Feshbach-Villars formalism and pion-nucleon scattering

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The Feshbach-Villars Hamiltonian formulation of the Klein-Gordon equation is used as the basis for a scattering theory for relativistic, spin-zero, particles. Time dependent and time independent scattering theories are developed. The time independent theory is used as the framework for a separable potential model for πN elastic scattering in P waves. This potential leads to a πN T matrix which has the right and left hand cuts associated with two-particle unitarity and crossing, respectively, as well as the direct and crossed nucleon poles with the proper residues. The effects of nucleon recoil are included. With a particular choice of the form factors in the potential, the Chew-Low T matrix is reproduced.

[NUCLEAR REACTIONS Relativistic scattering theory, pion-nucleon T matrix.]

I. INTRODUCTION

Over two decades ago Feshbach and Villars[1] developed a Hamiltonian formulation of the Klein-Gordon equation in order to show that, within proper limits, a relativistic wave-mechanical description of a single charged, spin-zero particle exists. They also dealt with the relativistic description of neutral particles, as well as charge multiplets such as the pion. Even though this formalism is described in textbooks[2,3] it does not appear to be well known, and certainly has not been widely used. This author knows of only two applications of it; one to nuclear motion effects in pionic atoms,[4] and another to π-nucleus scattering.[5]

Here we shall use the Feshbach-Villars (FV) formalism to develop time dependent and time independent scattering theories for relativistic, spin-zero particles. The time dependent theory is an alternative to the usual propagator theory for Klein-Gordon particles[2] and provides the basis for the time independent theory. The formal equations of the time independent theory turn out to be almost identical to those of nonrelativistic potential scattering theory. We shall use the time independent theory to develop a simple, but rather realistic, model for the πN T matrix.

Over the years, several approaches have been developed in an attempt to determine the off-shell πN T matrix.[6-22] Potential theory, quantum field theory, and dispersion relations have all been used in this effort.

In the phenomenological approaches based on potential theory, a separable interaction has been assumed and its form has been determined from the on-shell πN elastic scattering data. Landau and Tabakin[7] have obtained the form factors of an energy independent separable potential with the help of the inverse solution of the scattering problem. With their method the form factors are complex, and the T matrices have rapid off-shell variations.[7,9] These features have been shown to be consequences of the inelasticity and have been avoided by the use of a separable potential with an energy dependent coupling constant.[10]

These separable models[7-11] do not contain the direct and crossed nucleon poles, and the left hand cut associated with crossing symmetry. Only the right hand or unitarity cut is taken into account. In order to partly remedy this, Liu and Shakin[12] have developed a phenomenological T matrix which is the sum of two terms. The first term contains the nucleon poles and some of the effects due to distant singularities, while the second term is a separable form which can be determined by the on-shell data.

The well known Chew-Low (CL) model [13-15] provides a field-theoretic foundation for the πN T matrix, and has been used extensively in this connection.[16-20] The CL T matrix describes P-wave πN elastic scattering in the static nucleon, one-meson approximation and contains a coupling constant and a form factor or cutoff function which are usually obtained from the $P33$ phase shifts. Improvements on the model have been suggested which include recoil and inelasticity effects,[16-18] and inversion procedures have been developed for determining the form factor from on-shell data.[16,18] The effect of two meson states has been investigated by approximating the two π system by the ρ meson.[19]

Miller[20] has shown that the CL T matrix can be obtained as the solution of a Lippmann-Schwinger type equation with an energy dependent potential and a somewhat unusual propagator. If crossing is neglected his equation is linear, otherwise it is implicitly nonlinear. His work provides a partial justification for the multiple-scattering theories which assume the existence of an underlying potential.

Recently, Banerjee and Cammarata[21] have updated the CL theory by deriving a once subtracted modified Low equation for the off mass shell πN amplitude. Their theory includes the effects of nucleon recoil and retains the seagull terms and antinucleon contributions. It successfully describes low energy S-wave πN elastic scattering for which the original CL theory[13] makes no prediction.

A dispersion theory for the πN T matrix which combines features of the potential and field-theoretic approaches has been put forth by Reiner.[22] It uses as input on-shell information, as in the potential theories, but appeals to the underlying field theory to obtain the additional information needed to continue off shell.

The T matrix we shall derive here is obtained from a separable potential which couples positive and negative energy meson states. This potential arises in a natural way in the context of our time independent scattering theory, and leads to a T matrix which contains the crossing cut and the nucleon poles. As pointed out above, the existing potential models,[7-11] excepting Miller's,[20] do not treat the crossing symmetry properly. Our model is quite different from Miller's[20] in that our potential is energy independent and our propagators are of the usual form.

