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Physical consequences of anomalies in nonlocal potential problems

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An s-wave two-body separable potential may give rise to several phenomena which are absent for nonsingular local potentials. We examine the physical implications of a well known example of such phenomena, the continuum bound state, as well as of two lesser known anomalies, the so-called positive energy spurious state and negative energy bound states with improper long-range behavior. By examining these anomalies in light of Levinson's theorem, Wigner's phase shift inequality, and the effect of a perturbation on the anomalous states by their insertion in a three-body scattering situation, we find in agreement with previous studies that the continuum bound state acts as a resonance of negligible width. However, we find it difficult to see how the presence of a spurious state can be detected experimentally.

[NUCLEAR REACTIONS Scattering by a nonlocal potential, continuum bound states, spurious states, wave functions with improper long-range behavior. Levinson's theorem. Three-particle scattering.]

I. INTRODUCTION

This paper discusses certain "anomalous" properties associated with *nonlocal* two-body potentials. The term "anomalous" is used here in the specific and limited sense that these properties are not possible in the case of sufficiently regular *local* potentials; it is not intended to cast any aspersion on the physical character or usefulness of any potential which exhibits them. Our concern will be directed primarily to separable potentials (though other nonlocal potentials may also share these properties) because they are more easily discussed and because they are being extensively employed to describe two-particle interactions in few- and many-body problems.

In a recent series of papers, Arnold, Bagchi, Krause, and Mulligan[1] have presented an elegant discussion of the behavior of solutions of the radial Schrödinger equation with a nonlocal two-body potential. These authors discussed a long-known anomaly (continuum bound states) and also discuss at length another anomaly known as a "spurious state." On the basis of their results they also propose a new definition of "absolute phase shift" differing from previously proposed definitions. The logic of their arguments for this change stems from certain mathematical considerations. We respectfully suggest from our examination that the proposed redefinition has little in the way of physics to recommend it, and that on the basis of Levinson's theorem and Wigner's inequality associated with causality, the benefits are at best questionable.

We also discuss briefly another anomaly which we discovered quite accidentally,[2] though its existence had been noted earlier,[3] namely, anomalous negative energy bound states with peculiar asymptotic behavior. On the basis of our calculations we discuss some questions concerning physical "observability" of some of these anomalies, while in the Appendix we give an elementary example which illustrates how some of these anomalies arise and at the same time illustrates the ambiguities involved in naive node-counting arguments in fixing absolute phase shifts.

For simplicity we shall consider only s states and use a notation to facilitate as far as possible a comparison with the papers of Ref. 1. Thus we shall write for the radial $l\equiv 0$ wave function multiplied by r, $u(r)$, and the form of the Schrödinger equation will then be (units: $2m/\hbar^2=1$)

$$d^2u/dr^2+k^2u=\int_0^\infty U(r,r')u(r')dr', \quad k=E^{1/2}, \tag{1}$$

with E the energy eigenvalue. For a local and for a separable potential we have respectively

$$U(r,r')=\delta(r-r')U(r), \tag{2a}$$

$$U(r,r')=\lambda g(r)g(r'). \tag{2b}$$

In momentum space we represent the transform of $u(r)$ by $v(q)$, of $g(r)$ by $h(q)$, and of $U(r,r')$ by

$V(q, q')$; the transformation equations are

$$v(q) = (2/\pi)^{1/2} \int_0^\infty u(r) \sin qr \, dr , \tag{3a}$$

$$h(q) = (2/\pi)^{1/2} \int_0^\infty g(r) \sin qr \, dr , \tag{3b}$$

$$V(q, q') = (2/\pi) \int_0^\infty \int_0^\infty \sin qr \, U(r, r') \sin q'r' \, dr' \, dr, \tag{3c}$$

together with the same equations under the transformation

$$u(r) \rightleftarrows v(q), \quad g(r) \rightleftarrows h(q), \quad U(r, r') \rightleftarrows V(q, q'),$$

