Sixth-order boson expansion calculations applied to samarium isotopes

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In previous publications we reported on calculations based on the boson expansion method, and showed that good overall fits can be obtained to levels of a number of collective nuclei chosen over a wide range of the periodic table. In the present paper we concentrate on Sm isotopes and obtain much improved results, the fits to experiment being very good in all cases. In achieving this, the calculations are made by first removing a few errors committed in the previous work and also by including all terms up to sixth order in the Hamiltonian. In addition to calculating energy levels, $B(E2)$ values, and quadrupole moments as in our previous work, we also calculated isomer shifts and spectroscopic amplitudes for two-nucleon transfer reactions.

[NUCLEAR STRUCTURE ^{148,150,152,154}Sm microscopic calculation of energy levels, $B(E2)$'s, static quadrupole moments, isomer shifts, two-nucleon transfer spectroscopic amplitudes, boson expansion.]

I. INTRODUCTION

In the past few years, two of us (T.K. and T.T.) have been engaged in describing nuclear collective motions in terms of a boson expansion technique. A paper[1] which dealt with the formulation (here called I) was published some time ago. The second paper (here called II) which dealt with additional formulation, as well as numerical calculations, was published recently.[2] (References to a number of earlier publications made by other authors can be found in these two papers.) In this second paper, about ten nuclei were chosen over a wide range of the periodic table, and it was shown that our calculations successfully reproduced important characteristic features of all the nuclei considered.

These calculations were nevertheless made to serve primarily as a general survey of the applicability of our boson-expansion method. Therefore, little effort was expended to obtain further improved fits to data for individual nuclei or isotopes. Encouraged by the success achieved in II, we have since been engaged in performing improved calculations, and to report on the results obtained so far for Sm isotopes is the purpose of the present paper. The Sm isotopes were chosen because of the experimental fact that a transition from spherical to rotational character takes place in going from ^{148}Sm to ^{154}Sm. Because of this, the Sm isotopes have been used as an example by a number of authors in the past to test various theories of nuclear collective motions.[3-5]

In Sec. II we, very briefly, summarize the formulation given in I and II and used in the present calculations. The results of the new calculations are given in Sec. III. As will be remarked in Sec. II, a few errors were committed in I and II which have now been corrected. Also the calculation was restricted in II to fourth order, but it has now been extended fully to sixth order. It will be evident from Sec. III that these modifications allow us to get much better agreement of the obtained results with experiment than was possible in II. It will also be seen in Secs. II and III that the present work has been extended beyond that of I and II, in that it now includes calculations of isomer shifts and of spectroscopic amplitudes for two-nucleon transfer reactions. The results obtained for these quantities also agree rather well with experiment. Discussion of the present work is given in Sec. IV.

II. BRIEF SUMMARY OF FORMULATION

The Hamiltonian that we take as our starting point is given as a sum of a single-particle Hamiltonian, a particle-hole type quadrupole-quadrupole interaction, and a pairing interaction of both monopole and quadrupole types.[1,2] Taking first the single-particle Hamiltonian and the monopole-type pairing interaction, the Bogoliubov transformation is made, so that the original, shell-model type single-particle system is replaced by a system of

quasiparticles. As mentioned in II, we did not solve the BCS equation, but rather took the gap parameters from the experimental binding energies and determined the Fermi energies λ_p and λ_n so that the occupation number given by $\sum_j (2j+1)v_j^2$ agrees with the proton and neutron numbers for the nucleus under consideration. As discussed in Bohr-Mottelson,[6] it is not always meaningful to use the third difference formula (quoted in II) to calculate the gap parameter $\Delta_{p,n}$. In particular, in the case of ^{154}Sm irregularities in the experimental binding energies give a value of $\Delta_p = 0.86$ MeV which is much lower than that obtained for the other isotopes. For this nucleus we therefore took Δ_p as determined from a second difference in binding energies as given in Bohr-Mottelson.[6] This gave a result of $\Delta_p = 1.17$ MeV and produced a small but noticeable improvement in the energy spectrum we obtained.