At the present time our potential accounts for P-wave scattering in the four isospin-spin channels. For each channel there are two form factors, one associated with the unitarity cut and the other associated with the crossing cut. These form factors can be chosen so that our T matrix is of exactly the same form as the CL T matrix, although they need not be. In our model the nucleon poles occur with the residue properly related to the renormalized pion-nucleon coupling constant.

The outline of the paper is as follows. In Sec. II we summarize the FV formalism, with emphasis on those results that are essential for the development of the scattering theory. The time dependent and time independent scattering theories are developed in Sec. III. Here the treatment is sketchy since the development closely parallels the usual potential theory. Section IV presents the model for the πN T matrix. We discuss the results and make suggestions for future work in Sec. V. Throughout we work in natural units ($\hbar = c = 1$).

II. FESHBACH-VILLARS FORMALISM

Here we briefly summarize the results of Feshbach and Villars[1] that are necessary for the development presented in the following sections. In this section we consider a spinless, relativistic particle interacting with an external electromagnetic field whose four potential is denoted by $A^\mu = (A^0, \vec{A})$. Such a system can be described by

$$H\Psi = i\frac{\partial\Psi}{\partial t}, \tag{2.1}$$

where the two component wave function Ψ is given by

$$\Psi = \begin{bmatrix}\theta \\ \chi\end{bmatrix}, \tag{2.2}$$

and the Hamiltonian is

$$H = (\tau_3 + i\tau_2)\frac{|-i\vec{\nabla} - e\vec{A}|^2}{2\mu} + \mu\tau_3 + eA^0. \tag{2.3}$$

Here μ is the mass of the particle, e is its charge, and the τ's are the usual 2×2 Pauli matrices. Explicitly,

$$\tau_3 = \begin{vmatrix}1 & 0 \\ 0 & -1\end{vmatrix}, \quad \tau_3 + i\tau_2 = \begin{vmatrix}1 & 1 \\ -1 & -1\end{vmatrix}, \tag{2.4}$$

and we note that $\tau_3 + i\tau_2$ is not Hermitian. The continuity equation for the system is

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{2.5}$$

with

$$\rho = \Psi^\dagger\tau_3\Psi = |\theta|^2 - |\chi|^2, \tag{2.6}$$

$$\vec{j} = (2i\mu)^{-1}[\Psi^\dagger\tau_3(\tau_3 + i\tau_2)\vec{\nabla}\Psi - (\vec{\nabla}\Psi^\dagger)\tau_3(\tau_3 + i\tau_2)\Psi] - (e/\mu)\vec{A}\Psi^\dagger\tau_3(\tau_3 + i\tau_2)\Psi. \tag{2.7}$$

The nonHermitivity of the Hamiltonian (2.3) is responsible for the non-positive definiteness of the density ρ.

Because of this feature of the density it is not possible to normalize the wave functions in the usual way. Rather we have

$$\langle\Psi|\tau_3|\Psi\rangle = \pm 1, \tag{2.8}$$

where a solution of (2.1) which gives the upper sign is called a "positive" solution and that which gives the lower sign is called a "negative" solution. As discussed by Feshbach and Villars,[1] the positive and negative solutions describe particles of opposite charge.

In the FV formulation the expectation value of an operator Ω is given by

$$\langle\Omega\rangle = \langle\Psi|\tau_3\Omega|\Psi\rangle, \tag{2.9}$$

and the adjoint of an operator is defined by

$$\overline{\Omega} = \tau_3\Omega^\dagger\tau_3. \tag{2.10}$$

Here the bar indicates the FV adjoint and the dagger the usual adjoint. In this context an operator is said to be Hermitian if

$$\Omega = \bar{\Omega} . \tag{2.11}$$

It is easy to show that such an operator has real expectation values. The Hamiltonian (2.3) is Hermitian according to the FV definition (2.11).

The eigenvalue equation for an operator such as the Hamiltonian is of the usual form, i.e.,

$$H\Psi_\alpha = E_\alpha \Psi_\alpha . \tag{2.12}$$

It is straightforward to show that the eigenvalues E_α are real and that the eigenfunctions satisfy

$$\langle \Psi_\beta | \tau_3 | \Psi_\alpha \rangle = \pm \delta_{\alpha\beta} . \tag{2.13}$$

The free particle solutions of (2.1) with the Hamiltonian (2.3) ($A^\mu = 0$) can be written in the form

$$\psi_{\vec{p}}^\sigma(x) = w^\sigma(\omega) \frac{e^{i(\vec{p}\cdot\vec{x} - \sigma\omega t)}}{(2\pi)^{3/2}} , \quad \sigma = \pm 1 \tag{2.14}$$

where

$$\omega = +(p^2 + \mu^2)^{1/2} , \tag{2.15}$$

and

$$w^\sigma(\omega) = \frac{1}{2(\mu\omega)^{1/2}} \begin{bmatrix} \mu + \sigma\omega \\ \mu - \sigma\omega \end{bmatrix} . \tag{2.16}$$