$$q \rightleftarrows r, \quad q' \rightleftarrows r' .$$

The Schrödinger equation in this representation has the form

$$(k^2 - q^2) v(q) = \int_0^\infty V(q, q') v(q') dq' , \tag{4}$$

and in particular, for the separable potential where

$$V(q, q') = \lambda h(q) h(q') , \tag{5}$$

it reduces to

$$(k^2 - q^2) v(q) = \lambda h(q) \int_0^\infty h(q') v(q') dq' . \tag{6}$$

The scattering solution in this case is given by

$$\langle \vec{q}' | \psi_E^\pm \rangle = \langle \vec{q}' | \phi_E^\pm \rangle + \int d^3q \, G_0^\pm(\vec{q}') \langle \vec{q}' | t(E) | \vec{q} \rangle \langle \vec{q} | \phi_E^\pm \rangle , \tag{7}$$

where the off-shell s-wave t-matrix element is given by

$$\langle q' | t(E) | q \rangle = \frac{\lambda}{4\pi q' q} h(q') h(q) \tau(E) , \tag{8}$$

where

$$\tau(E) = \left\{ 1 - \lambda \int_0^\infty \frac{[h(q'')]^2}{E - q''^2 + i\epsilon} dq'' \right\}^{-1} . \tag{9}$$

II. ANOMALOUS NEGATIVE ENERGY BOUND STATES

It is well known from elementary quantum mechanics that if one has a short-range local potential, then for a bound state the energy eigenvalue E_B must be negative (assuming the potential vanishes at infinity), and the negative energy bound state wave function has the asymptotic form (for an s wave)

$$u(r) \sim A e^{-\kappa r} \quad (\kappa^2 = -E_B) , \tag{10}$$

with A a constant.

It is also quite well known that for a simple nonlocal potential of the separable Yamaguchi type[4] with $g(r) = e^{-\alpha r}$ so $U(r, r') = -\gamma e^{-\alpha r} e^{-\alpha r'}$ with $\gamma \equiv -\lambda$ positive, if a bound state exists (at a negative energy) its wave function has the Hulthén form

$$u(r) = e^{-\kappa r} - e^{-\alpha r} , \tag{11}$$

with

$$\gamma = 2\alpha (\kappa + \alpha)^2 , \tag{12}$$

as may be verified by direct substitution into the Schrödinger equation (1).

We note that since κ and α are necessarily positive, Eq. (12) guarantees the existence of a positive γ such that the Schrödinger equation is satisfied. Now so long as $\kappa < \alpha$ or, equivalently, $\gamma < 8\alpha^3$, the asymptotic form of $u(r)$ will be of the same form, $e^{-\kappa r}$, as in the case of a local potential, with the asymptotic rate of falloff of the bound state wave function determined by the binding energy alone. On the other hand, if $\kappa > \alpha$ or $\gamma > 8\alpha^3$, $u(r)$ is asymptotic to $e^{-\alpha r}$ which is determined by the potential parameters but not by the binding energy.

It may be objected that the Yamaguchi potential with $\gamma > 8\alpha^3$ may be masquerading as a short-range potential of range α^{-1} but that it is in actuality a long-range potential and α^{-1} is not a proper measure of its range. It is difficult to respond unequivocally to this in that the concept of range of a potential is not that well defined. However, one can show that for $k^2 = E > 0$, the purely real regular solution of the Schrödinger equation is asymptotic to $\sin(kr + \delta)$ where $\delta (\bmod \pi)$ is a well-defined phase shift (independent of r) which suggests that the potential is not active at large radii for positive energies. Furthermore, on semiquantitative grounds, Wigner[5] has shown that for a potential of range R

$$d\delta/dk > -R, \tag{13}$$

using an argument that the time advance of the outgoing wave packet relative to the ingoing one is limited by the range of the interaction. Hence if the range is indeed infinite, one would expect Wigner's inequality to be violated for arbitrarily large R at some k; investigation shows this not to be the case. Thus in spite of the anomalous behavior of the bound state wave function, no physical anomaly seems to be associated with it, and the situation is perhaps best described by saying the range is energy dependent and becomes more long-ranged as the energy becomes more and more negative.

It should further be noted that this anomalous behavior occurs not only for the Yamaguchi separable potential but is possible for bound states derived from any separable potential which possesses a pole in $h(q)$ for some complex q.

