to Eqs. (2.9)–(2.11), the various transition operators can also be represented in the explicit forms

$$U^{(R)}_{\beta\alpha} = \bar\delta_{\beta\alpha} G^{(R)-1}_\alpha + \bar V^{(R)}_\beta + \bar V^{(R)}_\beta G^{(R)} \bar V^{(R)}_\alpha , \tag{3.10}$$

$$U^{SR}_{\beta\alpha} = \bar\delta_{\beta\alpha} (g^R_\alpha)^{-1} + (\bar V^{(R)}_\beta - v^R_\beta) + (\bar V^{(R)}_\beta - v^R_\beta) G^{(R)} (\bar V^{(R)}_\alpha - v^R_\alpha) , \tag{3.11}$$

$$t^R_\alpha = v^R_\alpha + v^R_\alpha g^R_\alpha v^R_\alpha . \tag{3.12}$$

In (3.10) and (3.11) the channel potentials $\bar V^{(R)}_\alpha$ acting between the colliding fragments are

$$\bar V^{(R)}_\alpha = V^S + V^R - V^S_\alpha - V^R_\alpha = \sum_{\gamma\neq\alpha} V^S_\gamma + \sum_{\gamma\neq\alpha} V^R_\gamma . \tag{3.13}$$

And, as usual, we have $\bar\delta_{\beta\alpha} = 1 - \delta_{\beta\alpha}$. Note, that the lack of symmetry of (3.10) under interchange of α and β is only apparent. For, we have $\bar\delta_{\beta\alpha} G^{(R)-1}_\alpha + \bar V^{(R)}_\beta = \bar\delta_{\beta\alpha} G^{(R)-1}_\beta + \bar V^{(R)}_\alpha$. A similar remark applies to (3.11). Of course, as in any short-range theory we could work also with unsymmetric post and prior transition operators[24] $U^{(R)\pm}_{\beta\alpha}$, which are related to $U^{(R)}_{\beta\alpha}$ via $U^{(R)}_{\beta\alpha} = \bar\delta_{\beta\alpha} G^{(R)-1}_\alpha + U^{(R)+}_{\beta\alpha} = \bar\delta_{\beta\alpha} G^{(R)-1}_\beta + U^{(R)-}_{\beta\alpha}$. Analogously, post and prior operators $U^{SR\pm}_{\beta\alpha}$ can be defined according to $U^{SR}_{\beta\alpha} = \bar\delta_{\beta\alpha}(g^R_\alpha)^{-1} + U^{SR+}_{\beta\alpha} = \bar\delta_{\beta\alpha}(g^R_\beta)^{-1} + U^{SR-}_{\beta\alpha}$. All these operators yield the same on-shell amplitudes when introduced in the respective expressions of relation (3.22) given below.

B. Zero-screening limit

The limit $R \to \infty$ is now investigated for the on-shell matrix elements of (3.9) between channel states[25]

$$|\Phi_{\alpha m}\rangle = |\psi_{\alpha m}\rangle |\vec q_\alpha\rangle \tag{3.14}$$

belonging to the energy $E_{\alpha m} = q_\alpha^2/2M_\alpha + \hat E_{\alpha m}$. Here, $|\psi_{\alpha m}\rangle$ is the mth bound state wave function of the pair (β,γ) with binding energy $\hat E_{\alpha m}$, and the plane wave $|\vec q_\alpha\rangle$ describes the free motion of particle α relative to this bound state (M_α is the corresponding reduced mass). To simplify the notation the label R is suppressed on the states (3.14) and on the energies to which they belong.

Let us first consider the action of

$$[1 + G^{(R)}_\alpha(E_{\alpha m} \pm i0) t^R_\alpha(E_{\alpha m} \pm i0)] = [1 + g^R_\alpha(E_{\alpha m} \pm i0) v^R_\alpha] \tag{3.15}$$

on $|\Phi_{\alpha m}\rangle$. For this purpose we associate with the operators v^R_α, $G^{(R)}_\alpha(z)$, and $g^R_\alpha(z)$, which act in the three-body space, potentials $\mathfrak{v}^R_\alpha$, "free Green's functions" $\mathfrak{g}_{0,\alpha}(z)$, and "Coulomb Green's functions" $\mathfrak{g}^R_\alpha(z)$, respectively, which are restricted to the relative momentum states $|\vec q_\alpha\rangle$. I.e., we define[26]

$$v^R_\alpha |\psi_{\alpha m}\rangle |\vec q_\alpha\rangle = |\psi_{\alpha m}\rangle \mathfrak{v}^R_\alpha |\vec q_\alpha\rangle , \tag{3.16}$$

$$\mathfrak{g}_{0,\alpha}(z) = \left(z - \frac{Q_\alpha{}^2}{2M_\alpha}\right)^{-1} , \tag{3.17}$$

$$\mathfrak{g}^R_\alpha(z) = \left(z - \frac{Q_\alpha{}^2}{2M_\alpha} - \mathfrak{v}^R_\alpha\right)^{-1} , \tag{3.18}$$

where $\vec Q_\alpha$ are the relative momentum operators with eigenstates $|\vec q_\alpha\rangle$. Moreover, two-body Coulomb transition operators are defined by

$$\mathfrak{t}^R_\alpha(z) = \mathfrak{v}^R_\alpha + \mathfrak{v}^R_\alpha \mathfrak{g}^R_\alpha(z) \mathfrak{v}^R_\alpha . \tag{3.19}$$

All these quantities satisfy, of course, the usual resolvent and transition operator LS equations as known from the two-body theory.

With these definitions we find

$$\begin{aligned}
[1 + g^R_\alpha(E_{\alpha m} \pm i0) v^R_\alpha] |\psi_{\alpha m}\rangle |\vec q_\alpha\rangle
&= |\psi_{\alpha m}\rangle \left[1 + \mathfrak{g}^R_\alpha\left(\frac{q_\alpha{}^2}{2M_\alpha} \pm i0\right) \mathfrak{v}^R_\alpha\right] |\vec q_\alpha\rangle \\
&= |\psi_{\alpha m}\rangle \left[1 + \mathfrak{g}_{0,\alpha}\left(\frac{q_\alpha{}^2}{2M_\alpha} \pm i0\right) \mathfrak{t}^R_\alpha\left(\frac{q_\alpha{}^2}{2M_\alpha} \pm i0\right)\right] |\vec q_\alpha\rangle \\
&= |\psi_{\alpha m}\rangle \omega^{(\pm)}_{\alpha,R} |\vec q_\alpha\rangle = |\psi_{\alpha m}\rangle |\vec q^{\,(\pm)}_{\alpha,R}\rangle .
\end{aligned} \tag{3.20}$$

That is, $[1 + G^{(R)}_\alpha(E_{\alpha m} \pm i0) t^R_\alpha(E_{\alpha m} \pm i0)]$ acts on $|\Phi_{\alpha m}\rangle$ as a Møller operator $\omega^{(\pm)}_{\alpha,R}$ with respect to $|\vec q_\alpha\rangle$, mapping the latter onto screened Coulomb scattering states $|\vec q^{\,(\pm)}_{\alpha,R}\rangle$ which characterize the Coulomb-distorted free movement of the two fragments. We furthermore see that

$$\begin{aligned}
\langle \vec q_\alpha{}' | \langle \psi_{\alpha n} | t^R_\alpha(E_{\alpha m} + i0) | \psi_{\alpha m}\rangle |\vec q_\alpha\rangle
&= \delta_{nm} \langle \vec q_\alpha{}' | \mathfrak{t}^R_\alpha(q_\alpha{}^2/2M_\alpha + i0) | \vec q_\alpha\rangle \\
&= \delta_{nm} \langle \vec q_\alpha{}' | \mathfrak{v}^R_\alpha | \vec q^{\,(\pm)}_{\alpha,R}\rangle \\
&= \delta_{nm} t^R_\alpha(\vec q_\alpha{}', \vec q_\alpha)
\end{aligned} \tag{3.21}$$

represents the screened two-body Coulomb scattering amplitude for particle α off the center of mass of particles β and γ. Sandwiched between channel states (3.14), our basic relation (3.9) consequently reads on the energy shell

$$\begin{aligned}
\langle \vec q_\beta{}' | \langle \psi_{\beta n} | U^{(R)}_{\beta\alpha}(E_{\alpha m} + i0) | \psi_{\alpha m}\rangle |\vec q_\alpha\rangle
&= \delta_{\beta\alpha} \delta_{nm} t^R_\alpha(\vec q_\alpha{}', \vec q_\alpha) \\
&\quad + \langle \vec q_{\beta,R}{}'^{(-)} | \langle \psi_{\beta n} | U^{SR}_{\beta\alpha}(E_{\alpha m} + i0) | \psi_{\alpha m}\rangle |\vec q^{\,(+)}_{\alpha,R}\rangle .
\end{aligned} \tag{3.22}$$

