PHYSICAL REVIEW C VOLUME 20, NUMBER 3 SEPTEMBER 1979

Cluster expansions of the three-body problem

D. Eyre and T. A. Osborn
Cyclotron Laboratory, Department of Physics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
(Received 7 May 1979)

This paper derives cluster expansions for the three-body scattering problem. We determine, by computation, the utility of the exact and approximate descriptions that emerge from the cluster approach. In general, cluster expansions can give simple approximate solutions to the scattering process that are accurate whenever clusters dominate the underlying physical states of the system. The approach to the problem taken here is to employ the Karlsson-Zeiger integral equations to provide a theoretical framework that is natural for a cluster expansion. Eventually one can restate the scattering problem in terms of effective intercluster potentials. We construct integral equations whose solutions are the effective potentials. The cluster expansion for this problem leads to successively more exact effective potentials. For systems composed of either three bosons or fermions and interacting through separable potentials we compare exact three-body solutions in the bound state and elastic scattering sectors with those obtained by the cluster-expansion techniques.

[NUCLEAR REACTIONS Three-body problem. Cluster representations. Approximation schemes.]

I. INTRODUCTION

The objective of this paper is to describe cluster expansions of the three-body problem and to assess, by computation, the utility of both the exact and approximate descriptions of the three-body scattering theory that emerge from the cluster approach. In part this work is motivated by the success in nuclear collisions of the resonating group theory to approximately describe the elastic scattering of two tightly bound clusters. However, the resonating group theory treatment of the scattering process[1] is not an exact description of the N-body scattering process and a number of attempts to bring it within the framework of a complete and exact scattering formalism have been made.[2] It is now understood that considerable care is needed, as Adhikari and Glöckle have recently shown,[3] if one is to avoid both spurious solutions of the scattering problem and internal contradictions within the formalism.

The basic goal of any cluster expansion is to arrive at a simple approximate description of the scattering process that is accurate when clustering dominates the underlying physical states of the system. In this paper we obtain several sets of integral equations that feature in a natural way two-cluster states and at the same time remain exact descriptions of three-body scattering. The approximate solutions of these equations obtained by low order iterations define the cluster expansion of the three-body problem and enable us to quantitatively investigate the accuracy of this description and its rate of convergence to the exact three-body scattering solutions.

Our approach to cluster representations is set in the context of the Karlsson-Zeiger (hereafter KZ) integral equations.[4] Generally, integral equations for the three-body problem require a choice of a complete basis in moment space. In the approach of Faddeev[5] the basis is taken to be plane waves of three free particles, whereas in the KZ equations the basis is taken as one free particle adjoined to an interacting pair that is either in a scattering state or a bound state. The resultant KZ equations are structurally different from those found by Faddeev. In particular, the complex three-body energy z appears only in the propagators of the equations. The simplicity of the propagators in the KZ equations allows us to decompose these Green's functions into a sum of two propagators—one contains only a two-cluster state and the other the orthogonal three-particle continuum state. This decomposition in turn allows us to decouple the integral equations so that the continuum and the two-cluster contributions are treated separately. In fact, two sets of cluster expansions result, depending on the order in which one turns on the continuum and cluster states.

The method outlined above for decoupling the KZ equations has been investigated at a formal level by Bollé,[6] and given an explicit kernel form using a three-body scattering wave functions approach by Kuzmichev.[7] The equations outlined in the next two sections in part consolidate and extend the results of these two authors. A feature of considerable interest is the emergence of an effective channel potential that arises when the three-body degrees of freedom are decoupled from the problem. This effective intercluster potential is given as

the solution of an integral equation. Section V describes iterative approximations for the effective channel potentials. The final section describes numerical solutions of the cluster expansions for three equal mass bosons or fermions that interact via identical pairwise separable potentials. We obtain results for the three-body bound state sector as well as the scattering sector below three-body breakup.

II. DECOUPLED KARLSSON-ZEIGER EQUATIONS

In this section we recall the form of the KZ equation, introduce the cluster decomposition of the propagators, and give the operator form of the cluster expansion. We adhere to the choice of three-body momentum variables given by Karlsson and Zeiger, namely, the two independent Jacobi momentum variables in the three-body c.m. system are denoted by $(\vec{p}_\alpha, \vec{q}_\alpha)$. The momentum of particle α having mass m_α is $\vec{p}_\alpha$, and the relative momentum of the pair β and γ is $\vec{q}_\alpha$. The reduced mass of pair $\beta\gamma$ is $\mu_\alpha = m_\beta m_\gamma (m_\beta + m_\gamma)^{-1}$, and the reduced mass of cluster $\beta\gamma$ relative to particle α is $n_\alpha = m_\alpha(m_\beta + m_\gamma)(m_\alpha + m_\beta + m_\gamma)^{-1}$. We often will represent the pair of vectors $(\vec{p}_\alpha, \vec{q}_\alpha)$ by a single six dimensional vector $\vec{p}_0$ and associate with $\vec{p}_0$ a reduced mass $n_0^2 = n_\alpha \mu_\alpha$. In this case the three-body Hamiltonian for kinetic energy H_0 may be represented by

$$H_0 = \frac{p_\alpha^2}{2n_\alpha} + \frac{q_\alpha^2}{2\mu_\alpha} = \frac{p_0^2}{2n_0} . \tag{2.1}$$

The scattering problem is assumed to be initiated in channel α. Particle α, with initial momentum $\vec{p}'_\alpha$, is incident on the pair $\beta\gamma$. The amplitudes for elastic and rearrangement scattering are denoted by $H^+_{\beta\alpha}(\vec{p}_\beta; \vec{p}'_\alpha)$, where $\beta = 1, 2, 3$ and labels the final channel. The breakup amplitude will be the sum over β of the three functions $E^+_{\beta\alpha}(\vec{p}_0; \vec{p}'_\alpha)$. As Karlsson and Zeiger have shown, these six functions satisfy a set of coupled integral equations, whose kernels are given by the set of energy-independent rearrangement potentials,

$$\begin{aligned}
V^{bb}_{\beta\alpha}(\vec{p}_\beta, \vec{p}'_\alpha) &= -\bar{\delta}_{\beta\alpha}\phi_\beta(\vec{q}^{\,(1)}_\beta)\psi_\alpha(\vec{q}^{\,(2)}_\alpha) , \\
V^{bc}_{\beta\alpha}(\vec{p}_\beta, \vec{p}'_0) &= -\bar{\delta}_{\beta\alpha}\phi_\beta(\vec{q}^{\,(1)}_\beta)\Omega^-_\alpha(\vec{q}^{\,(2)}_\alpha, \vec{q}'_\alpha) , \\
V^{cb}_{\beta\alpha}(\vec{p}_0, \vec{p}'_\alpha) &= \bar{\delta}_{\beta\alpha} t^+_\beta(\vec{q}^{\,(1)}_\beta, \vec{q}_\beta)\psi_\alpha(\vec{q}^{\,(2)}_\alpha) , \\
V^{cc}_{\beta\alpha}(\vec{p}_0, \vec{p}'_0) &= \bar{\delta}_{\beta\alpha} t^+_\beta(\vec{q}^{\,(1)}_\beta, \vec{q}_\beta)\Omega^-_\alpha(\vec{q}^{\,(2)}_\alpha, \vec{q}'_\alpha) ,
\end{aligned} \tag{2.2}$$

where $\bar{\delta}_{\beta\alpha} = 1 - \delta_{\beta\alpha}$. The momenta $\vec{q}^{\,(1)}_\beta$ and $\vec{q}^{\,(2)}_\alpha$ are

$$\vec{q}^{\,(1)}_\beta = \vec{p}'_\alpha + \frac{\mu_\beta}{m_\gamma}\vec{p}_\beta , \quad \vec{q}^{\,(2)}_\alpha = -\frac{\mu_\alpha}{m_\gamma}\vec{p}'_\alpha - \vec{p}_\beta . \tag{2.3}$$