III. CONTINUUM BOUND STATES

A long-recognized anomaly of some separable potentials is the existence of bound states (states described by square-integrable wave functions) for positive energies and hence lying in the continuum. We call these *continuum bound states* (CBS's). The condition for the existence of a CBS at $k_0^2 = E_{CBS}$ is the vanishing of the (Fredholm) denominator in the on-shell t-matrix element as given in Eq. (9). Since the real and imaginary parts must both vanish, this requires

$$h(k_0) = 0 \tag{14}$$

and

$$1 - \lambda P \int \frac{h(q'')^2}{k_0^2 - q''^2}\, dq'' = 0 \,. \tag{15}$$

The elimination of k_0 between these two equations yields a condition which the potential parameters must satisfy (see Appendix, for example) and hence the occurrence of a CBS is accidental. If the condition is nearly, but not quite satisfied, the CBS is transmuted into a narrow width resonance in the scattering amplitude whose width narrows as the parameters approach satisfaction of the condition. When a CBS exists, there is also a scattering state solution[1] of the Schrödinger equation at the same energy corresponding to a phase shift of zero (modπ) which can be chosen to be orthogonal to the CBS wave function.

A useful application of CBS's lies in obtaining improved fits of two-body t-matrix data for p-wave π-N and π-π scattering by the use of separable potentials.[6] The technique depends on starting from a CBS or infinitely narrow resonance at the position of the resonance in the appropriate scattering channel, and then varying the potential parameters to fit the phase shift behavior both below and above resonance.

We shall come back to the phase shift behavior later but add two more points, the first of which is not emphasized in Ref. 1. These are the fact that there may exist more than one linearly independent CBS at a particular energy and that a CBS may satisfy the initial conditions $u(0) = u'(0) = 0$. Examples are discussed in the Appendix.

IV. SPURIOUS STATES

In Ref. 1 another anomaly of nonlocal potentials is recognized and emphasized. While for a *nonsingular local* potential the initial conditions $u(0) = u'(0) = 0$ imply that the solution is everywhere zero, it follows from the work of Mulligan *et al.* and the illustration in the Appendix that this need not be the case for nonlocal potentials. In particular, it can fail to be the case for a continuum bound state (see Appendix). Mulligan *et al.* further point out that for some nonlocal potentials a nontrivial solution with $u(0) = u'(0) = 0$ can exist at some energies even when there is not a CBS at these energies. In such a case, for the energy in question, what they define as a regular solution, i.e., one satisfying the conditions $u(0) = 0$, $u'(0) = 1$, does not exist. This satisfies the condition for what we have called an "anomaly," although it is important to note that there exist perfectly respectable *singular* local potentials for which no regular solutions exist at any energy. For these potentials, of which repulsive r^{-n} potentials with $n > 2$ are examples, perfectly reasonable scattering solutions exist. A non-normalizable state whose wave function satisfies the condition $u(0) = u'(0) = 0$ has been named[7] a "spurious state."

Up to this point we feel that the analysis of these authors is useful and informative. However, on the basis of the behavior of nodes of the regular solution of the Schrödinger equation as the energy varies through an energy associated with a spurious state, the authors go on to propose a new definition of the absolute phase shift which we believe has no real advantages and several disadvantages. We shall comment on this in detail in the next section.

However, we should remark on another feature of spurious states which seems to have been unnoticed by Mulligan *et al.* As we indicate by an example in the Appendix, one can construct nonlocal potentials having the property that at some particular value of the energy, two orthogonal and hence linearly independent solutions may exist with the possibility that one may be a CBS [with $u(0) = u'(0) = 0$] and the other may be a spurious state. In such a case the spurious state is not unique since one can add to it any multiple of the CBS and still satisfy the conditions for a spurious state. We show also that one may construct examples in which several CBS's and perhaps a spurious state may be linearly independent solutions at a given energy. These situations add to the problems associated with the alternative definition of the absolute phase shift as proposed by Mulligan *et al.*

V. ABSOLUTE PHASE SHIFTS

For scattering in a spherically symmetric potential, the scattering amplitude and cross section in a particular angular momentum state are expressible in terms of a phase shift δ determined only *modulo* π. If the phase shift is calculated from the asymptotic form of the radial wave function which is regular at the origin, it is also determined only *modulo* π. It is usually assumed that the phase shift is a continuous function of energy and the possibility of introducing discontinuities of multiples of π has not been exploited. In particular, in the case of local potentials there appears no need or advantage to do otherwise; rather, there are good reasons not to.