Proceeding as in Sec. II, we first demonstrate that the limit $R\to\infty$ can be performed without any problems in the *effective two-body operators* $\langle\psi_{\beta n}|U^{SR}_{\beta\alpha}|\psi_{\alpha m}\rangle$, the resulting unscreened quantities being denoted by $\langle\psi_{\beta n}|U^{SC}_{\beta\alpha}|\psi_{\alpha m}\rangle$. This is easily demonstrated with the help of the explicit representation (3.11) which yields

$$\langle\vec{q}_\beta'|\langle\psi_{\beta n}|U^{SR}_{\beta\alpha}|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle \doteq \langle\vec{q}_\beta'|\langle\psi_{\beta n}|[\bar{\delta}_{\beta\alpha}(g^R_\alpha)^{-1}+(\bar{V}^{(R)}_\beta - v^R_\beta)]|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle + \langle\vec{q}_\beta'|\langle\psi_{\beta n}|(\bar{V}^{(R)}_\beta - v^R_\beta)G^{(R)}(\bar{V}^{(R)}_\alpha - v^R_\alpha)|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle\,. \tag{3.23}$$

As in the momentum representation of the corresponding two-body equation (2.10), the full Green's function $G^{(R)}$ occurs here between states $(\bar{V}^{(R)}_\alpha - v^R_\alpha)|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle$ which are *normalizable*, a property which is shared even by their zero-screening limits $(\bar{V}_\alpha - v^C_\alpha)|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle$. This is obvious for the short-ranged part $\bar{V}^S_\alpha$ of

$$\bar{V}_\alpha - v^C_\alpha = \sum_{\gamma\neq\alpha} V^S_\gamma + \sum_{\gamma\neq\alpha} V^C_\gamma - v^C_\alpha = \bar{V}^S_\alpha + \bar{V}^C_\alpha - v^C_\alpha\,, \tag{3.24}$$

but holds for the Coulomb terms, too. For, in position space the contribution of the latter is

$$\left\langle\vec{\rho}_\alpha,\vec{r}_\alpha\right|\left(\sum_{\delta\neq\alpha} V^C_\delta - v^C_\alpha\right)\left|\psi_{\alpha m}\rangle|\vec{q}_\alpha\right\rangle = [e_\alpha e_\beta(r_\gamma^{-1}-\rho_\alpha^{-1})+e_\alpha e_\gamma(r_\beta^{-1}-\rho_\alpha^{-1})] \times\psi_{\alpha m}(\vec{r}_\alpha)\frac{e^{i\vec{q}_\alpha\cdot\vec{\rho}_\alpha}}{(2\pi)^{3/2}}\,, \tag{3.25}$$

with the relative coordinate $\vec{r}_\gamma$ between particles α and β being expressed as a linear combination of $\vec{\rho}_\alpha$ and $\vec{r}_\alpha$,

$$\vec{r}_\gamma = \vec{\rho}_\alpha - \frac{m_\gamma}{m_\beta + m_\gamma}\vec{r}_\alpha \quad (\alpha,\beta,\gamma \text{ cyclic}) \tag{3.26}$$

and similarly for $\vec{r}_\beta$ (see Fig. 1). For sufficiently fast (e.g., exponentially) decreasing bound state wave functions $\psi_{\alpha m}(\vec{r}_\alpha)$, a multipole expansion of r_γ^{-1} and r_β^{-1} in powers of ρ_α^{-1} is justified. Thus, in fact, the first nonvanishing terms of $(r_\gamma^{-1} - \rho_\alpha^{-1})$ and $(r_\beta^{-1}-\rho_\alpha^{-1})$ are of the order ρ_α^{-2}. Therewith the proof of our statement is completed.

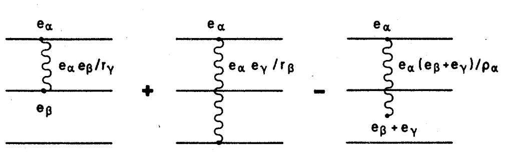

FIG. 1. Graphical representation of the difference between the Coulomb channel interaction $\bar{V}^C_\alpha$ and the Coulomb potential v^C_α acting between particle α and the center of mass of particles β and γ, in position space [cf. Eq. (3.25)].

Consequently, when the screening radius goes to infinity, divergencies can originate in (3.22) only from the *two-body* amplitudes $t^R_\alpha(\vec{q}_\alpha',\vec{q}_\alpha)$ and scattering states $|\vec{q}^{(\pm)}_{\alpha,R}\rangle$. These divergencies, however, can be eliminated by the renormalization procedure of Sec. II. In fact, with the renormalization factor $Z_{\alpha,R}(q_\alpha)$ defined as in the two-body case,[27] we obtain the on-shell two-body Coulomb amplitude for scattering of particle α off the center of mass of particles β and γ via

$$t^R_\alpha(\vec{q}_\alpha',\vec{q}_\alpha)Z_{\alpha,R}^{-1}(q_\alpha)\underset{R\to\infty}{\sim} t^C_\alpha(\vec{q}_\alpha',\vec{q}_\alpha)\,, \tag{3.27}$$

and the corresponding Coulomb scattering states via

$$|\vec{q}^{(\pm)}_{\alpha,R}\rangle Z^{\mp 1/2}_{\alpha,R}(q_\alpha)\underset{R\to\infty}{\sim}|\vec{q}^{(\pm)}_{\alpha,C}\rangle\,. \tag{3.28}$$

Thus, multiplying the whole equation (3.22) by $Z^{-1/2}_{\beta,R}(q_\beta')Z^{-1/2}_{\alpha,R}(q_\alpha)$, we arrive in the limit $R\to\infty$ at the representation

$$\lim_{R\to\infty} Z^{-1/2}_{\beta,R}(q_\beta')\langle\vec{q}_\beta'|\langle\psi_{\beta n}|U^{(R)}_{\beta\alpha}(E_{\alpha m}+i0)|\psi_{\alpha m}\rangle|\vec{q}_\alpha\rangle \times Z^{-1/2}_{\alpha,R}(q_\alpha) = \delta_{\beta\alpha}\delta_{nm}t^C_\alpha(\vec{q}_\alpha',\vec{q}_\alpha) + \langle\vec{q}'^{(-)}_{\beta,C}|\langle\psi_{\beta n}|U^{SC}_{\beta\alpha}(E_{\alpha m}+i0)|\psi_{\alpha m}\rangle|\vec{q}^{(+)}_{\alpha,C}\rangle = T_{\beta n,\alpha m}(\vec{q}_\beta',\vec{q}_\alpha) \tag{3.29}$$

for the transition amplitudes describing two-fragment processes of three charged particles.

Of course, our formalism remains valid if one of the bound particles in the incoming and outgoing states is neutral. This case has been discussed in detail in Ref. 3. Only the corresponding charge has to be set equal to zero in the above formulas. Furthermore, Eq. (3.29) simplifies considerably if the bound state or the elementary particle in the initial or in the final state is neutral. Then, the pure Coulomb amplitude is absent and the corresponding Coulomb scattering state has to be replaced by a plane wave. If, in particular, both in the initial and in the final state, one of the fragments is neutral, the limit $R\to\infty$ exists in the arrangement amplitudes without renormalization, i.e., no screening is necessary at all and conventional short-range scattering theory is applicable.[28]

Let us emphasize that the basic result (3.29) is valid also for pure Coulomb scattering, i.e., if $V^S_1 = V^S_2 = V^S_3 = 0$. For, our treatment yields also in this case a splitting of the interaction between the two fragments into a part which is of short range [cf. Eq. (3.25)] and the long-ranged Coulomb potential v^C_α. In other words, even *without* short-ranged pair potentials V^S_γ our formulation of two-fragment collisions of three particles is structurally equivalent to the genuine two-body problem with a Coulomb *plus* a short-ranged interaction, as discussed in Sec. II.