The functions $\Omega^\pm_\beta$, $t^\pm_\beta$, ψ_β, and ϕ_β are all constructed from solution of the α-γ two-body problem. If v_β is the two-body interaction and $h^0_\beta = q_\beta^2/2\mu_\beta$ the kinetic energy operator, then $\psi_\beta(\vec{q})$ is the bound state eigenfunction with binding energy ϵ_β, viz.,

$$h_\beta\psi_\beta = (h^0_\beta + v_\beta)\psi_\beta = -\epsilon_\beta\psi_\beta . \tag{2.4}$$

The vertex function ϕ_β is constructed from ψ_β by the equation

$$\phi_\beta(\vec{q}) = \left(\frac{q^2}{2\mu_\beta} + \epsilon_\beta\right)\psi_\beta(\vec{q}) . \tag{2.5}$$

The scattering wave function solutions of h_β are the momentum space matrix elements of the wave operator defined by the time limit

$$\Omega^\pm_\beta = s - \lim_{-t \to \pm\infty} e^{ih_\beta t} e^{-ih^0_\beta t} . \tag{2.6}$$

In terms of $\Omega^\pm_\beta$ the half-on-shell t matrices are

$$t^\pm_\beta(\vec{q}, \vec{q}') = \langle\vec{q}|\, v_\beta\Omega^\pm_\beta|\vec{q}'\rangle = \langle\vec{q}'|\Omega^{\pm\,\dagger}_\beta v_\beta|\vec{q}\rangle . \tag{2.7}$$

The six functions $H^+_{\beta\alpha}$ and $E^+_{\beta\alpha}$ satisfy the coupled integral equations

$$H^+_{\beta\alpha}(\vec{p}_\beta; \vec{p}'_\alpha) = V^{bb}_{\beta\alpha}(\vec{p}_\beta, \vec{p}'_\alpha) - \sum_{\gamma>0} 2n_\gamma \int \frac{V^{bb}_{\beta\gamma}(\vec{p}_\beta, \vec{p}''_\gamma)H^+_{\gamma\alpha}(\vec{p}''_\gamma; \vec{p}'_\alpha)}{p''^2_\gamma - k_\gamma^2 - i0} d\vec{p}''_\gamma - \sum_{\gamma>0} 2n_0 \int \frac{V^{bc}_{\beta\gamma}(\vec{p}_\beta, \vec{p}''_0)E^+_{\gamma\alpha}(\vec{p}''_0; \vec{p}'_\alpha)}{p''^2_0 - k_0^2 - i0} d\vec{p}''_0 , \tag{2.8}$$

$$E^+_{\beta\alpha}(\vec{p}_0; \vec{p}'_\alpha) = V^{cb}_{\beta\alpha}(\vec{p}_0, \vec{p}'_\alpha) - \sum_{\gamma>0} 2n_\gamma \int \frac{V^{cb}_{\beta\gamma}(\vec{p}_0, \vec{p}''_\gamma)H^+_{\gamma\alpha}(\vec{p}''_\gamma; \vec{p}'_\alpha)}{p''^2_\gamma - k_\gamma^2 - i0} d\vec{p}''_\gamma - \sum_{\gamma>0} 2n_0 \int \frac{V^{cc}_{\beta\gamma}(\vec{p}_0, \vec{p}''_0)E^+_{\gamma\alpha}(\vec{p}''_0; \vec{p}'_\alpha)}{p''^2_0 - k_0^2 - i0} d\vec{p}''_0 . \tag{2.9}$$

In describing an arbitrary final state channel γ it is useful to know the magnitude of the momenta allowed by energy conservation. We have reserved the symbol k_γ for this momenta which is determined by $\vec{p}'_\alpha$, the eigenvalues ϵ_γ of Eq. (2.4), and where by convention we set $\epsilon_0 = 0$:

$$k_\gamma = (2n_\gamma)^{1/2}\left(\frac{p'^2_\alpha}{2n_\alpha} - \epsilon_\alpha + \epsilon_\gamma\right)^{1/2} , \quad \gamma = 0, 1, 2, 3 . \tag{2.10}$$

Examining Eqs. (2.8) and (2.9) reveals the basic structure of the KZ equations. The equations form a set of connected scattering equations for the physical amplitudes $H^+_{\beta\alpha}$ and $E^+_{\beta\alpha}$. The kernels depend on simple energy-independent potential-like functions given in Eq. (2.2). These potentials utilize only half-off-shell two-body t-matrix information determined at positive two-body scattering energies. In Eqs. (2.8) and (2.9) the singu-

lar denominator $(p_\gamma''^2 - k_\gamma^2 - i0)^{-1}$ describes propagation of the cluster ψ_γ; the denominator $(p_0''^2 - k_0^2 - i0)^{-1}$ gives the propagation of an intermediate state that is composed of a free three-particle continuum.

We now turn our attention to the problem of decoupling these equations. In the process of solving this problem we will automatically construct a cluster expansion of the three-body problem. Define for complex z the Green's functions $G(z)$ and $G_\alpha(z)$ and $G_\alpha(z)$ by $(H-z)^{-1}$ and $(H_\alpha - z)^{-1}$, respectively. The transition operator $U_{\beta\alpha}(z)$ of Alt, Grassberger, and Sandhas[8] is determined by

$$G(z) = \delta_{\alpha\beta} G_\beta(z) - G_\beta(z) U_{\beta\alpha}(z) G_\alpha(z) . \tag{2.11}$$

Use $U_{\beta\alpha}(z)$ to define a related operator $T_{\beta\alpha}(z)$ by

$$T_{\beta\alpha}(z) = V_\beta G_0(z) U_{\beta\alpha}(z) G_0(z) V_\alpha, \quad \alpha > 0, \ \beta > 0 . \tag{2.12}$$

It is a simple matter to show that $T_{\beta\alpha}(z)$ satisfies the equation

$$T_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} V_\beta G_0(z) V_\alpha - \sum_\gamma \bar{\delta}_{\beta\gamma} V_\beta G_\gamma(z) T_{\gamma\alpha}(z) , \tag{2.13}$$

where γ is summed over $1, 2, 3$. It is well known[4] that the interacting matrix elements of $T_{\beta\alpha}(z)$ give representations of the physical transition amplitudes in momentum space. Specifically, if $s_\alpha = p_\alpha'^2/2n_\alpha - \epsilon_\alpha$ is the available incident energy, then

$$H^+_{\beta\alpha}(\vec{p}_\beta; \vec{p}'_\alpha) = \langle \vec{p}_\beta \psi_\beta | T_{\beta\alpha}(s_\alpha + i0) | \vec{p}'_\alpha \psi_\alpha \rangle \tag{2.14}$$

$$E^+_{\beta\alpha}(\vec{p}_0; \vec{p}'_\alpha) = \langle \vec{p}_\beta \Omega^-_\beta(\cdot, \vec{q}_\beta) | T_{\beta\alpha}(s_\alpha + i0) | \vec{p}'_\alpha \psi_\alpha \rangle . \tag{2.15}$$

The half-off-shell function $H^+_{\beta\alpha}$ is identical to the function appearing in Faddeev's description of the elastic and rearrangement amplitude, whereas $E^+_{\beta\alpha}$—the β component of the breakup amplitude —differs from Faddeev's breakup representation and the two are only equal for on-shell kinematics.