The first of these is to preserve the validity of Levinson's theorem

$$\delta(0) - \delta(\infty) = N \; , \qquad (16)$$

where N is the number of negative energy bound states of the particular angular momentum. The proof is based on phase shift continuity as a function of energy for local potentials. The second is the preservation of the Wigner inequality

$$d\delta/dk > -R \; , \qquad (17)$$

where R is the "range" of the potential beyond which its effect on the phase shift is negligible; its derivation also assumes the continuity of the phase shift. However, its validity would not be impaired by upward jumps of multiples of π with increasing energy, but similar downward discontinuities would make the left side of the inequality infinitely negative at such energies and destroy the inequality.

The situation is not so clear in the case of nonlocal potentials as has been pointed out by Bolsterli[8] and others. In particular, if a potential admits a CBS at some energy then, as noted earlier, this is the limit of a situation where a resonance in the scattering becomes infinitely narrow. Infinitely narrow resonances in a scattering channel are not directly observable. In fact even very narrow resonances must be inferred from the decay of the compound system rather than by direct observation of them in the scattering amplitude. For a very narrow resonance, the phase rises steeply by about π as one increases the energy over the width of the resonance. If one carries this behavior to the limit of a CBS then one expects to find a discontinuity of $+\pi$ in the phase shift at such a zero width resonance. Contrariwise virtually any perturbation on a CBS dissolves the discontinuity into a narrow resonance with a steep rise in δ of π. It is thus natural, as emphasized by Bolsterli, to assign a discontinuity of $+\pi$ to δ with increasing energy at a CBS. This does not contradict the Wigner "theorem" and yet preserves the Levinson theorem if the number N is taken to be the number of *negative energy* bound states.

On the other hand, Mulligan *et al.* suggest on the basis of arguments concerning the behavior of wave function nodes in the regular solution of the Schrödinger equation (arguments best understood by reading their discussion) that the absolute phase shift be defined so that it is continuous at CBS's but shows a discontinuity of $-\pi$ as the energy increases through each spurious state. They then write a modified form of Levinson's theorem to fit their definition. It is on this point that we have serious reservations concerning the utility of such a definition which we shall now try to explicate.

If a CBS should happen to exist, its existence can be detected or inferred experimentally by the simple medium of introducing a slight perturbation on the system. This we illustrate in the next section by calculating the effect of a slight binding of the target particle in an external field which shows that a finite width resonance like anomaly appears in the total cross section at an energy related to that of the CBS. Beregi[9] has shown that the same occurs if one couples the elastic channel in which the CBS occurs to a second channel. To assign a physical test for a spurious state should require that one be able to make some perturbation on the system which makes its existence physically apparent. In the following section we show that target particle binding will not do this, while an examination of Beregi's results show that coupling to another channel is equally impotent. The effect of a shift in the parameters of the potential which possesses a spurious state (generally[1]) is simply to shift the energy at which the spurious state occurs but does not generate any quantitative changes. Thus it appears dubious that the existence of a spurious state can be detected or inferred from experiment, and this would make the proposed modification of Levinson's theorem empty of any physical content. A downward discontinuity of the phase shift with energy would also represent a failure of Wigner's "theorem" since at a spurious state the left side of Eq. (17) would be negatively infinite.

Even the reliance on node-counting is somewhat suspect since nodal behavior can be quite peculiar for nonlocal potentials. In examples in the Appendix it is shown that one can have finite intervals over which the wave function is identically zero at some energy, and clarification is required as to how such occurrences should be counted. And there are singular but not particularly unphysical local potentials where spurious states exist at all energies as indicated earlier. For this reason we feel that it is dangerous to rely on node-counting or node behavior for determining whether the mathematical concept of a spurious state possesses any physical motivation for its being singled out for special consideration. This is not meant to imply that there are no special cases in which the presence of a node may be relevant to some physical effect and hence worthy of study, but we are skeptical about general arguments with respect to nodes which are used as a basis for a change in the definition of "absolute" phase shift from what has been commonly accepted. Another requirement, though less directly physically motivated, favoring the present choice for the definition of the absolute phase shift is the analyticity of the t matrix in the cut complex k plane. Thus we feel that the proposed choice of absolute phase shift of Mulligan *et al.*, their δ_L, has less physical content than their unfavored choice δ_D.