IV. GENERALIZATION TO N-PARTICLE SYSTEMS

For two-fragment collisions the formalism developed in Sec. III can be extended to arbitrary particle numbers in a straightforward manner.[6]

A. Operator relations

We denote by $a, b, \ldots$ two-fragment partitions of N distinguishable particles $1, 2, \ldots, N$. Then, the total Hamiltonian

$$H^{(R)} = H_0 + \sum_{i<j}^{N} V_{ij}^S + \sum_{i<j}^{N} V_{ij}^R = H_0 + V^S + V^R \tag{4.1}$$

with the short-ranged potentials $V_{ij}^S = V_{ji}^S$ and the screened Coulomb potentials V_{ij}^R is split, according to

$$H^{(R)} = H_a^{(R)} + \overline{V}_a^{(R)}, \tag{4.2}$$

into a channel Hamiltonian

$$H_a^{(R)} = H_0 + \sum_{\substack{i<j\\(ij)\subset a}} (V_{ij}^S + V_{ij}^R) = H_0 + V_a^S + V_a^R \tag{4.3}$$

and the channel interaction

$$\overline{V}_a^{(R)} = \sum_{\substack{i<j\\(ij)\not\subset a}} (V_{ij}^S + V_{ij}^R) = \overline{V}_a^S + \overline{V}_a^R. \tag{4.4}$$

Here, as usual, $(ij)\subset a$ means that both particles i and j are contained in the same fragment of the partition a.

Denoting by $\vec{\rho}_a$ the relative coordinate between the centers of mass of the two clusters of fragmentation a, we define as in (3.2) the Coulomb interaction v_a^R between them. In position space it reads explicitly

$$v_a^R(\vec{\rho}_a) = \sum_{\substack{i<j\\(ij)\not\subset a}} \frac{e_i e_j}{\rho_a} f_a^R(\rho_a). \tag{4.5}$$

That is, it is obtained by replacing in all potentials V_{ij}^R which occur in the Coulomb part $\overline{V}_a^R$ of the channel interaction, the relative coordinates between particles i and j by the center-of-mass variable $\vec{\rho}_a$. This allows us to introduce, besides the resolvents $G^{(R)}$ of $H^{(R)}$, and $G_a^{(R)}$ of $H_a^{(R)}$, the generalization of the Coulomb-distorted channel resolvent (3.5),

$$g_a^R(z) = (z - H_a^{(R)} - v_a^R)^{-1}. \tag{4.6}$$

Then, analogously to (3.6)–(3.8), transition operators are defined as follows:

$$G^{(R)} = \delta_{ba} G_a^{(R)} + G_b^{(R)} U_{ba}^{(R)} G_a^{(R)}, \tag{4.7}$$

$$G^{(R)} = \delta_{ba} g_a^R + g_b^R U_{ba}^{SR} g_a^R, \tag{4.8}$$

$$g_a^R = G_a^{(R)} + G_a^{(R)} t_a^R G_a^{(R)}. \tag{4.9}$$

Their interrelationship is expressed by an equation similar to (3.9),

$$U_{ba}^{(R)} = \delta_{ba} t_a^R + (1 + t_b^R G_b^{(R)}) U_{ba}^{SR} (1 + G_a^{(R)} t_a^R). \tag{4.10}$$

For the various transition operators, explicit representations can again be derived which are of the form (3.10)–(3.12), with α, β replaced by a, b.

B. Zero-screening limit

Sandwiched between channel states belonging to the energy E_a, Eq. (4.10) becomes, by means of a generalization of the arguments leading to Eq. (3.22),

$$\begin{aligned}\langle \vec{q}_b' | \langle \psi_b | U_{ba}^{(R)}(E_a + i0) | \psi_a \rangle | \vec{q}_a \rangle &= \delta_{ba} t_a^R(\vec{q}_a', \vec{q}_a)\\ &+ \langle \vec{q}_{b,R}'^{(-)} | \langle \psi_b | U_{ba}^{SR}(E_a + i0) | \psi_a \rangle | \vec{q}_{a,R}^{(+)} \rangle.\end{aligned} \tag{4.11}$$

Here $|\psi_a\rangle = |\psi_a^{(1)}\rangle |\psi_a^{(2)}\rangle$ is a product of the bound state wave functions of the two asymptotically free fragments of partition a, and $\vec{q}_a$ their relative momentum. In order to simplify the notation the quantum numbers characterizing the bound states are suppressed. Furthermore, $t_a^R(\vec{q}_a', \vec{q}_a)$ denotes the *two-body* on-shell amplitudes for screened Coulomb scattering of the total charges of the two clusters situated in their respective centers of mass, and $|\vec{q}_{a,R}^{(\pm)}\rangle$ are the corresponding scattering states. Note that both quantities go over for infinite screening radius into their unscreened counterparts $t_a^C(\vec{q}_a', \vec{q}_a)$ and $|\vec{q}_{a,C}^{(\pm)}\rangle$, respectively, after having been subjected to the usual *two-body* renormalization procedure [compare Eqs. (3.27) and (3.28)].

Thus, it remains to be shown that the zero-screening limit exists for the matrix elements of the Coulomb-modified short-range transition operator between channel states,

$$\langle \vec{q}_b' | \langle \psi_b^{(2)} | \langle \psi_b^{(1)} | U_{ba}^{SR} | \psi_a^{(1)} \rangle | \psi_a^{(2)} \rangle | \vec{q}_a \rangle. \tag{4.12}$$

Making use of their explicit representation, Eq. (3.11) with a, b substituting α, β, we recognize that for $R\to\infty$ the full Green's function G occurs sandwiched between normalizable states which in position space read as

$$\begin{aligned}\langle \vec{\rho}_a | \langle \vec{x}_a^{(2)} | \langle \vec{x}_a^{(1)} | (\overline{V}_a - v_a^C) | \psi_a^{(1)} \rangle | \psi_a^{(2)} \rangle | \vec{q}_a \rangle &= \sum_{\substack{i<j\\(ij)\not\subset a}} [V_{ij}^S(\vec{r}_{ij}) + e_i e_j (r_{ij}^{-1} - \rho_a^{-1})]\\ &\times \psi_a^{(1)}(\vec{x}_a^{(1)}) \psi_a^{(2)}(\vec{x}_a^{(2)}) \frac{e^{i\vec{q}_a\cdot\vec{\rho}_a}}{(2\pi)^{3/2}}.\end{aligned} \tag{4.13}$$

Here, $\vec{x}_a^{(k)}$ collectively denotes the internal variables of cluster k of the partition a. Note that the relative coordinates $\vec{r}_{ij}$ between particles i and j

are given as a sum of $\vec{\rho}_a$ and a linear combination $l(\vec{x}_a^{(2)}, \vec{x}_a^{(1)})$ of the internal variables,

$$\vec{r}_{ij} = \pm\vec{\rho}_a + l(\vec{x}_a^{(2)}, \vec{x}_a^{(1)}) . \quad (4.14)$$

In view of the strong decrease of the bound state wave functions $\psi_a^{(k)}(\vec{x}_a^{(k)})$, again a multipole expansion of r_{ij}^{-1} in powers of ρ_a^{-1} is justified, so that the first nonvanishing term of $(r_{ij}^{-1} - \rho_a^{-1})$ is of the order ρ_a^{-2}. This, however, implies that U_{ba}^{SC} *sandwiched between channel states exists without screening*. Consequently, after renormalization the zero-screening limit of both terms on the right-hand side of Eq. (4.11) exists, providing us with the two-fragment transition amplitude

$$T_{ba}(\vec{q}_b', \vec{q}_a) = \delta_{ba} t_a^C(\vec{q}_a', \vec{q}_a) + \langle \vec{q}_{b,C}'^{(-)} | \langle \psi_b | U_{ba}^{SC}(E_a + i0) | \psi_a \rangle | \vec{q}_{a,C}^{(+)} \rangle , \quad (4.15)$$

represented as a superposition of a pure Coulomb two-body amplitude and a Coulomb-modified short-range amplitude in the Coulomb representation.

We finally mention that the discussion at the end of Sec. III concerning neutral fragments and the pure Coulomb problem carries over directly to the present general case.