The single-particle energies, which were previously taken from the Nilsson Hamiltonian[7] with the deformation parameter $\beta = 0$, are here taken from results of shell-model studies of Klinkenberg.[8] The parameters in the Nilsson Hamiltonian were originally chosen so as to reproduce Klinkenberg's levels over all the major shells. As was shown (cf. Fig. 2 of Ref. 7), however, it was impossible to get exact agreement everywhere. In the course of the present calculations, we found that some aspects of the results of our calculation depended rather sensitively on the detail of the positions of the single-particle levels. We therefore decided to choose the single-particle energies to be close to what experiment shows, and thus choose Klinkenberg's levels. For all the nuclei, the same 11 neutron orbits, used in II, were again used. As for proton orbits, the calculation of II used 8 of them, and that is also the case here for $^{148\text{-}152}$Sm. For ^{154}Sm, however, three more levels ($2p_{1/2}$, $2p_{3/2}$, and $1f_{5/2}$) were added, because this gave a result significantly better than what could be obtained otherwise.

After the Bogoliubov transformation is made, the Hamiltonian is written in a form which is quadratic in the quasiparticle pair creation and scattering operators. As done in I, an orthogonal transformation is then made so as to isolate a collective particle-hole operator.

These fermion-pair operators are expanded in an infinite series of boson operator products. The coefficients of this expansion are then determined so that all the commutation relations, satisfied by the fermion pairs, are satisfied by their boson-expanded form as well. Since the Boson-expanded form is infinitely long, satisfaction of the commutation relations is required to hold order by order. After the expansion coefficients are obtained, the fermion-pair operators in the Hamiltonian are replaced by their boson-expanded forms, and thus the boson Hamiltonian emerges. As we remarked in the introduction, the present calculations employ the boson Hamiltonian to sixth order. In II, which used the fourth-order Hamiltonian, an explanation was given of how to make every calculation consistently to fourth order. The same guiding principle is again used here, except that it is now used to make every calculation consistently to sixth order.

Probably one of the most important ingredients, that was introduced in I and II, was to take into account the collective-noncollective coupling, still allowing the final form of the Hamiltonian to be written entirely in terms of collective bosons. Previously reported boson expansion calculations[3,9] had suffered from the fact that the level spacings were too large by a factor of 1.5 to 2, compared with experiment, even when the theoretical spectrum resembled experiment very closely. In II, it was shown that the collective-noncollective coupling all but removed this difficulty, making it possible for the first time to compare the theoretical spectra in *absolute scale* with experiment. In the present calculation, this coupling was taken into account throughout.

The boson Hamiltonian was diagonalized in a space spanned by products of the collective bosons. The basis states in this space are classified[10] by irreducible representations of the chain of groups $SU(5) \supset R(5) \supset R(3)$, i.e., by the set of quantum numbers N, v, γ, and I. Here N is the number of collective bosons, v is the seniority (i.e., the number of bosons that are not coupled pairwise to spin zero), and I is the angular momentum. The extra quantum number γ enumerates $R(5)$ states that are degenerate for a pair of values of v and I. In the actual calculations we truncated the boson space such that $N \leq 19$ and $v \leq 17$. For higher spins ($I \geq 10$), we further made a truncation with respect to seniority such that the dimension of the space did not exceed 100.

At this stage we remark that a few errors and misprints were detected in I and II, and we want to correct them here. The quantity $r_{\alpha;\beta}$ (and r_π) defined in Eq. (2.5) of II should have a sign opposite from what was given in this equation. The correction of this sign error is one of the major reasons why the present results fit data much better than did those of II (although the agreement with experiment obtained in II was already fairly good). Errata found in I for the sixth-order Hamiltonian are summarized in Appendix A.

The formulation and corresponding numerical calculations reported in I and II were restricted to those of energy levels, $B(E2)$ values and quadrupole

moments. In the present paper, an extension has been made so as to calculate isomer shifts and also the spectroscopic amplitudes for two-nucleon transfer reactions. Formulas that are needed to calculate these quantities are summarized in Appendices B and C, respectively.

III. RESULTS OF CALCULATIONS

A. Preliminary

Before presenting the results, we emphasize that we have only two adjustable parameters in the Hamiltonian, as was also emphasized in II. They are f_2 and g_2 which determine the strengths χ_2 and G_2 of the quadrupole-particle-hole and quadrupole-pairing interactions, respectively. They are defined as $\chi_2 = f_2 \chi_{sc}$ and $G_2 = g_2 \chi_{sc}$, where $\chi_{sc} = 240A^{-5/3}$ MeV is the so-called self-consistent value[11] of χ_2. We expect f_2 and g_2 to be roughly unity. Their values fixed to give the best overall results were found to be $f_2 = (0.845, 0.865, 0.955, 0.974)$ and $g_2 = (0.964, 1.047, 1.026, 1.01)$, corresponding to $\chi_2 = (0.049, 0.049, 0.053, 0.053)$ MeV and $G_2 = (0.056, 0.059, 0.057, 0.055)$ Mev, for ^{148,150,152,154}Sm respectively. Note that these values of χ_2 and G_2 lie within 4% of their respective median values. The use of the median values throughout retains the gross features of the results given below. Thus it is legitimate to call our theory microscopic. Compare this with the situation in some other boson theories,[5,32] where the Hamiltonian is parametrized 100%, without going through any step of its derivation.