To proceed with the decoupling of the KZ equations, it is convenient to write Eq. (2.13) in a matrix form. Let V, $\bar{V}$, $R_0(z)$, $R(z)$, and $T(z)$ denote the 3×3 matrices with operator elements $\delta_{\alpha\beta} V_\alpha$, $V_\alpha \bar{\delta}_{\alpha\beta}$, $\delta_{\alpha\beta} G_0(z)$, $\delta_{\alpha\beta} G_\alpha(z)$, and $T_{\alpha\beta}(z)$. Then Eq. (2.13) can be written

$$T(z) = \bar{V}[-R_0(z) V] - \bar{V} R(z) T(z) . \tag{2.16}$$

Now expand $G_\alpha(z)$ into cluster and continuum parts. Let the projection operators P^b_α and P^c_α be defined by

$$\langle \vec{p}_\alpha \vec{q}_\alpha | P^b_\alpha | \vec{p}''_\alpha \vec{q}''_\alpha \rangle = \delta(\vec{p}_\alpha - \vec{p}''_\alpha) \psi_\alpha(\vec{q}_\alpha) \psi_\alpha(\vec{q}''_\alpha)^* , \tag{2.17}$$

$$\langle \vec{p}_\alpha \vec{q}_\alpha | P^c_\alpha | \vec{p}''_\alpha \vec{q}''_\alpha \rangle = \delta(\vec{p}_\alpha - \vec{p}''_\alpha) \langle \vec{q}_\alpha | \Omega^-_\alpha \Omega^{-\dagger}_\alpha | \vec{q}''_\alpha \rangle . \tag{2.18}$$

The projectors P^b_α and P^c_α have orthogonal ranges, diagonalize H_α, and span the three-body Hilbert space. Thus one can write

$$G_\alpha(z) = G^b_\alpha(z) + G^c_\alpha(z) , \tag{2.19}$$

$$G^b_\alpha(z) = P^b_\alpha G_\alpha(z) P^b_\alpha, \quad G^c_\alpha(z) = P^c_\alpha G_\alpha(z) P^c_\alpha . \tag{2.20}$$

Note that the product $G^b_\alpha(z) G^c_\alpha(z) = 0$. Using this decomposition to write $R(z)$ as the sum of a cluster state and a continuum state gives us

$$R(z) = R^b(z) + R^c(z) , \tag{2.21}$$

where $[R^{b,c}(z)]_{\alpha\beta} = \delta_{\alpha\beta} G^{b,c}_\alpha(z)$.

The next stage is to decouple the equations for $H^+_{\beta\alpha}$ and $E^+_{\beta\alpha}$. We assume that $[1 + \bar{V} R^c(z)]^{-1}$ exists. This is very probable since the operator $\bar{V} R^c(z)$ is completely connected. In this circumstance the equation

$$T^c(z) = \bar{V} - \bar{V} R^c(z) T^c(z) \tag{2.22}$$

has the formal solution

$$T^c(z) = [1 + \bar{V} R^c(z)]^{-1} \bar{V} . \tag{2.23}$$

Using Eq. (2.21) in (2.16) leads to

$$[1 + \bar{V} R^c(z)] T(z) = \bar{V}[-R_0(z) V] - \bar{V} R^b(z) T(z) . \tag{2.24}$$

Applying the inverse of the left factor gives

$$T(z) = T^c(z)[-R_0(z) V] - T^c(z) R^b(z) T(z) . \tag{2.25}$$

At a formal level Eqs. (2.22) and (2.25) constitute a decoupling of the scattering problem. In Eq. (2.22) the only intermediate states that can propagate are continuum states. Given knowledge of $T^c(z)$, then Eq. (2.25) essentially turns on the cluster features of the scattering problem.

We note that the order in which cluster structure may be turned on is arbitrary. So assuming $[1 + \bar{V} R^b(z)]^{-1}$ exists, we can define $T^b(z)$ as given by the solution of

$$T^b(z) = \bar{V} - \bar{V} R^b(z) T^b(z) \tag{2.26}$$

and

$$T^b(z) = [1 + \bar{V} R^b(z)]^{-1} \bar{V} . \tag{2.27}$$

Then, arguing as above, we have

$$T(z) = T^b(z)[-R_0(z) V] - T^b(z) R^c(z) T(z) . \tag{2.28}$$

In this order, one solves for the effect of the clusters first and then, in Eq. (2.28), takes into account the changes in the system due to the continuum states. The statements (2.25) and (2.28) reproduce the results first obtained by Bollé.[6] However, in order to see the full physical content of the operator identities in Eqs. (2.22)–(2.28), one must take matrix elements of these equations with respect to the interacting two-particle basis sets. This is the task of the next section.

III. INTERACTING REPRESENTATIONS

We investigate now the consequence of the identities (2.22) and (2.25). In component form these two equations read:

$$T^c_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} V_\beta - \sum_\gamma \bar{\delta}_{\beta\gamma} V_\beta G^c_\gamma(z) T^c_{\gamma\alpha}(z) , \tag{3.1}$$

$$T_{\beta\alpha}(z) = T^c_{\beta\alpha}(z)[-G_0(z) V_\alpha] - \sum_\gamma T^c_{\beta\gamma}(z) G^b_\gamma(z) T_{\gamma\alpha}(z) . \tag{3.2}$$

After appropriate matrix elements are formed, and the completeness of the two-body wave operators Ω^-_γ is utilized, operator identities (3.1) and (3.2) will give us four independent equations—two integral equations and two quadrature relations. We define a quasibreakup amplitude $E^c_{\beta\alpha}$ from the operator $T^c_{\beta\alpha}(z)$ by

$$E^c_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha) = \langle \vec{p}_\beta \Omega^-_\beta(\cdot, \vec{q}_\beta) | T^c_{\beta\alpha}(s_\alpha + i0) | \vec{p}'_\alpha \psi_\alpha \rangle . \tag{3.3}$$

This definition clearly parallels that of Eq. (2.15) for the β component of the physical breakup amplitude. Now multiply Eq. (3.1) from the left by $\langle \vec{p}_\beta \Omega^-_\beta(\cdot, \vec{q}_\beta)|$, from the right by $|\vec{p}'_\alpha \psi_\alpha\rangle$, and set $z = s_\alpha + i0$. It is readily found that

$$E^c_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha) = V^{cb}_{\beta\alpha}(\vec{p}_0,\vec{p}'_\alpha) - \sum_{\gamma>0} 2n_0 \int \frac{V^{cc}_{\beta\gamma}(\vec{p}_0,\vec{p}''_0) E^c_{\gamma\alpha}(\vec{p}''_0;\vec{p}'_\alpha)}{p''^2_0 - k_0^2 - i0} d\vec{p}''_0 . \tag{3.4}$$

This is an integral equation for $E^c_{\beta\alpha}$. The kernel and driving terms are constructed from the KZ potentials $V^{cc}_{\beta\alpha}$ and $V^{cb}_{\beta\alpha}$.

The second relation to be extracted from Eq. (3.1) is obtained by using states $\langle \vec{p}_\beta \psi_\beta|$ and $|\vec{p}'_\alpha \psi_\alpha\rangle$ to construct matrix elements. We shall need the two-clusterlike amplitude

$$H^c_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha) = \langle \vec{p}_\beta \psi_\beta | T^c_{\beta\alpha}(s_\alpha + i0) | \vec{p}'_\alpha \psi_\alpha \rangle ; \tag{3.5}$$

then Eq. (3.1) becomes

$$H^c_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha) = V^{bb}_{\beta\alpha}(\vec{p}_\beta,\vec{p}'_\alpha) - \sum_{\gamma>0} 2n_0 \int \frac{V^{bc}_{\beta\gamma}(\vec{p}_\beta,\vec{p}''_0) E^c_{\gamma\alpha}(\vec{p}''_0,\vec{p}'_\alpha)}{p''^2_0 - k_0^2 - i0} d\vec{p}''_0 . \tag{3.6}$$

Given $E^c_{\gamma\alpha}$, Eq. (3.6) is a quadrature relation for $H^c_{\beta\alpha}$. Finally, turning to Eq. (3.2), its two-cluster matrix elements are obviously

$$H^+_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha) = H^c_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha) - \sum_{\gamma>0} 2n_\gamma \int \frac{H^c_{\beta\gamma}(\vec{p}_\beta;\vec{p}''_\gamma) H^+_{\gamma\alpha}(\vec{p}''_\gamma;\vec{p}'_\alpha)}{p''^2_\gamma - k_\gamma^2 - i0} d\vec{p}''_\gamma . \tag{3.7}$$

In obtaining Eq. (3.7) we have employed the identity

$$-G_0(s_\alpha + i0) V_\alpha |\vec{p}'_\alpha \psi_\alpha\rangle = |\vec{p}'_\alpha \psi_\alpha\rangle . \tag{3.8}$$

This latter result is nothing more than the Schrödinger equation for the state $|\vec{p}'_\alpha \psi_\alpha\rangle$.