Direct computation shows that

$$w^{\sigma\dagger}(\omega)\tau_3 w^{\sigma'}(\omega) = \sigma\delta_{\sigma\sigma'} , \tag{2.17}$$

and hence

$$\langle \psi_{\vec{p}}^\sigma | \tau_3 | \psi_{\vec{p}'}^{\sigma'} \rangle = \sigma\delta_{\sigma\sigma'}\delta(\vec{p} - \vec{p}') . \tag{2.18}$$

It is straightforward to verify that

$$\sum_\sigma w^\sigma(\omega)\sigma w^{\sigma\dagger}(\omega)\tau_3 = 1 , \tag{2.19}$$

which leads to the completeness relation for the free particle states,

$$\sum_\sigma \int d^3p \, |\psi_{\vec{p}}^\sigma(t)\rangle \, \sigma \, \langle \psi_{\vec{p}}^\sigma(t) | \tau_3 = 1 . \tag{2.20}$$

In (2.19) the one is a 2×2 unit matrix, whereas in (2.20) it is a unit operator in the space of the free particle states. These states play a very important role in the development of the following sections.

III. SCATTERING THEORY

Here we shall develop a scattering theory based on the formalism of the preceding section; however, we shall not restrict ourselves to the interactions present in the Hamiltonian (2.3). We shall simply assume

$$H = H_0 + U , \tag{3.1}$$

where H_0 is given by (2.3) with $A_\mu = 0$, and U is any potential that acts in the space of the free particle states (2.14) and satisfies (2.11). Our treatment will be sketchy as the equations we shall present can be derived using textbook methods.[2,23] Except where otherwise indicated, we do not work in a specific representation.

We begin by introducing the propagator with interaction G and the free propagator G_0 as solutions of the equations

$$\left(i\frac{\partial}{\partial t} - H\right) G(t, t') = \delta(t - t') , \tag{3.2}$$

$$\left(i\frac{\partial}{\partial t} - H_0\right) G_0(t, t') = \delta(t - t') . \tag{3.3}$$

We impose the well known Feynman boundary conditions[2] which require that the positive energy states propagate forward in time and the negative energy states propagate backwards in time. This leads to the following expression for the free propagator,

$$\begin{aligned} G_0(t, t') = &-i\theta(t - t') \int d^3p \, |\psi_{\vec{p}}^{(+)}(t)\rangle\langle \psi_{\vec{p}}^{(+)}(t')|\tau_3 \\ &+ i\theta(t' - t) \int d^3p \, |\psi_{\vec{p}}^{(-)}(t)\rangle(-1)\langle\psi_{\vec{p}}^{(-)}(t')|\tau_3 , \end{aligned} \tag{3.4}$$

where $\theta(\tau)$ is the usual unit step function. It is easy to verify (3.4) by using

$$d\theta(\tau)/d\tau = \delta(\tau) \tag{3.5}$$

and the completeness relation (2.20). A similar expression can be written down for the interacting propagator $G(t, t')$ by simply replacing the free particle states by the continuum states of H and adding in a possible contribution from the bound states of H.

Using standard procedures[2,23] we can write down the following integral equations for $G(t, t')$ and the solution of (2.1),

$$G(t, t') = G_0(t, t') + \int_{-\infty}^{\infty} dt'' G_0(t, t'')U(t'')G(t'', t') , \tag{3.6}$$

$$G(t, t') = G_0(t, t') + \int_{-\infty}^{\infty} dt'' G(t, t'')U(t'')G_0(t'', t') , \tag{3.7}$$

$$|\Psi(t)\rangle = |\psi(t)\rangle + \int_{-\infty}^{\infty} dt' G_0(t, t')U(t')|\Psi(t')\rangle , \tag{3.8}$$

$$|\Psi(t)\rangle = |\psi(t)\rangle + \int_{-\infty}^{\infty} dt' G(t, t')U(t')|\psi(t')\rangle . \tag{3.9}$$

We note that (3.9) is not an equation to be solved, but is rather an expression for $|\Psi(t)\rangle$ in terms of the free particle state $|\psi(t)\rangle$ and the interacting propagator $G(t, t')$. From (3.4) and (3.8) it follows that

$$|\Psi(t)\rangle \underset{t\to\pm\infty}{\longrightarrow} |\psi(t)\rangle \mp i\int d^3p\, |\psi_{\vec{p}}^{(\pm)}(t)\rangle(\pm 1) \times \int_{-\infty}^{\infty} dt' \langle\psi_{\vec{p}}^{(\pm)}(t')|\tau_3 U(t')|\Psi(t')\rangle . \quad (3.10)$$