Finally it is important to remark that our recommendation with respect to retaining the earlier definition of absolute phase shift such as to maintain the physical usefulness of Levinson's theorem and Wigner's inequality are premised on the assumption that bound state eigenvalues (whether positive or negative energy) are simple and not degenerate. The problems of degeneracy were raised in the discussion of Martin and Gourdin,[10] but it is still not clear, to our knowledge, what is the proper disposition of them.

VI. CONTINUUM BOUND STATES AND SPURIOUS STATES IN THREE-BODY SCATTERING SITUATIONS

As an example of the above ideas concerning the observability of a CBS and a spurious state when a slight perturbation is applied to the system, we consider the situation of a projectile scattering from a target bound to an infinitely heavy force center under the assumptions that all the particles are spinless and the projectile and target are of equal mass, that the projectile and force center do not interact, and that the CBS or spurious state occurs in the projectile-target subsystem channel. The exact solution of this scattering situation requires the use of the Faddeev equations. However, for large projectile energies and when the projectile-target scattering length is much smaller than the target-force center scattering length, the driving term of the Faddeev equations, the impulse approximation, serves as a reliable approximation to the exact scattering.

For the present case, the impulse approximation to the elastic scattering amplitude is

$$T^{\rm el}_{\rm imp}(\vec{q}_p', \vec{q}_p) = \int d^3q_c''\, \phi^*_{\rm TC}(\vec{q}_p' + \vec{q}_c'')\phi_{\rm TC}(\vec{q}_p + \vec{q}_c'')\left\langle \vec{q}_p' + \tfrac{1}{2}\vec{q}_c'' \right| t_{\rm PT}\left(\frac{q_p^2}{2} - \frac{k_{\rm TC}^2}{2} - \frac{q_c''^2}{4}\right)\left| \vec{q}_p + \tfrac{1}{2}\vec{q}_c'' \right\rangle , \tag{18}$$

where $\phi_{\rm TC}$ is the target-core bound state wave function, $-k_{\rm TC}^2$ is the target-core binding energy, and $t_{\rm PT}$ is evaluated as in Eqs. (8) and (9). In the present calculations we employed $k_{\rm TC}^2 = 3.5$ eV and a Gaussian s-wave wave function with an rms momentum distribution of 20.47 Å^{-1} for $\phi_{\rm TC}$. Two projectile-target interactions were examined, the first with the momentum space form factor

$$h(q) = q^2(q^2+n_1^2)^{-1} - cq^2(q^2+n_2^2)^{-1} , \tag{19}$$

with $n_1 = 20.0$ Å^{-1}, $n_2 = 40.0$ Å^{-1}, and having a CBS at $q_0 = 100.0$ Å^{-1}, and the second with a momentum space form factor

$$h(q) = q(q^2+n_1^2)^{-1} , \tag{20}$$

with $n_1 = 80.0$ Å^{-1} and having a spurious state at $q_{\rm sp} = 100.0$ Å^{-1}. The target-force center interaction gives rise to bonding parameters typical of the binding of protons in hydrocarbon molecules.[2] There would be great physical motivation if the projectile-target interactions used in the present calculation mimicked some interaction found in nature. However, the purpose of the calculation being to explore the differences between the effects of incorporating a CBS and a spurious state in a three-body situation, the values for the potential parameters in Eqs. (19) and (20) were chosen to make very obvious the differences between the two cases.

In Fig. 1 is plotted the imaginary part of the forward elastic scattering amplitude in impulse approximation for the CBS and spurious state cases. Also shown is the on-energy-shell impulse approximation (OEA)

$$T^{\rm OEA}_{\rm imp}(\vec{q}_p, \vec{q}_p) = \langle \vec{q}_p/2 | t_{\rm PT}(q_p^2/4) | \vec{q}_p/2 \rangle , \tag{21}$$

obtained in the loose binding limit by setting

$$\phi_{\rm TC}(\vec{k}) \to \delta(\vec{k}) \tag{22}$$

and

$$q_p^2 \gg k_{\rm TC}^2 . \tag{23}$$

As seen in Fig. 1, the spurious state amplitude shows no special structure while the CBS amplitude shows a pronounced structure at $q_p \approx 208$ Å^{-1}. The location of this peak and its association with the CBS may be understood as follows. For the CBS, a pole occurs in $\mathrm{Re}\,\tau(q^2)$ at $q^2 = q_0^2$. In addition, the target-core wave functions peak at $\vec{q}_c'' = -\vec{q}_p$. This pole overlaps the peaking of the wave functions at

$$q_p = (4q_0^2 + 2k^2)^{1/2} = 208.2 \text{ Å}^{-1} , \tag{24}$$

for our example. Likewise, both the real and imaginary parts of $\tau(q^2)$ are smoothly varying at q^2