Equations (3.4), (3.6), and (3.7) form a solvable system. In Eq. (3.4) we solve a three-body problem that allows only continuum intermediate states. Then via Eq. (3.6) we take $E^c_{\beta\alpha}$ and by integration determine an effective potential function $H^c_{\beta\alpha}$. Lastly, this effective potential is used in the coupled cluster structure given by Eq. (3.7) to find the exact elastic and rearrangement amplitudes $H^+_{\beta\alpha}$. Taken together, this system of three equations provides a cluster description of three-body scattering which is exact. The effect of the intermediate three-particle continuum states is summed up into the form of the effective two-cluster potentials $H^c_{\beta\alpha}$. Iterative approximations to Eq. (3.4) will automatically define a cluster expansion of the three-body problem wherein one can take into account successively more accurate, but approximate effects of the three-body continuum. At each stage of the iteration of Eq. (3.4) one is able to construct a more accurate approximation to the exact effective potential $H^c_{\beta\alpha}$.

One additional equation is needed to make the system (3.4)–(3.7) complete. We must find the breakup function $E^+_{\beta\alpha}$. This is given by the matrix element of Eq. (3.2) with respect to $\langle \vec{p}_\beta \Omega^-_\beta(\cdot, \vec{q}_\beta)|$ and $|\vec{p}'_\alpha \psi_\alpha\rangle$. The result is

$$E^+_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha) = E^c_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha) - \sum_{\gamma>0} 2n_\gamma \int \frac{E^c_{\beta\gamma}(\vec{p}_0;\vec{p}''_\gamma) H^+_{\gamma\alpha}(\vec{p}''_\gamma;\vec{p}'_\alpha)}{p''^2_\gamma - k_\gamma^2 - i0} d\vec{p}''_\gamma . \tag{3.9}$$

This is a quadrature relation for $E^+_{\beta\alpha}$. In order to use it we must first have solved for $H^+_{\gamma\alpha}$. Essentially the integral on the right of Eq. (3.9) computes for us the difference between the quasibreakup component $E^c_{\beta\alpha}$ and exact breakup function $E^+_{\beta\alpha}$. This difference is just due to the effect of including the influence of the cluster structure on the breakup amplitude.

Our second set of decoupled equations are derived by taking matrix elements of the identities (2.26) and (2.28). The component form of Eqs. (2.26) and (2.28) reads

$$T^b_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} V_\beta - \sum_\gamma \bar{\delta}_{\beta\gamma} V_\beta G^b_\gamma(z) T^b_{\gamma\alpha}(z) , \tag{3.10}$$

$$T_{\beta\alpha}(z) = T^b_{\beta\alpha}(z)[-G_0(z) V_\alpha] - \sum_\gamma T^b_{\beta\gamma}(z) G^c_\gamma(z) T_{\gamma\alpha}(z) . \tag{3.11}$$

Equation (3.10) leads to two integral equations,

one for the case when the right-hand state is a cluster and one for the continuum case. We define

$$H^b_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)=\langle \vec{p}_\beta\psi_\beta|\,T^b_{\beta\alpha}(s_\alpha+i0)|\vec{p}'_\alpha\psi_\alpha\rangle\,, \tag{3.12}$$

$$D^b_{\beta\alpha}(\vec{p}_\beta;\vec{p}''_0;s_\alpha+i0)=\langle \vec{p}_\beta\psi_\beta|\,T^b_{\beta\alpha}(s_\alpha+i0)|\vec{p}''_\alpha\Omega^-_\alpha(\cdot,\vec{q}''_\alpha)\rangle\,. \tag{3.13}$$

These amplitudes are the solutions of

$$H^b_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)=V^{bb}_{\beta\alpha}(\vec{p}_\beta,\vec{p}'_\alpha)-\sum_{\gamma>0}2n_\gamma\int\frac{V^{bb}_{\beta\gamma}(\vec{p}_\beta,\vec{r}_\gamma)H^b_{\gamma\alpha}(\vec{r}_\gamma;\vec{p}'_\alpha)}{r_\gamma^{\,2}-k_\gamma^{\,2}-i0}\,d\vec{r}_\gamma\,, \tag{3.14}$$

$$D^b_{\beta\alpha}(\vec{p}_\beta;\vec{p}''_0;s_\alpha+i0)=V^{bc}_{\beta\alpha}(\vec{p}_\beta,\vec{p}''_0)-\sum_{\gamma>0}2n_\gamma\int\frac{V^{bb}_{\beta\gamma}(\vec{p}_\beta,\vec{r}_\gamma)D^b_{\gamma\alpha}(\vec{r}_\gamma;\vec{p}''_0;s_\alpha+i0)}{r_\gamma^{\,2}-k_\gamma^{\,2}-i0}\,d\vec{r}_\gamma\,. \tag{3.15}$$

Associated with Eq. (3.10) are two quadrature relations that determine the functions

$$E^b_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)=\langle \vec{p}_\beta\Omega^-_\beta(\cdot,\vec{q}_\beta)|\,T^b_{\beta\alpha}(s_\alpha+i0)|\vec{p}'_\alpha\psi_\alpha\rangle \tag{3.16}$$

$$F^b_{\beta\alpha}(\vec{p}_0;\vec{p}''_0;s_\alpha+i0)=\langle \vec{p}_\beta\Omega^-_\beta(\cdot,\vec{q}_\beta)|\,T^b_{\beta\alpha}(s_\alpha+i0)|\vec{p}''_\alpha\Omega^-_\alpha(\cdot,\vec{q}''_\alpha)\rangle\,. \tag{3.17}$$

These two functions are computed from $H^b_{\beta\alpha}$ and $D^b_{\beta\alpha}$ by

$$E^b_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)=V^{cb}_{\beta\alpha}(\vec{p}_0,\vec{p}'_\alpha)-\sum_{\gamma>0}2n_\gamma\int\frac{V^{cb}_{\beta\gamma}(\vec{p}_0,\vec{r}_\gamma)H^b_{\gamma\alpha}(\vec{r}_\gamma;\vec{p}'_\alpha)}{r_\gamma^{\,2}-k_\gamma^{\,2}-i0}\,d\vec{r}_\gamma\,, \tag{3.18}$$

$$F^b_{\beta\alpha}(\vec{p}_0;\vec{p}''_0;s_\alpha+i0)=V^{cc}_{\beta\alpha}(\vec{p}_0,\vec{p}''_0)-\sum_{\gamma>0}2n_\gamma\int\frac{V^{cb}_{\beta\gamma}(\vec{p}_0,\vec{r}_\gamma)D^b_{\gamma\alpha}(\vec{r}_\gamma;\vec{p}''_0;s_\alpha+i0)}{r_\gamma^{\,2}-k_\gamma^{\,2}-i0}\,d\vec{r}_\gamma\,. \tag{3.19}$$

The linkage of the four functions $H^b_{\beta\alpha}$, $D^b_{\beta\alpha}$, $E^b_{\beta\alpha}$, and $F^b_{\beta\alpha}$ to the physical observables is dictated by Eq. (3.11). For example, take the matrix elements of (3.11) that lead to the physical breakup amplitude (2.15). We obtain

$$E^+_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)=E^b_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)-\sum_{\gamma>0}2n_0\int\frac{F^b_{\beta\gamma}(\vec{p}_0;\vec{p}''_0;s_\alpha+i0)E^+_{\gamma\beta}(\vec{p}''_0;\vec{p}'_\alpha)}{p_0''^2-k_0^{\,2}-i0}\,d\vec{p}''_0\,. \tag{3.20}$$

This is an integral equation for the breakup amplitude with kernels given by the functions $E^b_{\beta\alpha}$, $F^b_{\beta\gamma}$. Finally, the set of rearrangement and elastic amplitudes is determined from Eq. (3.11) by taking two-cluster matrix elements. One finds that

$$H^+_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)=H^b_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)-\sum_{\gamma>0}2n_0\int\frac{D^b_{\beta\gamma}(\vec{p}_\beta;\vec{p}''_0;s_\alpha+i0)E^+_{\gamma\alpha}(\vec{p}''_0,\vec{p}'_\alpha)}{p_0''^2-k_0^{\,2}-i0}\,d\vec{p}''_0\,. \tag{3.21}$$

The set of six equations (3.10)–(3.11), (3.14)–(3.15), and (3.20)–(3.21) are a solvable system of three integral equations and three quadrature relations. Evidently this set of equations is more elaborate than those of the first decoupling scheme. However, the physical interpretation of our equations is again evident. The functions $H^b_{\beta\alpha}$ are the two-cluster scattering amplitudes in the situation where there are no intermediate three-particle continuum states. $D^b_{\beta\alpha}$ gives an off-shell 3-2 transition amplitude that results if all intermediate states are two-cluster states. The last pair of equations (3.20)–(3.21) allow one to compute the difference between the exact amplitudes $E^+_{\beta\alpha}$ and $H^+_{\beta\alpha}$ and the two-cluster approximations $E^b_{\beta\alpha}$ and $H^b_{\beta\alpha}$.