By projecting out a positive energy free state in the future or a negative energy free state in the past[2] and using (2.18), we find the following expression for the S matrix:

$$S_{\vec{p}\sigma,\vec{p}'\sigma'} = \sigma\delta_{\sigma\sigma'}\delta(\vec{p}-\vec{p}') - i\sigma\int_{-\infty}^{\infty} dt\langle\psi_{\vec{p}}^{\sigma}(t)|\tau_3 U(t)|\Psi_{\vec{p}'}^{\sigma'}(t)\rangle . \quad (3.11)$$

Here $|\Psi_{\vec{p}}^{\sigma'}(t)\rangle$ is the solution of (3.8) with $|\psi(t)\rangle = |\psi_{\vec{p}'}^{\sigma'}(t)\rangle$. By combining (3.9) and (3.11) we find the alternative expression

$$S_{\vec{p}\sigma,\vec{p}'\sigma'} = \sigma\delta_{\sigma\sigma'}\delta(\vec{p}-\vec{p}') - i\sigma\int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt'\langle\psi_{\vec{p}}^{\sigma}(t)|\tau_3[U(t)\delta(t-t') + U(t)G(t,t')U(t')]|\psi_{\vec{p}'}^{\sigma'}(t')\rangle . \quad (3.12)$$

The arguments that allow us to interpret (3.11) or (3.12) as the amplitudes for the various processes are the same as those used by Bjorken and Drell[2] in their development of the propagator theory for the Dirac and Klein-Gordon equations and therefore will not be repeated here. It should be noted in this connection that Bjorken and Drell[2] take their plane waves to be of the form

$$e^{\mp ip\cdot x} = e^{\mp i(\omega t - \vec{p}\cdot\vec{x})}$$

which gives a negative energy state a momentum $-\vec{p}$, rather than $+\vec{p}$, as we have in (2.14). Since $w^{\sigma}(\omega)$ only depends on $|\vec{p}|$ [see (2.15) and (2.16)], we could adopt their convention with no essential change in our formalism. Their choice is more convenient in working with Feynman diagrams, whereas ours is slightly more convenient for the application presented in Sec. IV. The S matrix which arises from the interactions in (2.3) gives the amplitudes for the following processes:

σ	σ'	process,
+1	+1	particle scattering,
−1	−1	antiparticle scattering,
+1	−1	pair production,
−1	+1	pair annihilation.

We now assume that the interaction U does not depend on the time, which allows us to develop a time independent scattering theory very much like that found in nonrelativistic quantum mechanics. With this assumption it follows from (3.2) that

$$G(t,t') = G(t-t') , \quad (3.13)$$

which makes it possible to define the Fourier transform

$$G(z) = \int_{-\infty}^{\infty} d\tau\, e^{iz\tau}G(\tau) . \quad (3.14)$$

The same equation holds for G_0. Using (3.4) we find

$$G_0(z) = \sum_{\sigma}\int d^3p \frac{|\psi_{\vec{p}}^{\sigma}\rangle\sigma\langle\psi_{\vec{p}}^{\sigma}|\tau_3}{z-\sigma\omega} , \quad (3.15)$$

with ω given by (2.15) and $\langle\vec{x}|\psi_{\vec{p}}^{\sigma}\rangle$ given by the time independent part of (2.14). We distinguish the time dependent and time independent states by the presence or absence of the time argument. We see that $G_0(z)$ has cuts on the real axis for $z>\mu$ and $z<-\mu$. In order for the Fourier inversion of (3.15) to give back (3.4), we must let

$$z \to z \pm i\epsilon, \quad z \gtrless \begin{matrix}\mu \\ -\mu\end{matrix}, \quad \epsilon > 0 . \quad (3.16)$$

Using (2.14) and (3.14) we find

$$S_{\vec{p}\sigma,\vec{p}'\sigma'} = \sigma\delta_{\sigma\sigma'}\delta(\vec{p}-\vec{p}') - 2\pi i\sigma\delta(\sigma\omega - \sigma'\omega')\langle\psi_{\vec{p}}^{\sigma}|\tau_3 T(z)|\psi_{\vec{p}'}^{\sigma'}\rangle, \quad z=\omega=\omega' , \quad (3.17)$$

with the T operator given by

$$T(z) = U + UG(z)U . \quad (3.18)$$

We see that there is no physical process if $\sigma \neq \sigma'$, which simply means that pair production or annihilation cannot be caused by a static potential. Fourier transforming (3.6) and (3.7) gives us

$$G(z) = G_0(z) + G_0(z)UG(z) , \quad (3.19)$$

$$G(z) = G_0(z) + G(z)UG_0(z) , \quad (3.20)$$

which combined with (3.18) leads to

$$G(z)U = G_0(z)T(z) , \quad (3.21)$$

$$UG(z) = T(z)G_0(z) , \quad (3.22)$$

$$T(z) = U + UG_0(z)T(z) , \quad (3.23)$$

$$T(z) = U + T(z)G_0(z)U . \quad (3.24)$$

Equations (3.18)–(3.24) are of exactly the same form as those that occur in the time independent theory for nonrelativistic potential scattering, however here negative energy states contribute to the propagators, and relativistic kinematics are used.