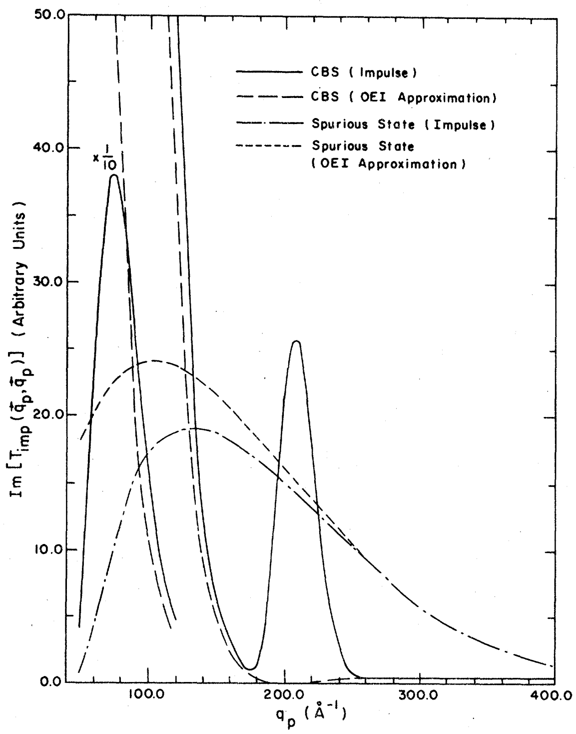

FIG. 1. The imaginary part of the forward elastic scattering amplitude in the impulse approximation as a function of the projectile initial momentum for the separable potential form factor of Eq. (19) possessing a continuum bound state (solid curve) and for the form factor of Eq. (20) possessing a spurious state (dot-dash curve). Also shown is the OEA approximation to the imaginary part of the forward elastic scattering amplitude for each of the form factors. The spurious state results have been scaled down by a factor of 10^3.

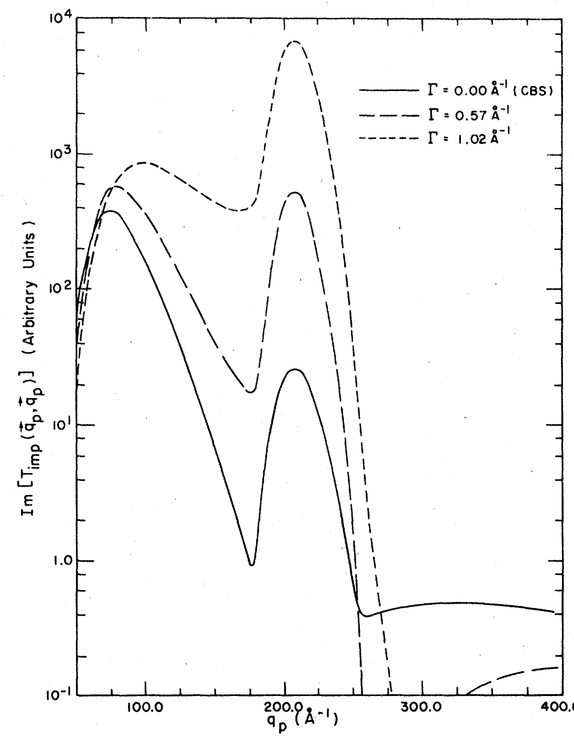

FIG. 2. The imaginary part of the forward elastic scattering amplitude in the impulse approximation as a function of the projectile initial momentum for the separable potential form factor of Eq. (19). Shown with the continuum bound state case are the corresponding amplitudes for resonances of widths $\Gamma = 0.57$ Å^{-1} and 1.02 Å^{-1} obtained by perturbations on the CBS conditions.

$=q_{sp}^2$ for the spurious state case allowing for no special enhancement when folded with target-core wave functions and integrated. Finally for the CBS case, one can show that the width of the pronounced structure at $q_p \approx 208$ Å^{-1} is due entirely to the rms momentum distribution in ϕ_{TC}.