Both sets of decoupled equations share the feature of unlinking the continuum and two-particle states. This structure then gives us the opportunity to treat either the continuum intermediate states or the two-cluster intermediate states in a perturbation expansion. An expansion of this sort is attractive because it has a well-defined physical meaning.

It is helpful to have simple names to distinguish the two different decoupling schemes we have derived in this section. Because the first method, Eqs. (3.1)–(3.9), reduces the three-body problem, via integral equation (3.7) to a set of coupled two-cluster channels interacting through the effective channel potentials $H^c_{\beta c}(\vec{p}_\beta,\vec{p}''_\gamma)$, we shall call this the effective potential representation (hereafter EPR). Our second decoupling scheme determines $H^+_{\beta\alpha}$ with the integral equation (3.20) which has only continuum intermediate states. So we shall denote this representation by the initials CIS.

Finally recall that the solubility of the decoupled schemes EPR and CIS requires that (3.2) and (3.10) have unique solutions. In a previous study of the KZ equations we have shown how to introduce an algebraic decomposition of the amplitudes that transforms the KZ equations into a Fredholm equation with nonsingular kernel.[9] Similarly

(3.4), (3.14), and (3.15) can be brought into a Fredholm form. Thus Eqs. (3.6), (3.14), and (3.15) obey the Fredholm alternative and have unique solution, provided that the corresponding homogeneous equations have no solution. A complete mathematical proof of uniqueness would now show that these homogeneous equations can have no solutions or only have solutions in exceptional situations. We have dealt with this problem by solving the numerical equivalents of (3.6), (3.14), and (3.15). For three-body energies in the bound state and two-cluster scattering sector (i.e., below breakup) we have always found unique solutions.

IV. CLUSTER REPRESENTATIONS OF THE BOUND STATE

In this section we present the decoupled integral equations that occur in the EPR approach to the three-body bound state. Let the ith bound state wave function of the full three-particle Hamiltonian H have energy $-s_i<0$. Faddeev's solution to this problem is to represent the wave function as a sum of three terms, $|\Psi_\beta\rangle$. These three functions are the solution of the coupled homogeneous system

$$|\Psi_\beta\rangle=-\sum_\gamma G_\beta(z)V_\beta\bar{\delta}_{\beta\gamma}|\Psi_\gamma\rangle \tag{4.1}$$

for $z=-s_i$. We find it convenient to define a modified Faddeev component function given by

$$(H_\beta-z)|\Psi_\beta\rangle=|\Phi_\beta\rangle,\quad \beta=1,2,3. \tag{4.2}$$

In terms of $|\Phi_\beta\rangle$ the bound state analogs of the scattering amplitudes, $H^+_{\beta\alpha}$ and $E^+_{\beta\alpha}$, are

$$H^+_\beta(\vec{p}_\beta)=\langle\vec{p}_\beta\psi_\beta|\Phi_\beta\rangle,\quad E^+_\beta(\vec{p}_0)=\langle\vec{p}_\beta\Omega^-_\beta(\cdot,\vec{q}_\beta)|\Phi_\beta\rangle. \tag{4.3}$$

No label α occurs with these amplitudes since there is no incoming state. In terms of H^+_β and E^+_β the cluster and continuum projections of the Faddeev components $|\Psi_\beta\rangle$ are

$$\langle\vec{p}_\beta\psi_\beta|\Psi_\beta\rangle=(p_\beta{}^2/2n_\beta-\epsilon_\beta+s_i)^{-1}H^+_\beta(\vec{p}_\beta),$$
$$\langle\vec{p}_\beta\Omega^-_\beta(\cdot,\vec{q}_\beta)|\Psi_\beta\rangle=(p_\beta{}^2/2n_\beta+q_\beta{}^2/2\mu_\beta+s_i)^{-1}E^+_\beta(\vec{p}_0). \tag{4.4}$$

In many ways the functions $H^+_\beta(\vec{p}_\beta)$ and $E^+_\beta(\vec{p}_0)$ are the generalization of the two-body vertex function defined in Eq. (2.5). The conventional plane wave representation of $|\Psi_\beta\rangle$ may be determined from H^+_β and E^+_β by

$$\langle\vec{p}_0|\Psi_\beta\rangle=\frac{2n_\beta\psi_\beta(\vec{q}_\beta)H^+_\beta(\vec{p}_\beta)}{p_\beta{}^2-2n_\beta(\epsilon_\beta-s_i)}+\int\frac{\Omega^-_\beta(\vec{q}_\beta,\vec{q}_\beta'')E^+_\beta(\vec{p}_\beta,\vec{q}_\beta'')}{p_\beta{}^2/2n_\beta+q_\beta''{}^2/2\mu_\beta+s_i}\,d\vec{q}_\beta''. \tag{4.5}$$

This equation is the immediate consequence of the definitions (4.4) and the completeness of Ω^-_β and ψ_β.

The general relationships outlined above provide the bound state wave function once H^+_β and E^+_β are known. The first thing to note is that H^+_β and E^+_β are solutions of an appropriate homogeneous KZ equation. Equation (4.1) implies $|\Phi_\beta\rangle$ satisfies

$$|\Phi_\beta\rangle=-\sum_\gamma\bar{\delta}_{\beta\gamma}V_\beta G_\gamma(z)|\Phi_\gamma\rangle,\quad z=-s_i. \tag{4.6}$$

Taking the interacting matrix elements of this equation shows that H^+_β and E^+_β are solutions of the homogeneous form of Eqs. (2.8) and (2.9), with the propagators $(p_\gamma''{}^2-k_\gamma{}^2-i0)^{-1}$ and $(p_0''{}^2-k_0{}^2-i0)^{-1}$ replaced by $(p_\gamma''{}^2-2n_\gamma\epsilon_\gamma+2n_\gamma s_i)^{-1}$ and $(p_0''{}^2+2n_0s_i)^{-1}$, respectively.

The operator equations (4.6) may be decoupled by the same analysis as in Eqs. (2.19)–(2.25). It is readily shown that

$$|\Phi_\beta\rangle=-\sum_\gamma T^c_{\beta\gamma}(z)G^b_\gamma(z)|\Phi_\gamma\rangle,\quad z=-s_i \tag{4.7}$$

where $T^c_{\beta\gamma}(z)$ is given by the solution of Eq. (3.1). In fact, if $E^c_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha;-s_i)$ is the solution of Eq. (3.4) with $-k_0{}^2$ replaced by $2n_0s_i$, then the effective channel potential is

$$H^c_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha;-s_i)=V^{bb}_{\beta\alpha}(\vec{p}_\beta,\vec{p}'_\alpha)-\sum_{\gamma>0}2n_0\int\frac{V^{bc}_{\beta\gamma}(\vec{p}_\beta,\vec{p}_0'')E^c_{\gamma\alpha}(\vec{p}_0'';\vec{p}'_\alpha;-s_i)}{p_0''{}^2+2n_0s_i}\,d\vec{p}_0''. \tag{4.8}$$

The homogeneous integral equation for the bound state cluster amplitudes H^+_β is determined by taking two-cluster matrix elements of Eq. (4.7). One finds that

$$H^+_\beta(\vec{p}_\beta)=-\sum_{\gamma>0}2n_\gamma\int\frac{H^c_{\beta\gamma}(\vec{p}_\beta;\vec{p}_\gamma'';-s_i)H^+_\gamma(\vec{p}_\gamma'')}{p_\gamma''{}^2-2n_\gamma\epsilon_\gamma+2n_\gamma s_i}\,d\vec{p}_\gamma''. \tag{4.9}$$

Finally, the continuum component of $|\Phi_\beta\rangle$ is predicted by the quadrature relation

$$E^+_\beta(\vec{p}_0)=-\sum_{\gamma>0}2n_\gamma\int\frac{E^c_{\beta\alpha}(\vec{p}_0;\vec{p}_\gamma'';-s_i)H^+_\gamma(\vec{p}_\gamma'')}{p_\gamma''{}^2-2n_\gamma\epsilon_\gamma+2n_\gamma s_i}\,d\vec{p}_\gamma''. \tag{4.10}$$

This completes our cluster description of the three-body bound state problem. Note that one can obtain a quantitative definition of the fraction of the three-body bound state that is in a cluster form by computing the ratios of the norm of the first term on the right of Eq. (4.5) over that of the entire term $\langle\vec{p}_0|\Psi_\beta\rangle$.