Another expression for $G_0(z)$ can be obtained by Fourier transforming (3.3). We find

$$G_0(z)=(z-H_0)^{-1}. \tag{3.25}$$

From the properties of the Pauli matrices it follows from (2.3) that

$$H_0^2=p_0^2+m^2, \tag{3.26}$$

where $\vec{p}_0$ is the momentum operator. Combining (3.25) and (3.26) we arrive at

$$G_0(z)=\frac{z+H_0}{z^2-p_0^2-m^2}, \tag{3.27}$$

This expression has the advantage of separating the singularity structure of $G_0(z)$ from its matrix structure.

Integral equations for the T matrix are obtained by inserting (3.15) in (3.23) or (3.24) and defining matrix elements according to

$$T^{\sigma\sigma'}(\vec{p},\vec{p}\,';z)=\langle\psi^\sigma_{\vec{p}}|\tau_3 T(z)|\psi^{\sigma'}_{\vec{p}'}\rangle \tag{3.28}$$

and

$$U^{\sigma\sigma'}(\vec{p},\vec{p}\,')=\langle\psi^\sigma_{\vec{p}}|\tau_3 U|\psi^{\sigma'}_{\vec{p}'}\rangle. \tag{3.29}$$

We find

$$T^{\sigma\sigma'}(\vec{p},\vec{p}\,';z)=U^{\sigma\sigma'}(\vec{p},\vec{p}\,')+\sum_{\sigma''}\int U^{\sigma\sigma''}(\vec{p},\vec{p}\,'')\frac{\sigma''d^3p''}{z-\sigma''\omega''}T^{\sigma''\sigma'}\times(\vec{p}\,'',\vec{p}\,';z), \tag{3.30}$$

which has the form of a coupled channel Lippmann-Schwinger equation.

In Sec. IV we shall apply the time independent scattering theory to the important and interesting problem of pion-nucleon scattering.

IV. PION-NUCLEON SCATTERING

Here we consider pion-nucleon scattering in the four P wave channels distinguished by the total isospin T and total angular momentum J. We label the channels with a single index α according to

$$\alpha=1,2,3,4,\quad 2T,2J=11,13,31,33. \tag{4.1}$$

One of the purposes of this section is to show that there exists a separable potential which when used in the equations of the preceding section leads to the same T matrix as the CL theory[13-15] in the one-meson approximation. The potential is given by

$$U=-\sum_\alpha|G_\alpha\rangle|\alpha\rangle\zeta_\alpha\langle\alpha|\langle G_\alpha|\tau_3, \tag{4.2}$$

where $|\alpha\rangle$ is a product of the eigenstates of total isospin and angular momentum, i.e.,

$$|\alpha\rangle=|(1,\tfrac{1}{2})JM\rangle|(1,\tfrac{1}{2})TM'\rangle \tag{4.3}$$

and ζ_α is a strength parameter. The free particle representation of the form factor $|G_\alpha\rangle$ is defined by

$$\langle\psi^\sigma_{\vec{p}}|\tau_3|G_\alpha\rangle|\alpha\rangle=G^\sigma_\alpha(p)\langle\hat{p}|\alpha\rangle. \tag{4.4}$$

We shall show later how to choose $G^\sigma_\alpha(p)$ so as to reproduce the CL result for the T matrix. It is straightforward to show that (4.2) satisfies (2.11), i.e., our potential is Hermitian in the FV sense.