V. INTEGRAL EQUATIONS FOR TWO-FRAGMENT SCATTERING AMPLITUDES: THREE-BODY CASE

The screening procedure used in Sec. III to *define* scattering amplitudes for two charged particles, one of which is composite, can also serve as a means to perform *practical calculations*. We may, indeed, evaluate for finite R the on-shell amplitudes (3.22)

$$T_{\beta n,\alpha m}^{(R)}(\vec{q}_\beta', \vec{q}_\alpha) = \langle \vec{q}_\beta' | \langle \psi_{\beta n} | U_{\beta\alpha}^{(R)}(E_{\alpha m} + i0) | \psi_{\alpha m} \rangle | \vec{q}_\alpha \rangle , \quad (5.1)$$

with the help of *any* method known in short-range multichannel scattering theory. Subtracting from (5.1) the screened Coulomb amplitude (3.21) obtained by means of a two-body LS equation, and renormalizing this difference, the transition to infinite screening radius can be performed numerically (cf. the corresponding discussion at the end of Sec. II). In this way we arrive at the correct Coulomb-modified short-range transition amplitudes occurring in (3.29)

$$T_{\beta n,\alpha m}^{SC}(\vec{q}_\beta', \vec{q}_\alpha) = \langle \vec{q}_{\beta,C}'^{(-)} | \langle \psi_{\beta n} | U_{\beta\alpha}^{SC}(E_{\alpha m} + i0) | \psi_{\alpha m} \rangle | \vec{q}_{\alpha,C}^{(+)} \rangle . \quad (5.2)$$

Adding the explicitly known two-body center-of-mass Coulomb amplitude, the full transition amplitudes $T_{\beta n,\alpha m}(\vec{q}_\beta', \vec{q}_\alpha)$ are obtained according to (3.29).

One special method to calculate the two-fragment amplitudes (5.1) is provided by the *quasiparticle* approach[5,29] which is particularly well suited for the present problem. For, it yields *exact* integral equations directly for transition operators sandwiched between two-body bound states, i.e., for effective two-body amplitudes the on-shell elements of which coincide with the expressions (5.1). And just for such amplitudes our zero-screening procedure has been established.[25] We, therefore, describe in the following the application of the quasiparticle concept to $N = 3$.

A. Quasiparticle equations for the full amplitudes

The transition operators $U_{\beta\alpha}^{(R)}$ defined by (3.6) or (3.10) fulfill the Faddeev-type integral equations[5]

$$U_{\beta\alpha}^{(R)} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} T_\gamma^{(R)} G_0 U_{\gamma\alpha}^{(R)} . \quad (5.3)$$

Here, $T_\gamma^{(R)}$ is the two-body T matrix (2.9) for subsystem γ read in the three-body space. Following the treatment of Ref. 5 (see also Ref. 29) we decompose $T_\gamma^{(R)}$ into a sum of N_γ separable terms and the rest $T_\gamma'^{(R)}$:

$$T_\gamma^{(R)}(z) = \sum_{r,s=1}^{N_\gamma} | \varphi_{\gamma r}(z) \rangle\!\rangle \Delta_{\gamma,rs}(z) \langle\!\langle \varphi_{\gamma s}(z^*) | + T_\gamma'^{(R)}(z) . \quad (5.4)$$

Making use of this representation the three-body equations (5.3) are reduced *exactly* to *effective two two-body equations* of the Lippmann-Schwinger type which can be written in matrix notation as

$$\mathcal{T}^{(R)} = \mathcal{V}^{(R)} + \mathcal{V}^{(R)} \mathcal{G}_0 \mathcal{T}^{(R)} . \quad (5.5)$$

Here, the elements of the amplitude, potential, and free Green's function matrices are defined by

$$\mathcal{T}_{\beta n,\alpha m}^{(R)}(z) = \langle\!\langle \varphi_{\beta n}(z^*) | G_0(z) U_{\beta\alpha}^{(R)}(z) G_0(z) | \varphi_{\alpha m}(z) \rangle\!\rangle , \quad (5.6)$$

$$\mathcal{V}_{\beta n,\alpha m}^{(R)}(z) = \langle\!\langle \varphi_{\beta n}(z^*) | G_0(z) U_{\beta\alpha}'^{(R)}(z) G_0(z) | \varphi_{\alpha m}(z) \rangle\!\rangle , \quad (5.7)$$

$$\mathcal{G}_{0;\beta n,\alpha m}(z) = \delta_{\beta\alpha} \Delta_{\alpha,nm}(z) . \quad (5.8)$$

The operator $U_{\beta\alpha}'^{(R)}$ occurring in (5.7) satisfies Eq. (5.3) with $T_\gamma^{(R)}$ replaced by $T_\gamma'^{(R)}$:

$$U_{\beta\alpha}'^{(R)} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} T_\gamma'^{(R)} G_0 U_{\gamma\alpha}'^{(R)} . \quad (5.9)$$

In praxis, the splitting (5.4) is usually based on a splitting of the potential

$$V_\gamma^{(R)} = V_\gamma^S + V_\gamma^R = \sum_{r=1}^{N_\gamma} | \chi_{\gamma r} \rangle \lambda_{\gamma r} \langle \chi_{\gamma r} | + V_\gamma'^{(R)} . \quad (5.10)$$

Then, $| \varphi_{\gamma r}(z) \rangle\!\rangle$ is related to the form factor $| \chi_{\gamma r} \rangle$, which itself may be energy dependent, via

$$| \varphi_{\gamma r}(z) \rangle\!\rangle = [1 + T_\gamma'^{(R)}(z) G_0(z)] | \chi_{\gamma r} \rangle , \quad (5.11)$$

with

$$T_\gamma'^{(R)} = V_\gamma'^{(R)} + V_\gamma'^{(R)} G_0 T_\gamma'^{(R)}. \quad (5.12)$$

And the inverse of the matrix Δ_γ is given by

$$[\Delta_\gamma^{-1}(z)]_{rs} = \delta_{rs}\lambda_{\gamma r}^{-1} - \langle \chi_{\gamma r} | G_0(z) | \varphi_{\gamma s}(z)\rangle . \quad (5.13)$$

It is advantageous to choose the form factors $|\chi_{\gamma r}\rangle$ orthogonal to the bound states $|\psi_{\gamma m}\rangle$:

$$\lambda_{\gamma r}\langle \chi_{\gamma r} | \psi_{\gamma m}\rangle = \delta_{rm} \quad \text{for } r = 1, \ldots, N_\gamma \quad (5.14)$$

at the binding energies $\hat{E}_{\gamma m}$, $m = 1, \ldots, n_\gamma$ with $n_\gamma \leq N_\gamma$. Then the corresponding bound state poles show up only in the diagonal elements $\Delta_{\gamma, mm}$

$$\langle \vec{q}_\gamma' | \Delta_{\gamma, mm}(z) | \vec{q}_\gamma \rangle = \delta(\vec{q}_\gamma' - \vec{q}_\gamma) \frac{S_{\gamma, m}(z)}{z - q_\gamma^2/2M_\gamma - \hat{E}_{\gamma m}} \quad (5.15)$$

with $S_{\gamma,m}(z)$ being, in general, a complicated function of z which, however, as a consequence of (5.14), becomes unity for $z = q_\gamma^2/2M_\gamma + \hat{E}_{\gamma m}$. Consequently, the quantities

$$\mathcal{T}_{\beta n, \alpha m}^{(R)}(\vec{q}_\beta', \vec{q}_\alpha) = \langle \vec{q}_\beta' | \mathcal{T}_{\beta n, \alpha m}^{(R)}(E_{\alpha m} + i0) | \vec{q}_\alpha \rangle ,$$

with $n = 1, \ldots, n_\beta$ and $m = 1, \ldots, n_\alpha$ labeling two-body bound states, coincide on the energy shell with the transition amplitudes (5.1). In other words, solving Eq. (5.5) numerically and proceeding as described after (5.1), we end up with the desired arrangement scattering amplitudes for three charged particles. This procedure corresponds to the *first* practical approach described at the end of Sec. II.