V. EXPANSION APPROXIMATIONS

This section will develop the iterative expansion of the EPR system of equations. Iterative approximations to Eq. (3.4) will define approximations to the effective channel potentials $H^c_{\beta\alpha}(\vec{p}_\beta;\vec{p}\,'_\alpha)$. We establish that the first iterate effective potential contains the impulse approximation. At the close of this section we indicate how the system of equations simplifies if the basic pairwise interaction is separable.

We define the zeroth iterate of $E^c_{\beta\alpha}$ and $H^c_{\beta\alpha}$ by

$$E^{c,0}_{\beta\alpha}(\vec{p}_0;\vec{p}\,'_\alpha)=0\,,\quad H^{c,0}_{\beta\alpha}(\vec{p}_\beta;\vec{p}\,'_\alpha)=V^{bb}_{\beta\alpha}(\vec{p}_\beta,\vec{p}\,'_\alpha)\,. \tag{5.1}$$

The ith iterate of Eq. (3.4) is defined by the recursion relation

$$E^{c,i+1}_{\beta\alpha}(\vec{p}_0;\vec{p}\,'_\alpha)=V^{cb}_{\beta\alpha}(\vec{p}_0,\vec{p}\,'_\alpha)-\sum_{\gamma>0}2n_0\int\frac{V^{cc}_{\beta\gamma}(\vec{p}_0,\vec{p}\,''_0)E^{c,i}_{\gamma\alpha}(\vec{p}\,''_0,\vec{p}\,'_\alpha)}{p''^2_0-k_0^2-i0}\,d\vec{p}\,''_0\,. \tag{5.2}$$

In terms of the ith iterate for the pseudobreakup amplitude $E^{c,i}_{\beta\alpha}$ there is a corresponding effective channel potential given by

$$H^{c,i}_{\beta\alpha}(\vec{p}_\beta;\vec{p}\,'_\alpha)=V^{bb}_{\beta\alpha}(\vec{p}_\beta,\vec{p}\,'_\alpha)-\sum_{\gamma>0}2n_0\int\frac{V^{bc}_{\beta\gamma}(\vec{p}_\beta,\vec{p}\,''_0)E^{c,i}_{\gamma\alpha}(\vec{p}\,''_0;\vec{p}\,'_\alpha)}{p''^2_0-k_0^2-i0}\,d\vec{p}\,''_0\,. \tag{5.3}$$

The last step in obtaining an approximation for the physical scattering amplitude is to use the approximate potential $H^{c,i}_{\beta\alpha}$ in place of $H^c_{\beta\alpha}$ in Eq. (3.7). The corresponding solution of Eq. (3.7) we will denote $H^{+,i}_{\beta\alpha}$. In all the examples studied numerically in this paper

$$H^{c,i}_{\beta\alpha}\to H^c_{\beta\alpha}\,,\quad H^{+,i}_{\beta\alpha}\to H^+_{\beta\alpha}\ \text{as}\ i\to\infty\,. \tag{5.4}$$

The convergence property reflects the fact that Eq. (3.4) may be solved by iteration. This is characteristic of many physical circumstances where it occurs that the coupling of the three-particle continuum to itself is weak. Technically this means that the operator $\overline{V}R^c(z)$ has all of its eigenvalues inside the unit circle of the complex plane. This will not always be the case and alternate approximate methods can be devised to deal with this situation. The last stage of obtaining a solution is to solve the coupled channel problem given in Eq. (3.7). Essentially Eq. (3.7) is a kinematic copy of the two-body problem and is always easy to solve whatever the strength of the coupling.

We turn now to the question of whether or not the approximate solutions obtained by iteration are self-consistent with the impulse approximation. We prove that the first iterate contains the impulse approximation. To this end we establish that

$$H^{\rm imp}_{\alpha\alpha}(\vec{p}_\alpha;\vec{p}\,'_\alpha)=H^{c,1}_{\alpha\alpha}(\vec{p}_\alpha;\vec{p}\,'_\alpha)-\sum_{\gamma>0}2n_\gamma\int\frac{V^{bb}_{\alpha\gamma}(\vec{p}_\alpha,\vec{p}\,''_\gamma)H^{c,0}_{\gamma\alpha}(\vec{p}\,''_\gamma;\vec{p}\,'_\alpha)}{p''^2_\gamma-k_\gamma^2-i0}\,d\vec{p}\,''_\gamma\,, \tag{5.5}$$

where the left-hand side is the standard expression for the impulse approximation given by

$$H^{\rm imp}_{\alpha\alpha}(\vec{p}_\alpha;\vec{p}\,'_\alpha)=\sum_{\gamma\neq0}\int\frac{\phi_\alpha(\vec{q}^{\,(1)}_\alpha)\langle\vec{q}^{\,(2)}_\gamma|\,t_\gamma(\tilde{p}^{\,2}_\alpha-\epsilon_\alpha-\tilde{p}''^2_\gamma+i0)\,|\vec{q}^{\,(1)}_\gamma\rangle\psi_\alpha(\vec{q}^{\,(2)}_\alpha)}{\tilde{q}^{(2)2}_\gamma+\tilde{p}''^2_\gamma-\tilde{p}^{\,2}_\alpha+\epsilon_\alpha-i0}\,d\vec{p}\,''_\gamma\,, \tag{5.6}$$

where $\tilde{p}_\gamma^{\,2}=p_\gamma^{\,2}/2n_\gamma$ and $\tilde{q}_\gamma^{\,2}=q_\gamma^{\,2}/2\mu_\gamma$. The vectors $\vec{q}^{\,(1,2)}_\gamma$ are

$$\vec{q}^{\,(1)}_\gamma=-\vec{p}\,'_\alpha-\frac{\mu_\gamma}{m_\beta}\vec{p}\,''_\gamma\,,\quad \vec{q}^{\,(2)}_\gamma=-\vec{p}_\alpha-\frac{\mu_\gamma}{m_\beta}\vec{p}\,''_\gamma\,. \tag{5.7}$$

To prove Eq. (5.5), substitute the definitions into the right-hand side of Eq. (5.5). The right-hand side is then equal to

$$\sum_{\gamma\neq\alpha}\int\phi_\alpha(\vec{q}^{\,(1)}_\alpha)\left[\int\frac{\Omega^-_\gamma(\vec{q}^{\,(2)}_\gamma,\vec{q}\,''_\gamma)t^+_\gamma(\vec{q}^{\,(1)}_\gamma,\vec{q}\,''_\gamma)}{\tilde{p}''^2_\gamma+\tilde{q}''^2_\gamma-\tilde{p}^{\,\prime2}_\alpha+\epsilon_\alpha-i0}\,d\vec{q}\,''_\gamma-\frac{\psi_\gamma(q^{(2)}_\gamma)\phi_\gamma(q^{(1)}_\gamma)}{\tilde{p}''^2_\gamma-\epsilon_\gamma-\tilde{p}^{\,\prime2}_\alpha+\epsilon_\alpha-i0}\right]\psi_\alpha(\vec{q}^{\,(2)}_\alpha)\,d\vec{p}\,''_\gamma\,. \tag{5.8}$$