If we put (4.2) into (3.23) or (3.24) we find

$$T(z)=-\sum_\alpha|G_\alpha\rangle\frac{|\alpha\rangle\langle\alpha|}{d_\alpha(z)}\langle G_\alpha|\tau_3, \tag{4.5}$$

where

$$d_\alpha(z)=\zeta_\alpha^{-1}+\langle\alpha|\langle G_\alpha|\tau_3 G_0(z)|G_\alpha\rangle|\alpha\rangle=\zeta_\alpha^{-1}+\sum_\sigma\int_0^\infty dp\,p^2\frac{\sigma G^{\sigma 2}_\alpha(p)}{z-\sigma\omega}. \tag{4.6}$$

In arriving at this result we have used the fact that the states $|\alpha\rangle$ are orthonormal, and have assumed that $G^\sigma_\alpha(p)$ is real. Since the CL T matrix has a simple pole at $z=0$, we choose ζ_α so that $d_\alpha(0)=0$. When we do this and put the expression for ζ_α in (4.6) we find

$$d_\alpha(z)=z\sum_\sigma\int_0^\infty\frac{dp p^2}{\omega}\frac{G^{\sigma 2}_\alpha(p)}{z-\sigma\omega}. \tag{4.7}$$

We denote the residue of the pole at $z=0$ in $d_\alpha^{-1}(z)$ by λ_α, i.e.,

$$\lambda_\alpha^{-1}=-\sum_\sigma\int_0^\infty\frac{dp p^2}{\omega^2}\sigma G^{\sigma 2}_\alpha(p). \tag{4.8}$$

Combining (4.7) and (4.8) we arrive at

$$d_\alpha(z)=z\lambda_\alpha^{-1}\left(1+z\lambda_\alpha\sum_\sigma\int_0^\infty\frac{dp p^2\sigma G^{\sigma 2}_\alpha(p)}{\omega^2(z-\sigma\omega)}\right). \tag{4.9}$$

The CL T matrix is given by

$$T_{\rm CL}(\vec{p},\vec{p}\,';z)=-\frac{2}{\pi}\frac{pv(p)}{(2\omega)^{1/2}}\frac{p'v(p')}{(2\omega')^{1/2}}\sum_\alpha h_\alpha(z)P_\alpha(\hat{p},\hat{p}'), \tag{4.10}$$

where $v(p)$ is the cutoff function, P_α is a projection operator given by

$$P_\alpha(\hat{p},\hat{p}')=\langle\hat{p}|\alpha\rangle\langle\alpha|\hat{p}'\rangle, \tag{4.11}$$

and

$$h_\alpha^{-1}(z)=z\lambda_\alpha^{-1}\left[1-\frac{z}{\pi}\int_\mu^\infty d\omega\frac{p^3v^2(p)}{\omega^2}\left(\frac{\lambda_\alpha}{\omega-z}+\frac{H_\alpha(\omega)}{\omega+z}\right)\right]. \tag{4.12}$$

Here $H_\alpha(\omega)$ is a function to be determined by crossing symmetry, and

$$\lambda_\alpha=\frac{2}{3}\left(\frac{f}{\mu}\right)^2\times\begin{cases}-4, & \alpha=1\\ -1, & \alpha=2\\ -1, & \alpha=3\\ 2, & \alpha=4\end{cases}, \tag{4.13}$$

where f is the renormalized pion-nucleon coupling constant ($f^2\simeq 0.08$). By using (3.28) and (4.4) it is straightforward to show that

$$T^{(+,+)}(\vec{p},\vec{p}\,';z)=T_{CL}(\vec{p},\vec{p}\,';z)\,, \tag{4.14}$$

if we choose

$$G_\alpha^{(+)}(p)=\left(\frac{2}{\pi}\right)^{1/2}\frac{pv(p)}{(2\omega)^{1/2}}\,, \tag{4.15}$$

$$G_\alpha^{(-)}(p)=G_\alpha^{(+)}(p)\left(\frac{H_\alpha(\omega)}{\lambda_\alpha}\right)^{1/2}, \tag{4.16}$$

and take λ_α in (4.8) to be given by (4.13).

We see from (4.16) that in order for our model to make sense we must have $H_\alpha(\omega)/\lambda_\alpha$ positive for $\omega\geq\mu$. In order to see if this is so, we consider the CL crossing relation[13-15]

$$h_\alpha(-z)=\sum_\beta A_{\alpha\beta}h_\beta(z)\,, \tag{4.17}$$

where the crossing matrix is given by

$$A=\frac{1}{9}\begin{pmatrix}1 & -4 & -4 & 16\\ -2 & -1 & 8 & 4\\ -2 & 8 & -1 & 4\\ 4 & 2 & 2 & 1\end{pmatrix}. \tag{4.18}$$

If we take the imaginary part of (4.17) with $z=\omega+i\varepsilon$, where ω is real and greater than μ, and use (4.12) we find

$$H_\alpha(\omega)/\lambda_\alpha=\left|\sum_\gamma A_{\alpha\gamma}h_\gamma(z)\right|^{-2}\times\sum_\beta A_{\alpha\beta}|h_\beta(z)|^2,\quad z=\omega+i\epsilon\,. \tag{4.19}$$

Clearly $H_4(\omega)/\lambda_4$ is positive for $\omega\geq\mu$. Based on (4.19) alone we cannot make such a statement for the other channels; however, we can obtain an approximate result. For ω near the $P33$ resonance the fourth term in each of the sums in (4.19) should dominate, so we have

$$H_\alpha(\omega)/\lambda_\alpha\approx A_{\alpha 4}^{-1}\quad(\omega \text{ near resonance}) \tag{4.20}$$

which is positive for all α. This result indicates that our model is sensible and furthermore suggests [see (4.18) and (4.12)] that the crossing singularity is most important in the $P33$ state.