As expressed in a previous section, the CBS may be thought of as being the limit of a resonance of vanishingly narrow width and that a perturbation of the CBS conditions produces a narrow resonance. This is demonstrated in Fig. 2. This figure shows the imaginary part of the forward bound-target scattering amplitude in impulse approximation for the CBS of Eq. (19) and Fig. 1 and for two perturbations upon the CBS conditions yielding subchannel resonances with $k_{res} = 100.0$ Å^{-1} and with widths $\Gamma = 0.57$ Å^{-1}, 1.02 Å^{-1}. Again in this numerical way, it may be seen that the CBS acts as a resonance of zero width and may be observed when coupled to another channel or when target binding is allowed.

Lastly, we would claim that an exact solution via the Faddeev equations to the scattering problem posed in this section would yield no new results. It is known that the impulse approximation is accurate in the "quasiclassical binding" case where the target-core potential is slowly varying over the projectile-target interaction volume.[2,11] This is the case for the interactions of Eqs. (19) and (20), the ratio of the projectile-target scattering length to target-core scattering length being 0.33 in the former case and 0.036 in the latter case.

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APPENDIX

In this appendix we discuss some very simple examples of nonlocal potentials which clearly illuminate the phenomena of CBS's and "spurious states" without any direct reference to the theory of integral equations and exhibit some of the special phenomena referred to earlier. We consider first a separable potential with the "form factor"

$$g(r)=\Delta_a^R(r)=\begin{cases}0 & (r<a),\\ 1 & (a<r<R+a)\ ,\\ 0 & (r>R+a)\ .\end{cases} \tag{A1}$$

We first take $a=0$ and in that case drop the subscript and superscript so $\Delta(r)=\Delta_0^R(r)$. The Schrödinger equation for the s state then takes the form

$$d^2u/dr^2+k^2u-k^{-1}\lambda\Delta(r)I=0\ , \tag{A2}$$

with

$$I=k\int_0^\infty \Delta(r')u(r')dr'=k\int_0^R u(r')dr'\ . \tag{A3}$$

The solution which vanishes at $r=0$ has the form

$$u(r)=A(1-\cos kr)+B\sin kr, \tag{A4}$$

with

$$A=\lambda k^{-3}I\ . \tag{A5}$$

Substitution of (A4) with (A5) into (A3) gives the relation between B and I:

$$[1-\lambda k^{-3}(kR-\sin kR)]I=B(1-\cos kR)\ . \tag{A6}$$

These results hold for $r<R$, while for $r>R$ we have

$$u(r)=C\sin(kr+\delta)\ . \tag{A7}$$

Matching logarithmic derivatives at $r=R$ yields

$$\cot(kR+\delta)=\frac{\sigma\sin kR(1-\cos kR)+[(kR)^3-\sigma(kR-\sin kR)]\cos kR}{\sigma(1-\cos kR)^2+[(kR)^3-\sigma(kR-\sin kR)]\sin kR}\ , \tag{A8}$$

with

$$\sigma\equiv\lambda R^3\ . \tag{A9}$$

For a continuum bound state we must have $C=0$ whence (A4) and its derivative must vanish at $r=R$; a nontrivial solution requires $\cos kR=1$ or $k=2\pi n/R$ with n a positive integer together with $B=0$ which in turn requires that

$$1-\lambda k^{-3}(2\pi n)=0\ . \tag{A10}$$

Thus the condition on the potential parameters for a CBS to exist takes the form

$$\sigma=\lambda R^3=(kR)^3/(2\pi n)=(2\pi n)^2,\quad n=1,2,3,\ \dots\ , \tag{A11}$$

and for a given n the state occurs at the energy $E=k^2=(2\pi n/R)^2$. Since for a bound state $B=0$, we see from Eq. (A4) that $u(0)=u'(0)=0$. The wave functions when $n=1$ and 3 are shown in Figs. 3(a) and 3(b). Had we not taken a to be zero, the treatment would have been similar and the condition for a CBS would still have been (A11) but the wave function would now be identically zero over the interval from $r=0$ to a; they would then have the form for $n=1$ and 3 shown in Figs. 3(c) and 3(d).