B. Modified off-shell extension of the full amplitude

For the generalization of the *second* practical approach it proves convenient to introduce another off-shell continuation of the amplitude (5.1). It is defined by a LS equation of the form (5.5), but with $\mathcal{G}_0$ replaced by a more appropriate free matrix Green's function $\tilde{\mathfrak{g}}_0$. Namely, for indices enumerating the two-body bound states, $\tilde{\mathfrak{g}}_0$ is chosen to have only diagonal elements which are of the simple structure of a genuine two-body free Green's function, i.e., of the form (5.15) with $S_{\gamma,m}(z) = 1$ for *all* values of z. In other words, we define the elements of $\tilde{\mathfrak{g}}_0$ via

$$\tilde{\mathfrak{g}}_{0;\beta n, \alpha m}(z) = \begin{cases} \delta_{\beta\alpha}\delta_{nm}\mathfrak{g}_{0,\alpha}(z - \hat{E}_{\alpha m}), & \text{for } n = 1, \ldots, n_\beta \\ & \text{or } m = 1, \ldots, n_\alpha \\ \mathcal{G}_{0;\beta n, \alpha m}(z), & \text{otherwise}, \end{cases} \quad (5.16)$$

with $\mathfrak{g}_{0,\alpha}$ given by (3.17). Multiplying (5.5) from the right by $\mathcal{G}_0\tilde{\mathfrak{g}}_0^{-1}$, we obtain again a matrix equation of the Lippmann-Schwinger structure

$$\tilde{\mathcal{T}}^{(R)} = \tilde{\mathcal{V}}^{(R)} + \tilde{\mathcal{V}}^{(R)}\tilde{\mathfrak{g}}_0\tilde{\mathcal{T}}^{(R)} \quad (5.17)$$

for the matrix operator

$$\tilde{\mathcal{T}}^{(R)} = \mathcal{T}^{(R)}\mathcal{G}_0\tilde{\mathfrak{g}}_0^{-1}, \quad (5.18)$$

which contains the simpler free matrix Green's function $\tilde{\mathfrak{g}}_0$ instead of $\mathcal{G}_0$.

From the above mentioned properties of $\mathcal{G}_0$ and $\tilde{\mathfrak{g}}_0$ it follows that all elements $(\mathcal{G}_0\tilde{\mathfrak{g}}_0^{-1})_{nm}$ with m denoting a two body bound state vanish, when applied onto $|\vec{q}_\alpha\rangle$ at the corresponding channel energy $E_{\alpha m}$, except the one with $n = m$ which, in fact, has the value one. Therefore, on the right-hand energy shell, $\tilde{\mathcal{T}}^{(R)}$ coincides with $\mathcal{T}^{(R)}$,

$$\tilde{\mathcal{T}}_{\beta n, \alpha m}^{(R)}(E_{\alpha m} + i0) | \vec{q}_\alpha \rangle = \mathcal{T}_{\beta n, \alpha m}^{(R)}(E_{\alpha m} + i0) | \vec{q}_\alpha \rangle \quad \text{for } m = 1, \ldots, n_\alpha . \quad (5.19)$$

In other words, $\tilde{\mathcal{T}}^{(R)}$ and $\mathcal{T}^{(R)}$ represent different off-shell continuations of the amplitude (5.1). A relation similar to (5.19) holds true also between $\tilde{\mathcal{V}}^{(R)}$ and $\mathcal{V}^{(R)}$. Henceforth, we will work with Eq. (5.17) instead of (5.5).

C. Quasiparticle equations for the unscreened Coulomb-modified short-range amplitudes

The two-body structure of the quasiparticle equations (5.5) and (5.17) suggests generalizing also the *second* practical approach described at the end of Sec. II. As mentioned above, the form (5.17) is particularly convenient for this purpose. In order to be able to perform all algebraic manipulations as in the genuine two-body case, an effective two-body Coulomb potential matrix $\tilde{\mathfrak{v}}^R$ and an amplitude matrix $\tilde{\mathfrak{t}}^R$ are associated with the operators (3.16) and (3.19) by means of

$$\tilde{\mathfrak{v}}_{\beta n, \alpha m}^R = \begin{cases} \delta_{\beta\alpha}\delta_{nm}\mathfrak{v}_\alpha^R, & \text{for } n \text{ and } m \\ & \text{bound state indices} \\ 0, & \text{otherwise} \end{cases} \quad (5.20)$$

and

$$\tilde{\mathfrak{t}}_{\beta n, \alpha m}^R(z) = \begin{cases} \delta_{\beta\alpha}\delta_{nm}\mathfrak{t}_\alpha^R(z - \hat{E}_{\alpha m}), & \text{for } n \text{ and } m \\ & \text{bound state indices} \\ 0, & \text{otherwise}. \end{cases} \quad (5.21)$$

To introduce nonvanishing elements in these matrices only for bound state indices is suggested by the desire to reproduce the on-shell relation (3.22) for the physical two-fragment amplitudes which are characterized by these indices [compare Eq. (5.27)]. The other $\mathcal{T}_{\beta n, \alpha m}$ in (5.6) for which n and m do *not* correspond to bound state indices, are purely auxiliary quantities, introduced in order to improve the accuracy. Hence, no renormalization procedure, and consequently no splitting of the form (3.22), is needed for these terms. In this context it may be helpful to recall the argumentation employed in Refs. 3 and 4. There, by inspecting the kernels of the effective two-body integral equations, it was found that, due to the absence of physical-sheet poles in the effective propagators for nonbound state indices, no overlap with the singularity of $\mathfrak{v}_\alpha^R$ in the zero-screening limit can occur. Consequently, no renormalization procedure is necessary in these unphysical channels.

The Coulomb resolvent matrix corresponding to the definitions (5.16), (5.20), and (5.21) is then introduced by the conventional two-body relation

$$\tilde{\mathcal{G}}^R = \tilde{\mathcal{G}}_0 + \tilde{\mathcal{G}}_0 \tilde{\mathfrak{t}}^R \tilde{\mathcal{G}}_0 , \qquad (5.22)$$

or equivalently by

$$\tilde{\mathcal{G}}^R = \tilde{\mathcal{G}}_0 + \tilde{\mathcal{G}}_0 \tilde{\mathfrak{v}}^R \tilde{\mathcal{G}}^R . \qquad (5.23)$$

Then, in complete analogy to (2.8), a Coulomb-distorted short-range amplitude $\mathcal{T}^{SR}$ is defined via

$$\tilde{\mathcal{T}}^{(R)} = \tilde{\mathfrak{t}}^R + (1 + \tilde{\mathfrak{t}}^R \tilde{\mathcal{G}}_0)\mathcal{T}^{SR}(1 + \tilde{\mathcal{G}}_0 \tilde{\mathfrak{t}}^R) . \qquad (5.24)$$

Introducing a potential matrix $\mathcal{V}^{SR}$ by means of

$$\tilde{\mathcal{V}}^{(R)} = \mathcal{V}^{SR} + \tilde{\mathfrak{v}}^R \qquad (5.25)$$

we find that this amplitude satisfies the integral equation

$$\mathcal{T}^{SR} = \mathcal{V}^{SR} + \mathcal{V}^{SR} \tilde{\mathcal{G}}^R \mathcal{T}^{SR} . \qquad (5.26)$$

The definition (5.24) is closely related to (3.22), as becomes transparent by going over to the explicit notation. Indeed, for bound state indices $n = 1, \ldots, n_\beta$ and $m = 1, \ldots, n_\alpha$ it reads

$$\begin{aligned} \tilde{\mathcal{T}}^{(R)}_{\beta n,\alpha m}(z) = {} & \delta_{\beta\alpha}\delta_{nm} t_\alpha^R(z - \hat{E}_{\alpha m}) \\ & + [(1 + t_\beta^R(z - \hat{E}_{\beta n}) \mathcal{G}_{0,\beta}(z - \hat{E}_{\beta n})] \\ & \times \mathcal{T}^{SR}_{\beta n,\alpha m}(z)[1 + \mathcal{G}_{0,\alpha}(z - \hat{E}_{\alpha m}) t_\alpha^R(z - \hat{E}_{\alpha m})] . \end{aligned} \qquad (5.27)$$

Taking into account (3.20) and (3.21), we see that both the left-hand side and the first term on the right-hand side of (5.27), when sandwiched between plane waves $\langle \vec{q}_\beta' |$ and $| \vec{q}_\alpha \rangle$, coincide on the energy shell with the corresponding terms of (3.22). This implies on-shell the equality of the respective second terms, too, namely

$$\begin{aligned} \langle \vec{q}_{\beta,R}'^{(-)} | \langle \psi_{\beta n} | U_{\beta\alpha}^{SR}(E_{\alpha m} + i0) | \psi_{\alpha m} \rangle | \vec{q}_{\alpha,R}^{(+)} \rangle \\ = \langle \vec{q}_{\beta,R}'^{(-)} | \mathcal{T}^{SR}_{\beta n,\alpha m}(E_{\alpha m} + i0) | \vec{q}_{\alpha,R}^{(+)} \rangle . \end{aligned} \qquad (5.28)$$