The potential model presented here is actually more flexible than the CL model, in that it is not necessary to choose the form factor $G_\alpha^{(+)}(p)$ to be the same in all of the channels. This will make it easier to fit the experimental phase shifts. Also, it is straightforward to extend our model to the nonstatic situation, as we now show.

Here, we shall be content to treat the nucleon nonrelativistically. We take for the free Hamiltonian

$$H_0=\sum_\sigma\int|\sigma\vec{p}\vec{P}\rangle d^3p\,d^3P\left(\omega+\frac{P^2}{2m}\right)\langle\vec{P}\vec{p}\sigma|\tau_3\,, \tag{4.21}$$

where $\vec{p}$ and $\vec{P}$ are the momentum of the pion and nucleon, respectively, and m is the nucleon mass. Here

$$|\sigma\vec{p}\vec{P}\rangle=|\psi_{\vec{p}}^\sigma\rangle|\vec{P}\rangle=w^\sigma(\omega)|\vec{p}\rangle|\vec{P}\rangle\,, \tag{4.22}$$

with $|\vec{p}\rangle$ and $|\vec{P}\rangle$ the usual plane wave states. The first term in (4.21) is simply the eigenfunction expansion of the pion's kinetic energy operator; however, the second term is not the usual nonrelativistic kinetic energy operator for the nucleon. The usual choice for the nucleon's kinetic energy operator does not give a reasonable crossing singularity. Using (2.18) we find

$$H_0|\sigma\vec{p}\vec{P}\rangle=\sigma\left(\omega+\frac{P^2}{2m}\right)|\sigma\vec{p}\vec{P}\rangle\,. \tag{4.23}$$

From (3.25) and the completeness relation for the states (4.22), it follows that

$$G_0(z)=\sum_\sigma\int\frac{|\sigma\vec{p}\vec{P}\rangle\sigma d^3p d^3P\langle\vec{P}\vec{p}\sigma|}{z-\sigma(\omega+P^2/2m)}\,. \tag{4.24}$$

We introduce the relative momentum $\vec{q}$ and the total momentum $\vec{Q}$ by

$$\vec{q}=\frac{1}{m+\mu}(m\vec{p}-\mu\vec{P})\,,\qquad \vec{Q}=\vec{p}+\vec{P}, \tag{4.25}$$

in terms of which we have

$$|\sigma\vec{p}\vec{P}\rangle=w^\sigma(\omega)|\vec{q}\rangle|\vec{Q}\rangle\,. \tag{4.26}$$

It should be kept in mind that ω is calculated from (2.15), by inverting (4.25); explicitly,

$$\vec{p}=\vec{q}+\frac{\mu}{m+\mu}\vec{Q}\,. \tag{4.27}$$

Assuming the interaction U depends only on the relative coordinates of the pion and nucleon, we have

$$\langle\vec{P}\vec{p}\sigma|\tau_3U|\sigma\vec{p}\,'\vec{P}\,'\rangle=\langle\vec{q}|w^{\sigma\dagger}(\omega)\tau_3Uw^{\sigma'}(\omega')|\vec{q}\,'\rangle\delta(\vec{Q}-\vec{Q}\,')\,,$$
$$=U^{\sigma\sigma'}(\vec{q},\vec{q}\,';\vec{Q})\delta(\vec{Q}-\vec{Q}\,')\,. \tag{4.28}$$

If we write out (3.23) in the basis (4.26), and use (4.23) and (4.28), we find

$$\langle\vec{P}\vec{p}\sigma|\tau_3T(z)|\sigma\vec{p}\,'\vec{P}\,'\rangle=T^{\sigma\sigma'}(\vec{q},\vec{q}\,';\vec{Q};z)\delta(\vec{Q}-\vec{Q}\,')\,, \tag{4.29}$$

where the T matrix is obtained by solving

$$T^{\sigma\sigma'}(\vec{q},\vec{q}\,';\vec{Q};z)=U^{\sigma\sigma'}(\vec{q},\vec{q}\,';\vec{Q})+\sum_{\sigma''}\int U^{\sigma\sigma''}(\vec{q},\vec{q}\,'';\vec{Q})\times\frac{\sigma''d^3q''T^{\sigma''\sigma'}(\vec{q}\,'',\vec{q}\,';\vec{Q};z)}{z-\sigma(\omega''+P''^2/2m)}\,. \tag{4.30}$$