The term inside the brackets can be reduced to the matrix elements of a single off-shell t matrix. Recall the standard two-body identity

$$g_0(z)t(z)=g(z)v\,. \tag{5.9}$$

Take the $\langle\vec{q}\,|$ and $|\vec{q}\,'\rangle$ matrix elements of Eq. (5.9) and use the completeness of the set Ω^- and ψ to evaluate the product $g(z)v$. The identity

$$\frac{\langle\vec{q}\,|\,t(z)\,|\vec{q}\,'\rangle}{\tilde{q}^2-z}=\frac{\psi(\vec{q}\,)\phi(\vec{q}\,')}{\epsilon+z}+\int\frac{\Omega^-(\vec{q},\vec{q}\,'')t^+(\vec{q}\,',\vec{q}\,'')}{\tilde{q}\,''^2-z}\,d\vec{q}\,'' \tag{5.10}$$

follows. If we employ Eq. (5.10) in the expression (5.8) we obtain the formula (5.6) for $H^{\rm imp}_{\alpha\alpha}(\vec{p}_\alpha;\vec{p}\,'_\alpha)$. To interpret this result correctly, we note that the right-hand side of Eq. (5.5) is not the definition of $H^{+,1}_{\beta\alpha}$. Recall that $H^{+,1}_{\beta\alpha}$ is the solution to the integral equation (3.7), with $H^c_{\beta\gamma}$ replaced by $H^{c,1}_{\beta\alpha}$. However, the solution of $H^{+,1}_{\beta\alpha}$ contains all the terms on the right-hand side of Eq. (5.5) and so includes the full impulse approximation.

The final topic of this section is to indicate the simplification that occurs when the pairwise in-

teractions are separable potentials. Consider the behavior of the amplitudes $E^{+}_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)$ and $E^{c}_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)$. The definition of these functions in Eqs. (2.15) and (3.3) have the common feature that they are matrix elements of an operator of the form $V_\beta A_{\beta\alpha}(s_\alpha+i0)$. The operator $A_{\beta\alpha}(z)$ simply takes a different form in the two cases $E^{+}_{\beta\alpha}$ and $E^{c}_{\beta\alpha}$. Using definitions (2.15) or (3.3) and inserting a complete set of momentum states between V_β and $A_{\beta\alpha}(s_\alpha+i0)$ gives us

$$E^{+,c}_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)=\int t^{+}_{\beta}(\vec{q}''_\beta,\vec{q}_\beta)\langle\vec{p}_\beta\vec{q}''_\beta|A_{\beta\alpha}(s_\alpha+i0)|\vec{p}'_\alpha\rangle\,d\vec{q}''_\beta\,. \tag{5.11}$$

In the case where the interaction is separable we may write t^{+} as the product of two factors

$$t^{+}(\vec{q}'',\vec{q})=\frac{t^{+}(\vec{q},\vec{q})}{\phi(\vec{q})}\,\phi(\vec{q}'')\,, \tag{5.12}$$

where ϕ is the vertex function defined in Eq. (2.5). Thus the right-hand side of Eq, (5.11) can be written

$$\frac{t^{+}_\beta(\vec{q}_\beta,\vec{q}_\beta)}{\phi_\beta(\vec{q}_\beta)}\int\phi_\beta(\vec{q}''_\beta)\langle\vec{p}_\beta\vec{q}''_\beta|A_{\beta\alpha}(s_\alpha+i0)|\vec{p}'_\alpha\rangle\,d\vec{q}''_\beta$$
$$=-\frac{t^{+}_\beta(\vec{q}_\beta,\vec{q}_\beta)}{\phi_\beta(\vec{q}_\beta)}\,\langle\vec{p}_\beta\psi_\beta|V_\beta A_{\beta\alpha}(s_\alpha+i0)|\vec{p}'_\alpha\psi_\alpha\rangle\,. \tag{5.13}$$

However, we note that the amplitude on the right of Eq. (5.13) is just $H^{+,c}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)$. Altogether one obtains the factorization property

$$E^{+,c}_{\beta\alpha}(\vec{p}_0;\vec{p}'_\alpha)=-\frac{t^{+}_\beta(\vec{q}_\beta,\vec{q}_\beta)}{\phi_\beta(\vec{q}_\beta)}\,H^{+,c}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)\,. \tag{5.14}$$

This property is easily seen to reduce the two-vector variable integral equations to equations in the one-vector variable. Consider the EPR system first. Then quadrature relation (3.6) and integral equation (3.4) reduce to the common form

$$H^{c}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)=V^{bb}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)-\sum_{\gamma>0}\int I_{\beta\gamma}(\vec{p}_\beta;\vec{p}''_\gamma;s_\alpha+i0)H^{c}_{\gamma\alpha}(\vec{p}''_\gamma;\vec{p}'_\alpha)d\vec{p}''_\gamma\,, \tag{5.15}$$

where the kernel $I_{\beta\gamma}$ is given by the integral

$$I_{\beta\gamma}(\vec{p}_\beta,\vec{p}'_\gamma;s_\alpha+i0)=-\int\frac{V^{bc}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma,\vec{q}''_\gamma)t^{+}_\gamma(\vec{q}''_\gamma,\vec{q}''_\gamma)}{(\tilde{p}''^{2}_\gamma+\tilde{q}''^{2}_\gamma-s_\alpha-i0)\phi_\gamma(\vec{q}''_\gamma)}\,d\vec{q}''_\gamma\,. \tag{5.16}$$

Finally we state the form of the integral equations in the CIS representation. Again the separability leads to factorization properties analogous to Eq. (5.14). In this case Eq. (3.20) simplifies to

$$H^{+}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)=H^{b}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)-\sum_{\gamma>0}\int J_{\beta\gamma}(\vec{p}_\beta;\vec{p}''_\gamma;s_\alpha+i0)H^{+}_{\gamma\alpha}(\vec{p}''_\gamma;\vec{p}'_\alpha)\,d\vec{p}''_\gamma\,. \tag{5.17}$$

The kernel $J_{\beta\gamma}$ is the solution of

$$J_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma;s_\alpha+i0)=I_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma;s_\alpha+i0)-\sum_{\lambda>0}2n_\lambda\int\frac{V^{bb}_{\beta\lambda}(\vec{p}_\beta;\vec{p}''_\lambda)J_{\lambda\gamma}(\vec{p}''_\lambda;\vec{p}'_\gamma;s_\alpha+i0)}{p''^{2}_\lambda-k_\lambda^{2}-i0}\,d\vec{p}''_\lambda\,. \tag{5.18}$$

The equation for $J_{\beta\gamma}$ is the one-variable form of Eq. (3.15). After the separable reduction is effected, it is seen that the CIS system is no more difficult to solve than the EPR system.

VI. CONCLUSIONS

We report here the computational results we have obtained using the decoupling schemes EPR and CIS. Basically we have compared the approximate solutions found by iterative expansions for the effective potentials with those determined from an exact solution of the Amado-Lovelace equations for the separable potential three-body problem. To begin with, we quote[10] the form taken by the Amado-Lovelace equation. The two-cluster amplitudes $H^{+}_{\beta\alpha}$ satisfy

$$H^{+}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha)=V^{A}_{\beta\alpha}(\vec{p}_\beta;\vec{p}'_\alpha;s_\alpha+i0)-\sum_{\gamma>0}2n_\gamma\int\frac{V^{A}_{\beta\gamma}(\vec{p}_\beta;\vec{p}''_\gamma;s_\alpha+i0)H^{+}_{\gamma\alpha}(\vec{p}''_\gamma;\vec{p}'_\alpha)}{p''^{2}_\gamma-k^{2}_\gamma-i0}\,d\vec{p}''_\gamma\,. \tag{6.1}$$

The kernel $V^{A}_{\beta\alpha}$ is constructed from the two-body functions by

$$V^{A}_{\beta\gamma}(\vec{p}_\beta;\vec{p}''_\gamma;s_\alpha+i0)=-\bar{\delta}_{\beta\gamma}\frac{\phi_\beta(\vec{q}^{(1)}_\beta)\phi_\gamma(\vec{q}^{(2)}_\gamma)}{\tilde{p}''^{2}_\gamma+\tilde{q}^{(2)2}_\gamma-s_\alpha-i0}\,S_\gamma(s_\alpha+i0-\tilde{p}''^{2}_\gamma)\,. \tag{6.2}$$

The function S_γ is defined from the full-off-shell two-body t matrix[5,9] by

$$t_\gamma(\vec{q}_\gamma,\vec{q}''_\gamma;s_\alpha+i0-\tilde{p}''^{2}_\gamma)=-\frac{\phi_\gamma(\vec{q}_\gamma)S_\gamma(s_\alpha+i0-\tilde{p}''^{2}_\gamma)\phi_\gamma(\vec{q}''_\gamma)}{\tilde{p}''^{2}_\gamma-\tilde{k}_\gamma^{2}-i0}\,. \tag{6.3}$$

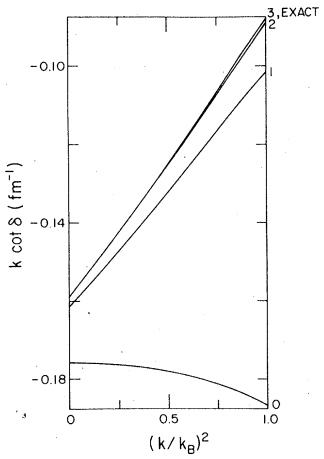

FIG. 1. Solution of the EPR for a system of fermions corresponding to n-d scattering in the ^4S state. Each curve is labeled on the right by the ith iterate of the EPR expansion.