From the investigation leading to Eq. (3.29) we know that the states $\langle \vec{q}_{\beta,R}'^{(-)} |$ and $| \vec{q}_{\alpha,R}^{(+)} \rangle$, multiplied by $Z_{\beta,R}^{-1/2}(q_\beta')$ and $Z_{\alpha,R}^{-1/2}(q_\alpha)$, respectively, go over for $R \to \infty$ into the unscreened Coulomb scattering states. Furthermore, as has been shown in Sec. III, this limit can be performed immediately in $\langle \psi_{\beta n} | U_{\beta\alpha}^{SR}(E_{\alpha m} + i0) | \psi_{\alpha m} \rangle$, and consequently due to the equality (5.28) the same holds true for the quantity $\mathcal{T}^{SR}_{\beta n,\alpha m}(E_{\alpha m} + i0)$. Denoting the zero-screening limit of the latter by $\mathcal{T}^{SC}_{\beta n,\alpha m}(E_{\alpha m} + i0)$ we, therefore, end up with the representation

$$\mathcal{T}^{SC}_{\beta n,\alpha m}(\vec{q}_\beta', \vec{q}_\alpha) = \langle \vec{q}_{\beta,C}'^{(-)} | \mathcal{T}^{SC}_{\beta n,\alpha m}(E_{\alpha m} + i0) | \vec{q}_{\alpha,C}^{(+)} \rangle \qquad (5.29)$$

for the *unscreened* Coulomb-modified short-range transition amplitude (5.2).

An integral equation for $\mathcal{T}^{SC}$ itself now follows directly by letting R go to infinity in Eq. (5.26). In this way, we arrive at the generalization of Eq. (2.15),

$$\mathcal{T}^{SC} = \mathcal{V}^{SC} + \mathcal{V}^{SC} \tilde{\mathcal{G}}^C \mathcal{T}^{SC} . \qquad (5.30)$$

Here, the Coulomb-modified short-ranged potential matrix

$$\mathcal{V}^{SC} = \tilde{\mathcal{V}} - \tilde{\mathfrak{v}}^C \qquad (5.31)$$

and the Coulomb Green's function matrix $\tilde{\mathcal{G}}^C$ are given by Eqs. (5.25) and (5.22), respectively, after having switched off the screening.

Let us briefly comment on the relation between the present approach and the one developed in Refs. 1–4. There, we demonstrated explicitly that $\mathcal{V}^{SC}$ does not contain contributions of infinite range. From this fact the validity of Eq. (5.30) and, as a consequence, the existence of $\mathcal{T}^{SC}$ could be inferred. Alternatively, in the present paper we started by directly proving the existence of $\mathcal{T}^{SC}$ from which then the validity of the integral equation (5.30) followed. However, whether Eq. (5.30) is also useful for practical calculations depends on properties of its kernel which go beyond those required for proving its validity. In fact, since $\mathcal{V}^{SC}$ is still of fairly long range it is even questionable whether the kernel $\mathcal{V}^{SC}\tilde{\mathcal{G}}^C$ falls into that class for which standard integral equations theory is applicable. Certainly further investigations are needed to clarify this point.

For convenience we present our result (5.30) in explicit notation,

$$\mathcal{T}^{SC}_{\beta n,\alpha m}(z) = \mathcal{V}^{SC}_{\beta n,\alpha m}(z) + \sum_{\gamma\delta, rs} \mathcal{V}^{SC}_{\beta n,\gamma r}(z) \times \tilde{\mathcal{G}}^C_{\gamma r,\delta s}(z) \mathcal{T}^{SC}_{\delta s,\alpha m}(z) , \qquad (5.32)$$

with

$$\mathcal{V}^{SC}_{\beta n,\alpha m}(z) = \begin{cases} \tilde{\mathcal{V}}_{\beta n,\alpha m}(z) - \delta_{\beta\alpha}\delta_{nm}\mathfrak{v}_\alpha^C , & \text{for } n \text{ or } m \text{ bound state indices} \\ \tilde{\mathcal{V}}_{\beta n,\alpha m}(z) , & \text{otherwise} , \end{cases} \qquad (5.33)$$

and

$$\tilde{\mathcal{G}}^C_{\beta n,\alpha m}(z) = \begin{cases} \delta_{\beta\alpha}\delta_{nm}\mathcal{G}_\alpha^C(z - \hat{E}_{\alpha m}) , & \text{for } n \text{ or } m \text{ bound state indices} \\ \mathcal{G}_{0;\beta n,\alpha m}(z) , & \text{otherwise} . \end{cases} \qquad (5.34)$$

We mention that from (5.21) and (5.27) it follows immediately that the full unscreened amplitude $\mathcal{T}_{\beta n,\alpha m}$ equals $\mathcal{T}^{SC}_{\beta n,\alpha m}$ if neither n nor m denote two-body bound states, i.e., if $n \neq 1, \ldots, n_\beta$ and $m \neq 1, \ldots, n_\alpha$. This is consistent with the fact that (5.32) goes over into (5.5) in this case. Solving Eq. (5.32) we therefore obtain, besides the full amplitudes for unphysical channels, the Coulomb-distorted amplitudes $\mathcal{T}^{SC}_{\beta n,\alpha m}$ for physical channels enumerated by $n(m) = 1, \ldots, n_\beta(n_\alpha)$. Sandwiching the latter ones between $\langle \vec{q}'^{(-)}_{\beta,C}|$ and $|\vec{q}^{(+)}_{\alpha,C}\rangle$, the full unscreened arrangement amplitudes are obtained via (5.29), (5.2), and (3.29).

This procedure is considerably simplified if the Coulomb interactions $\mathfrak{v}^C_\gamma$ between the colliding clusters are repulsive. For, inserting the spectral decomposition of $\mathfrak{g}^C_\gamma(z-\hat{E}_{\gamma r})$ occurring via (5.34) in Eq. (5.32), we obtain relations which directly yield off-shell continuations of the amplitudes (5.2). In fact, we have for $m=1, \ldots, n_\alpha$ and $n=1, \ldots, n_\beta$

$$\begin{aligned}\langle \vec{q}'^{(-)}_{\beta,C}|\mathcal{T}^{SC}_{\beta n,\alpha m}(z)|\vec{q}^{(+)}_{\alpha,C}\rangle &= \langle \vec{q}'^{(-)}_{\beta,C}|\mathcal{V}^{SC}_{\beta n,\alpha m}(z)|\vec{q}^{(+)}_{\alpha,C}\rangle \\ &+\sum_\gamma\sum_{r=1}^{n_\gamma}\int d^3q''_\gamma \langle \vec{q}'^{(-)}_{\beta,C}|\mathcal{V}^{SC}_{\beta n,\gamma r}(z)|\vec{q}''^{(-)}_{\gamma,C}\rangle \frac{1}{z-\hat{E}_{\gamma r}-q''^2_\gamma/2M_\gamma}\langle \vec{q}''^{(-)}_{\gamma,C}|\mathcal{T}^{SC}_{\gamma r,\alpha m}(z)|\vec{q}^{(+)}_{\alpha,C}\rangle \\ &+\sum_\gamma\sum_{r,s=n_\gamma+1}^{N_\gamma}\int d^3q''_\gamma \langle \vec{q}'^{(-)}_{\beta,C}|\tilde{\mathcal{V}}_{\beta n,\gamma r}(z)|\vec{q}''_\gamma\rangle \mathcal{G}_{0;\gamma r,\gamma s}(z)\langle \vec{q}''_\gamma|\mathcal{T}^{SC}_{\gamma s,\alpha m}(z)|\vec{q}^{(+)}_{\alpha,C}\rangle ,\end{aligned} \quad (5.35)$$

and a similar relation with $\langle \vec{q}'^{(-)}_{\beta,C}|$ replaced by $\langle \vec{q}'_\beta|$ for the elements with $n = n_{\beta+1}, \ldots, N_\beta$.