We now turn to the "spurious states" of Mulligan *et al.* For these $u(0)=u'(0)=0$ but the conditions for a CBS are not met. From (A4) these conditions clearly require $B=0$ whence for a nontrivial solution

$$1-\lambda k^{-3}(kR-\sin kR)=0\ , \tag{A12}$$

or

$$(kR)^3=\sigma(kR-\sin kR). \tag{A13}$$

This equation has a solution for k for any $\sigma>6$. One finds easily that the phase shift for this solution is given by

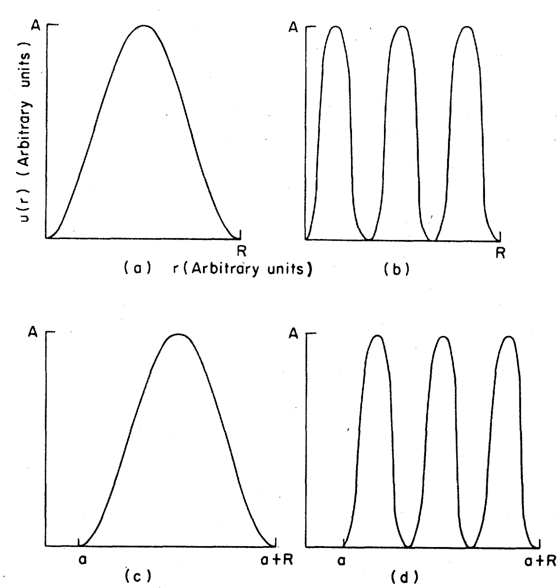

FIG. 3. The $n=1$ (a,c) and $n=3$ (b,d) continuum bound state solutions to the Schrödinger equation for the separable potential given in Eq. (A1).

$$\delta = -(kR)/2 \pmod{\pi} . \tag{A14}$$

Note that apart from the inequality $\sigma \equiv \lambda R^3 > 6$, no relation between the parameters of the potential need be satisfied for a spurious state to exist, and any change in the parameters which does not result in a violation of the above inequality will only change the energy at which the spurious state is located. If we examine what happens in the situation where $a \neq 0$, we find that for spurious states the solution has not a node but a nodal segment from 0 to a on which the wave function vanishes. We mention this to emphasize that the node-counting analysis of Refs. 1 requires some elaboration for special potentials of the type considered here.

It can further happen with a nonlocal potential that there exist two linearly independent solutions at a particular energy with one a CBS and the other a spurious state. This will occur in the case of a sum of two separable potentials of the form

$$U(r, r') = \lambda \Delta_0^R(r)\Delta_0^R(r') + \lambda' \Delta_a^{R'}(r)\Delta_a^{R'}(r') , \tag{A15}$$

with

$$0 < R < a < a + R' , \quad R \neq R' . \tag{A16}$$

In this case one sees that for $k = 2\pi n/R$ and $\lambda R^3 = (2\pi n)^2$ a CBS exists of the form

$$u_{\rm CBS}(r) = \begin{cases} A(1 - \cos kr), & 0 < r < R \\ 0, & r > R . \end{cases} \tag{A17}$$

On the other hand, for the same value of k, a spurious state exists of the form

$$u_{\rm sp}(r) = \begin{cases} 0, & r < a \\ A[1 - \cos k(r-a)], & a < r < a + R' \\ C \sin(kr + \delta), & r > a + R' \end{cases} \tag{A18}$$

with

$$\delta = -kR'/2 - ka \pmod{\pi} , \tag{A19}$$

$$C = 2A \sin(kR'/2) . \tag{A20}$$

In this case there is an ambiguity in the rule of Mulligan *et al.* as to how their rule for the absolute phase shift is to be applied, while Bolsterli's rule would simply require an upward jump of the phase shift by π at this energy.

Another case of interest is that in which one has the same potential (A14) but now with $R = R' < a$. In this case again with $k = 2\pi n/R$ and $\lambda R^3 = (2\pi n)^2$ there are now two linearly independent CBS's. For one of these

$$u_1(r) = A_1(1 - \cos kr) \tag{A21}$$

over the interval $0 < r < R$ and zero elsewhere; the second has the form

$$u_2(r) = A_2[1 - \cos k(r-a)] , \tag{A22}$$

over the interval $a < r < a + R$ and zero elsewhere. The two are clearly orthogonal to one another and there exists a regular solution as well which is orthogonal to both.

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