Finally, all the equations are reduced to diagonal form by taking appropriate linear combinations of $H^+_{\alpha\alpha}$, $H^+_{\beta\alpha}$, and $H^+_{\gamma\alpha}$.

The two-body interaction is chosen to be a Yamaguchi potential, viz.,

$$v(\vec{q},\vec{q}\,') = \lambda \frac{1}{q^2+\beta^2} \frac{1}{q'^2+\beta^2} . \tag{6.4}$$

The range parameter β is set to 1.444 01 fm^{-1}, and λ adjusted to reproduce a two-particle binding energy of $\epsilon = 0.053\,695$ fm^{-2}. The solution on Fig. 1 labeled EXACT is the solution of the Amado-Lovelace equations throughout the elastic scattering sector. This system corresponds to neutron-deuteron scattering in the spin quartet state. Also shown on Fig. 1 are the first four approximate solutions, namely the value of $k\cot\delta$ determined for $H^{+,i}_{\beta\alpha}$ where $i = 0,1,2,3$. The range of momentum plotted is the fraction of the momentum at breakup threshold which is given by $k_B = 0.2676$ fm^{-1}. When $i = 3$ the approximate solution is indistinguishable from the exact result on the graph. Note that the approximation for $i = 0$ is quite bad. This is in accord with the fact that only for $i \geq 1$ do our approximations become sufficiently self-consistent to contain the impulse approximation.

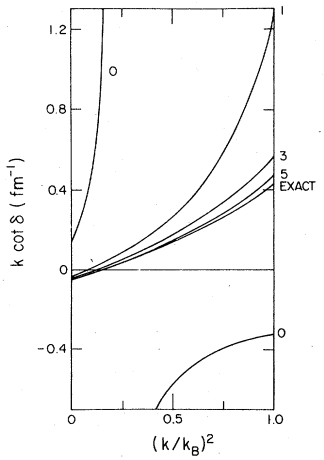

FIG. 2. Solution of the EPR for identical spin-0 and isospin-0 particles. Curves are labeled on the right by the ith iterate of the EPR expansion.

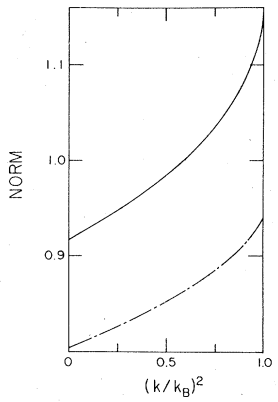

FIG. 3. Continuous curve is the L^2 norm of I and dotted curve is the square root of the largest eigenvalue of $I^\dagger I$. The curves are shown for the case of identical spin-0 and isospin-0 particles.

Figure 2 shows the behavior of $k\cot\delta$ in the elastic scattering sector for three identical bosons in the $J = 0$ state. This is a more strongly interacting system than the three-fermion system because of the different spin and isospin recoupling coefficients. Again for $i \geq 1$ there is rapid convergence of the approximate solutions. In fact a good measure of the degree to which the three-body continuum states in Eq. (5.15) couple strongly to each other is determined by the L^2 norm of the operator I

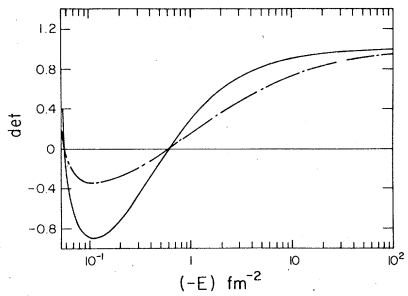

FIG. 4. Fredholm determinant at negative energies. The continuous curve is obtained from an exact EPR, and the dotted curve from the Amado-Lovelace equation.

TABLE I. Iterative solutions to CIS showing the convergence rate for $k \cot \delta$ at energy $(k/k_B)^2 = 0.5$.

Iterate	Fermions	Iterate	Bosons
0	−0.178 07	0	−0.580 27
1	−0.134 32	5	−0.145 78
2	−0.125 08	10	0.003 83
3	−0.124 74	15	0.065 20
4	−0.124 99	20	0.102 05
		21	0.105 93
		22	0.154 29
		23	0.177 62
		24	0.152 75
		25	0.124 19
		26	0.117 97
		27	0.137 83
		28	0.157 26
		29	0.152 16
		30	0.135 78
		35	0.140 56
		40	0.141 75
		45	0.141 49
∞	−0.125 03 fm^{-1}	∞	0.139 75 fm^{-1}

defined by Eq. (5.16). The top curve in Fig. 3 gives this norm and the lower curve is the value of the square root of the largest eigenvalue of $I^\dagger I$. The curves shown are for the boson case. The corresponding fermion results are obtained by dividing by 2.

Figure 4 shows the values of the Fredholm determinant for negative three-body energies in the three-boson system. The dotted curve gives the Fredholm determinant defined by the Amado-Lovelace Eqs. (6.1). The solid curve is that implied by the cluster integral equation for the bound state Eq. (4.9). It is seen that, although different, both Fredholm determinants have the same zeros, each zero corresponding to a three-body bound state pole.

Our last computations concern the CIS cluster description. We do not have as well developed a physical picture for this decoupling scheme and for this reason have not carried out such extensive calculations as in the EPR approach. Typical results are given in Table I. We have calculated $H^b_{\beta\alpha}$ and $J_{\beta\alpha}$. With these functions given we have solved Eq. (5.17) by iteration. Recall $H^b_{\beta\alpha}$ has the simple interpretation of the solution of the KZ integral equations with the three-particle continuum intermediate states turned off. Thus the iterative convergence of Eq. (5.17) gives a measure of the perturbation of the solution caused by the three-particle continuum states. For fermions the convergence is very rapid. The boson case shows a characteristic slow convergence which oscillates about the correct solution.

[1]K. Wildermuth and Y. C. Tang, *A Unified Theory of the Nucleus* (Vieweg, Braunschweig, 1977).
[2]J. Schwager, Ann. Phys. (N. Y.) 98, 14 (1976); R. Raphael, P. C. Tandy, and W. Tobocman, Phys. Rev. C 14, 1355 (1976); S. F. J. Wilk and T. A. Osborn, Ann. Phys. (N. Y.) 114, 410 (1978); E. W. Schmidt and M. L'Huillier, Tübingen report, 1979 (unpublished).
[3]S. K. Adhikari and W. Glöckle, to be published in Phys. Rev. C.
[4]B. R. Karlsson and E. M. Zeiger, Phys. Rev. D 11, 939 (1975).
[5]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory*, translated from Russian, the Israel Program for Scientific Translations, Jerusalem (Davey, New York, 1965).
[6]D. Bollé, Phys. Rev. D 13, 1809 (1976).
[7]V. E. Kuzmichev, Kiev reports, 1977 (unpublished).
[8]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).
[9]D. Eyre and T. A. Osborn, Nucl. Phys. A299, 301 (1978).
[10]C. Lovelace, Phys. Rev. 135, B1225 (1964); R. D. Amado, Annu. Rev. Nucl. Sci. 19, 61 (1969).