A final remark concerns the case of attractive Coulomb forces in one or more subsystems. Associating to each of the infinitely many two-body bound states a separable term in (5.4) would blow up the system of equations (5.5). In this case, it might be preferable to represent by separable terms in (5.4) only the dominant bound states, in particular those which are explicitly considered in the special physical situation, and to absorb the remaining ones via T'_γ into the definition of $U'_{\beta\alpha}$, cf. Eq. (5.9). Whether the quasi-Born series, which perturbatively determines the effective potential $\mathcal{V}$, and consequently also $\mathcal{V}^{SC}$, then still converges deserves further investigation.

D. Modified screening procedure

In the above definitions of three-body transition operators and effective two-body amplitudes *all* Coulomb potentials had been screened before applying the renormalization and limiting procedure. From the reasoning following Eq. (3.22) we know, on the other hand, that such a screening is actually required only for the center-of-mass Coulomb potential $\mathfrak{v}^C_\alpha$ when it is being used to define the two-body amplitudes $\mathfrak{t}^C_\alpha$ and the scattering states $|\vec{q}^{(\pm)}_{\alpha,C}\rangle$. In contrast, for the definition of the Coulomb-modified short-range amplitudes $\mathcal{T}^{SC}$ we could have worked from the very beginning with unscreened Coulomb potentials.

This fact has already been utilized in the *second* approach where, as shown in Sec. V C, the *unscreened* amplitude $\mathcal{T}^{SC}$ is obtained directly as solution of (5.30). But we expect that also the *first* practical approach, based on solving Eqs. (5.5) or (5.17), can be simplified in a corresponding manner. Indeed, according to the above considerations, the screening of the Coulomb interaction is expected to be unnecessary in most terms of the effective potential (5.7). To make this explicit we introduce the modified effective potential

$$\bar{\mathcal{V}}^{(R)} = \mathcal{V}^{SC} + \tilde{\mathfrak{v}}^R \quad (5.36)$$

which originates from (5.25) by replacing the Coulomb-modified short-ranged part $\mathcal{V}^{SR}$ by its zero-screening limit $\mathcal{V}^{SC}$. The representation (5.31) allows us to write (5.36) in the form

$$\bar{\mathcal{V}}^{(R)} = \tilde{\mathcal{V}} - \tilde{\mathfrak{v}}^C + \tilde{\mathfrak{v}}^R \quad (5.37)$$

which clearly shows that $\bar{\mathcal{V}}^{(R)}$ is obtained from the unscreened potential $\tilde{\mathcal{V}}$ by screening only its long-ranged contribution $\tilde{\mathfrak{v}}^C$.

Inserting $\bar{\mathcal{V}}^{(R)}$ instead of $\tilde{\mathcal{V}}^{(R)}$ in (5.17), a new screened amplitude $\bar{\mathcal{T}}^{(R)}$ is defined as

$$\bar{\mathcal{T}}^{(R)} = \bar{\mathcal{V}}^{(R)} + \bar{\mathcal{V}}^{(R)}\tilde{\mathfrak{g}}_0\bar{\mathcal{T}}^{(R)} . \quad (5.38)$$

Making use of the splitting (5.36) of $\bar{\mathcal{V}}^{(R)}$ we obtain the representation

$$\bar{\mathcal{T}}^{(R)} = \tilde{\mathfrak{t}}^R + (1+\tilde{\mathfrak{t}}^R\tilde{\mathfrak{g}}_0)\bar{\mathcal{T}}^{SR}(1+\tilde{\mathfrak{g}}_0\tilde{\mathfrak{t}}^R) , \quad (5.39)$$

where $\bar{\mathcal{T}}^{SR}$ is given as solution of

$$\bar{\mathcal{T}}^{SR} = \mathcal{V}^{SC} + \mathcal{V}^{SC}\tilde{\mathfrak{g}}^R\bar{\mathcal{T}}^{SR} . \quad (5.40)$$

Since this relation, which contains the screening only via $\tilde{\mathfrak{g}}^R$, goes over into (5.30) for $R\to\infty$, its solution becomes identical with $\mathcal{T}^{SC}$ in this limit. Comparison of (5.39) and (5.24) then reveals that $\tilde{\mathcal{T}}^{(R)}$ and $\bar{\mathcal{T}}^{(R)}$ lead to the *same* unscreened amplitude (3.29) when subjected to the renormalization and limiting procedure. In other words, the *first* practical approach discussed in Sec. V A yields the same on-shell arrangement amplitudes when being based on the solutions of (5.38) instead of (5.5) or (5.17). However, the amount of work required for the numerical solution is drastically different for both types of equations. For, when solving Eqs. (5.5) or (5.17) the full effective potential has to be recalculated every time the value of R is increased. This, however, is a most time-consuming task. In contrast, when Eq. (5.38) is used the bulk of $\bar{\mathcal{V}}^{(R)}$ namely, $\mathcal{V}^{SC}$, has to be computed only once (for a fixed energy) since the R dependence resides solely in the trivial term $\tilde{\mathfrak{v}}^R$. Therefore, the procedure described in the present section represents a considerable simplification of the first practical approach.

In addition, the present version is even more closely related to the corresponding method in the genuine two-body problem. Namely, as is the case there, only that part of the potential $\tilde{\mathcal{V}}$ has to be screened which behaves like a pure Coulomb potential in the *relative* two-fragment coordinate, whereas $\mathcal{V}^{SC}$, which is of shorter range with respect to this variable although it contains two-body Coulomb potentials, remains unscreened.

Sections V D and V C, therefore, contain the most natural generalizations of the first and the second practical approach discussed for the two-body case in Sec. II.

VI. INTEGRAL EQUATIONS FOR TWO-FRAGMENT AMPLITUDES: N-BODY CASE

The method discussed at the beginning of Sec. V for $N=3$ can be extended immediately to arbitrary particle numbers. First, the *screened* amplitude (4.11),

$$T_{ba}^{(R)}(\vec{q}_b', \vec{q}_a) = \langle \vec{q}_b' | \langle \psi_b | U_{ba}^{(R)}(E_a + i0) | \psi_a \rangle | \vec{q}_a \rangle, \quad (6.1)$$

is calculated for finite R by means of *any* method applicable for purely short-ranged interactions. Subtracting from it the center-of-mass Coulomb amplitude $t_a^R(\vec{q}_a', \vec{q}_a)$ yields, on account of (4.11), the screened Coulomb-modified short-range amplitude. Renormalizing the latter, and repeating the calculation for increasing values of R, the zero-screening limit is approached numerically. By this procedure we end up with the unscreened Coulomb-modified short-range amplitude

$$T_{ba}^{SC}(\vec{q}_b', \vec{q}_a) = \langle \vec{q}_{b,C}'^{(-)} | \langle \psi_b | U_{ba}^{SC}(E_a + i0) | \psi_a \rangle | \vec{q}_{a,C}^{(+)} \rangle, \quad (6.2)$$

and according to (4.15) with the full unscreened scattering amplitude $T_{ba}(\vec{q}_b', \vec{q}_a)$.

Particularly appropriate for the calculation of the screened amplitude (6.1) is the generalization of the three-body quasiparticle method to N-body problems proposed in Ref. 30. For, it replaces in $(N-2)$ steps the original operator identities for $U_{ba}^{(R)}$ by exact equations for off-shell extensions $\mathcal{T}_{ba}^{(R)}$ of the *effective two-body* amplitudes $\langle \psi_b | U_{ba}^{(R)}(E_a + i0) | \psi_a \rangle$,

$$\mathcal{T}_{ba}^{(R)} = \mathcal{V}_{ba}^{(R)} + \sum_{c,d} \mathcal{V}_{bc}^{(R)} \mathcal{G}_{0;cd} \mathcal{T}_{da}^{(R)}. \quad (6.3)$$

For details of the definitions of the effective potential $\mathcal{V}^{(R)}$ and "free Green's function" $\mathcal{G}_0$ we refer to Ref. 30. There it is also shown that on the energy shell the physical reaction amplitude (6.1) can be obtained via

$$T_{ba}^{(R)}(\vec{q}_b', \vec{q}_a) = \langle \vec{q}_b' | \mathcal{T}_{ba}^{(R)}(E_a + i0) | \vec{q}_a \rangle. \quad (6.4)$$

Therefore, solving Eq. (6.3) and proceeding as discussed after Eq. (6.1), represents the N-body generalization of the *first* practical approach described in Sec. II. We remark that this method may again be simplified along the lines described in Sec. V D.