If we work in the c.m. frame, we have $\vec{q}=\vec{p}$, $\vec{P}=-\vec{p}$, and

$$U^{\sigma\sigma'}(\vec{q},\vec{q}\,';o)=\langle\psi_{\vec{q}}^{\sigma}|\tau_3 U|\psi_{\vec{q}\,'}^{\sigma'}\rangle. \quad (4.31)$$

In the c.m. frame (4.30) is formally the same as (3.30), which means we can use the static limit results if we interpret the pion's momentum as its momentum in the c.m. frame and the energy z as the total pion-nucleon energy (minus the nucleon mass) in that frame. Our pion-nucleon T matrix with recoil effects included will have a pole at $z=0$, a right hand cut beginning at $z=\mu$, and a left hand cut beginning at $z=-\mu$. This singularity structure agrees with a form of the Low equation obtained by Miller,[20] in which terms of order $1/m$ are retained and it is assumed that scattering occurs in relative P waves only. This justifies the choice (4.21) for the free Hamiltonian.

V. DISCUSSION

It is clear that we have found a formulation of the πN interaction which to a large extent bridges the gap between the potential models and the field-theoretic results. In particular, our model has the analytic structure implied by field theory, i.e., right and left hand cuts associated with unitarity and crossing, respectively, and the direct and crossed nucleon poles. As pointed out above, the phenomenological potential models[7-11] do not include the nucleon poles and the crossing singularity. Reiner[22] has found that the separable potential models give half-shell functions which are qualitatively similar to his more fundamental results, except for the $P11$ state. According to him the difference between the results for this state is primarily due to the large contribution from the direct nucleon pole. Thus it is important to include this singularity. It will be interesting to see if the crossing singularity in our model leads to form factors that are qualitatively different from those of the existing separable potentials.

It should be possible to adapt the inversion procedures for the CL theory[16,18] to our model, so as to allow us to construct our form factors directly from the on-shell data. As pointed out above, our T matrix is more flexible than the CL one in that our form factors are state dependent. We are presently working on an inversion procedure.

It will be interesting to see if the techniques of Londergan *et al.*[10] can be adapted to our model. This would make it possible to replace our coupled channel potential (4.2) by a single channel potential with an energy dependent coupling constant. This could show how to extend their approach so as to include the singularities discussed above. In their model the channel coupling comes about because of inelasticity effects, not because of the coupling between positive and negative energy states as in ours. At the present time we are attempting to use their approach in order to include inelasticity effects in our model. Here we hope to make contact with the existing work on the inclusion of inelasticity effects in the CL theory.[16-19]

As pointed out in Sec. I, Miller[20] has developed a potential theory which leads to a T matrix of the CL form. It appears that our approaches agree in spirit, but not in detail. At present we are attempting to find a connection between the two formalisms other than the final results.

Since our πN T matrix is based on an underlying Hamiltonian formalism, it is fairly straightforward to include electromagnetic effects, and to use it in few or many particle problems. In particular, putting in the Coulomb force leads to a theory of the atomic states of the $\pi^- p$ system, which contains a reasonable description of strong interaction effects. With this model it is possible to calculate strong interaction level shifts, as well as widths due to the process in which the proton captures the pion and a photon is emited.

The FV Hamiltonian will lead to somewhat novel Faddeev equations for the $NN\pi$ and $N\pi\pi$ systems. Since our model for the πN T matrix has the crossing singularity built in, it will be possible to investigate some effects due to crossing in these three-particle systems. It is not yet clear whether our T matrix will lead to three-particle equations which are crossing symmetric. This point is presently under investigation. Here we hope to make contact with the existing work on the role of crossing in π-nucleus scattering.[24-26]

It is interesting to note that if one assumes a static nucleon and neglects the π-π interaction, the $N\pi\pi$ Hamiltonian will separate and the three-particle amplitudes will be the same on shell as the two-particle amplitudes. This suggests that a model for the $N\pi\pi$ systems based on our πN T matrix will treat the direct nucleon pole better than the Lovelace[27] and AAY[28] models. In the Lovelace model this pole is put into the equations in an *ad hoc* way, while in the AAY model only the crossed nucleon pole is included.

Finally, it will be interesting to see if our approach can be obtained as a limit of a covariant formulation. There is reason to believe that this is so. Celenza *et al.*[29] have examined relativistic T matrix equations for the πN system. Their Eq. (2.22) bears a strong resemblance to our Eq. (3.30) in that it involves both positive and negative energy meson states. At present we are attempting to use their work as a basis for a covariant generalization of ours.

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