Of course, the *second* approach developed in Sec. V C can be extended to arbitrary particle numbers, too. For this purpose, it is again advantageous to work with, instead of (6.3), an equation containing the simpler free Green's function matrix $\tilde{\mathfrak{g}}_0$, with elements

$$\tilde{\mathfrak{g}}_{0;ba} = \begin{cases} \delta_{ba}\, \mathfrak{g}_{0,a}(z - \hat{E}_a), & \text{for } b \text{ or } a \text{ denoting bound state channels} \\ \mathcal{G}_{0;ba}, & \text{otherwise.} \end{cases} \quad (6.5)$$

Here

$$\mathfrak{g}_{0,a}(z) = \left(z - \frac{Q_a^2}{2M_a}\right)^{-1} \quad (6.6)$$

is a genuine two-body free Green's function which acts on the relative momentum states $|\vec{q}_a\rangle$ only [cf. Eq. (3.17)]. Multiplication of (6.3) with $\mathcal{G}_0 \tilde{\mathfrak{g}}_0^{-1}$ results in the N-body analog of (5.17):

$$\tilde{\mathcal{T}}_{ba}^{(R)} = \tilde{\mathcal{V}}_{ba}^{(R)} + \sum_{c,d} \tilde{\mathcal{V}}_{bc}^{(R)} \tilde{\mathfrak{g}}_{0;cd} \tilde{\mathcal{T}}_{da}. \quad (6.7)$$

Since we assume $\mathcal{G}_0$ to be chosen such that it displays the same pole behavior as $\tilde{\mathfrak{g}}_0$ (compare the discussion in Secs. V A and V B for the three-particle case), the solution $\tilde{\mathcal{T}}^{(R)} = \mathcal{T}^{(R)} \mathcal{G}_0 \tilde{\mathfrak{g}}_0^{-1}$ of (6.7) equals $\mathcal{T}^{(R)}$ on the right-hand energy shell.

Next we introduce a two-body Coulomb potential $\mathfrak{v}_a^R$ describing the interaction between the charges of the two colliding fragments concentrated in their centers of mass, which is related to the N-body operator (4.5) according to

$$\langle \vec{q}_a' | \mathfrak{v}_a^R | \vec{q}_a \rangle = \langle \vec{q}_a' | \langle \psi_a | v_a^R | \psi_a \rangle | \vec{q}_a \rangle = \mathfrak{v}_a^R(\vec{q}_a' - \vec{q}_a). \quad (6.8)$$

The corresponding two-body Coulomb resolvent $\mathfrak{g}_a^R$ and Coulomb transition operator $\mathfrak{t}_a^R$ are

$$\mathfrak{g}_a^R(z) = \left(z - \frac{Q_a{}^2}{2M_a} - \mathfrak{v}_a^R\right)^{-1} \tag{6.9}$$

and

$$\mathfrak{t}_a^R = \mathfrak{v}_a^R + \mathfrak{v}_a^R \mathfrak{g}_a^R \mathfrak{v}_a^R , \tag{6.10}$$

respectively. With the help of these quantities we define for bound state channels a and b an effective two-body amplitude $\mathcal{T}_{ba}^{SR}$, in analogy to (5.27), via

$$\begin{aligned}\tilde{\mathcal{T}}_{ba}^{(R)}(z) = {} & \delta_{ba}\mathfrak{t}_a^R(z - \hat{E}_a) \\ & + [1 + \mathfrak{t}_b^R(z - \hat{E}_b)\mathfrak{g}_{0;b}(z - \hat{E}_b)] \\ & \times \mathcal{T}_{ba}^{SR}(z)[1 + \mathfrak{g}_{0;a}(z - \hat{E}_a)\mathfrak{t}_a^R(z - \hat{E}_a)] .\end{aligned} \tag{6.11}$$

For a or b *not* characterizing bound state channels, the first term on the right-hand side of (6.11) and either the right or the left bracket in the second term are missing. Sandwiching (6.11) between plane waves $\langle \vec{q}_b' |$ and $| \vec{q}_a \rangle$ and comparing with (4.11) we see that the left-hand sides and the first terms on the right-hand sides of both equations, respectively, coincide on the energy shell. Generalizing the argumentation following (5.28), we infer that the zero-screening limit of $\mathcal{T}^{SR}$, which we denote by $\mathcal{T}^{SC}$, is indeed related on the energy shell to the Coulomb-modified short-range amplitude (6.2) according to

$$T_{ba}^{SC}(\vec{q}_b', \vec{q}_a) = \langle \vec{q}_{b,C}'^{(-)} | \mathcal{T}_{ba}^{SC}(E_a + i0) | \vec{q}_{a,C}^{(+)} \rangle . \tag{6.12}$$

An integral equation directly for $\mathcal{T}^{SC}$ is now easily derived. For this purpose we introduce the zero-screening limits of the potentials $\tilde{\mathcal{V}}_{ba}^{(R)}$ and $\mathfrak{v}_a^R$, and of $\mathfrak{g}_a^R$, to be denoted by $\tilde{\mathcal{V}}_{ba}$, $\mathfrak{v}_a^C$, and $\mathfrak{g}_a^C$, respectively. As in (5.33), the short-ranged part $\mathcal{V}^{SC}$ of $\tilde{\mathcal{V}}$ is obtained by subtracting from the latter the long-ranged Coulomb contribution,

$$\mathcal{V}_{ba}^{SC} = \begin{cases} \tilde{\mathcal{V}}_{ba} - \delta_{ba}\mathfrak{v}_a^C , & \text{for } a \text{ or } b \text{ denoting bound state channels} \\ \tilde{\mathcal{V}}_{ba} , & \text{otherwise} . \end{cases} \tag{6.13}$$

Familiar algebra, then, leads immediately to the generalization of (5.32),

$$\mathcal{T}_{ba}^{SC} = \mathcal{V}_{ba}^{SC} + \sum_{c,d} \mathcal{V}_{bc}^{SC} \tilde{\mathfrak{g}}_{cd}^C \mathcal{T}_{da}^{SC} . \tag{6.14}$$

Here the elements of the matrix Coulomb Green's function $\tilde{\mathfrak{g}}^C(z)$ are defined by [cf. Eq. (5.34)]

$$\tilde{\mathfrak{g}}_{ba}^C(z) = \begin{cases} \delta_{ba}\, \mathfrak{g}_a^C(z - \hat{E}_a) , & \text{for } b \text{ or } a \text{ denoting bound state channels} \\ \mathfrak{g}_{0;ba}(z) , & \text{otherwise} . \end{cases} \tag{6.15}$$

Solving (6.14) we, therefore, get without any screening procedure an amplitude $\mathcal{T}_{ba}^{SC}$. When sandwiched between the explicitly known two-body Coulomb states $\langle \vec{q}_{b,C}'^{(-)} |$ and $| \vec{q}_{a,C}^{(+)} \rangle$, it yields via (6.12) the Coulomb-modified short-range amplitude (6.2), and then by means of (4.15) the full transition amplitude $T_{ba}(\vec{q}_b', \vec{q}_a)$.

As emphasized in Sec. V C this method is simplified considerably if the Coulomb potential $\mathfrak{v}_a^C$ is repulsive. For, insertion of the spectral decomposition of the Coulomb Green's function $\mathfrak{g}_a^C$ occurring in (6.15) results in an integral equation directly for

$$\mathcal{T}_{ba}^{SC}(\vec{q}_b', \vec{q}_a) = \langle \vec{q}_{b,C}'^{(-)} | \mathcal{T}_{ba}^{SC} | \vec{q}_{a,C}^{(+)} \rangle , \tag{6.16}$$

which we need not write down explicitly since it generalizes the three-body equation (5.35) in an obvious way.

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[26]We mention that in Refs. 3 and 4 the operators $\mathfrak{v}_\alpha^R$, $\mathcal{G}_{0,\alpha}$, etc. have been denoted by $V_R^{Q_\alpha}$, $G_0^{Q_\alpha}$, etc.

[27]In the special case of exponentially screened Coulomb potentials

$$v_\alpha^R(\vec{\rho}_\alpha) = e_\alpha(e_\beta + e_\gamma)\exp(-\rho_\alpha/R)/\rho_\alpha,$$

the renormalization factor has the form

$$Z_{\alpha,R}(q_\alpha) = \exp[-2ie_\alpha(e_\beta + e_\gamma)M_\alpha/q_\alpha(\ln 2q_\alpha R - C)],$$

cf. the two-body analog in Sec. II.

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