In calculating the $B(E2)$'s and the static quadrupole moments, we used the effective charge e_{eff}, as we did in II, which for simplicity was assumed to be the same for protons and neutrons. Its value was fixed to reproduce the experimental $B(E2; 2_g^+ \to 0_g^+)$ and was found to be $e_{\text{eff}} = (0.64, 1.08, 1.16, 0.66)e$ for the four isotopes. It is unfortunate to find that the values for ^{148}Sm and ^{154}Sm are somewhat smaller than those for ^{150}Sm and ^{152}Sm, although the small value for ^{154}Sm is largely due to the addition of three proton orbits, as we discussed in Sec. II.

B. Comparison of fourth- and sixth-order calculations

We first give in Fig. 1 the best sets of fourth- and sixth-order energy spectra obtained for $^{148\text{-}154}$Sm. In Sec. II C, we compare the sixth-order results with experiment, and it will be seen that the agreement is very good in general. The comparison made in Fig. 1 will then serve to give an idea of the extent to which the fourth-order results deviate from experiment, i.e., about what sort of improvement was achieved in going from fourth-order to sixth-order calculations.

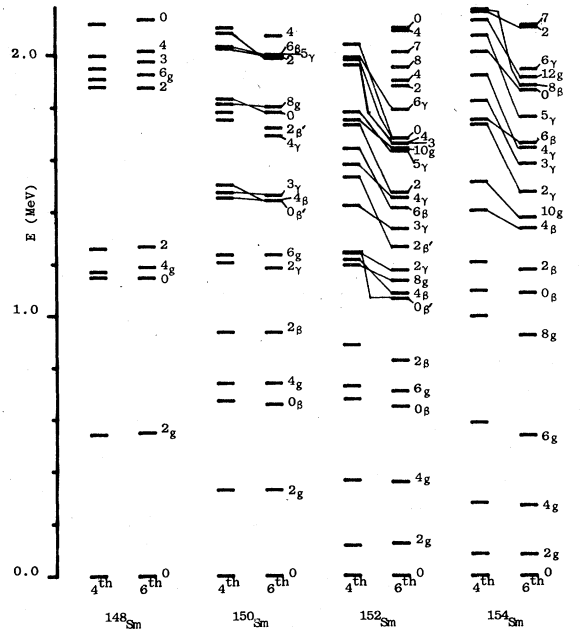

FIG. 1. Comparison of fourth- and sixth-order calculations for energy spectra of ^{148,150,152,154}Sm. Only the spins of the sixth-order spectra are labeled. The corresponding fourth-order levels, which differ significantly in energy or in ordering, are connected by straight lines to the corresponding sixth-order levels.

In Fig. 1, we labeled spins only for the sixth-order spectra. When marked difference appears in the energies between the fourth- and sixth-order calculations, the corresponding states are connected by lines. It is seen that essentially no difference took place between the fourth- and sixth-order results for ^{148}Sm. The same is true for ^{150}Sm, although a slight difference begins to be seen for states higher than 1.5 MeV.

In ^{152}Sm, the difference between the two results becomes more noticeable, having two characteristic features. The one is that, in going from fourth- to sixth-order calculations, the spacings between states that belong to the ground and the β bands get much narrower, and the other is that the head of the γ band is brought down. Essentially the same features are repeated in ^{154}Sm, with further enhancement.

It is easy to understand why the difference between the fourth- and sixth-order results is small for ^{148}Sm, but increases as the mass number is increased. Our boson expansion method begins by constructing a complete set in a spherically symmetric representation. Therefore, it is expected that the calculation converges at a lower order, when our method is applied to nuclei which are

basically spherical. The result shown in Fig. 1 for ^{148}Sm, in fact, shows that the calculation has converged at the fourth order. As the mass number (or more precisely the neutron number) is increased, the nuclei deviate more and more from the spherical shape, and call for calculations with increased order.

In II, we showed that our boson Hamiltonian could be expressed as

$$H = H(\pi_\mu) + H(\pi_\mu, \beta_\mu) + V(\beta_\mu), \tag{1}$$

where β_μ are the deformation parameters of the Bohr-Mottelson model,[12] while π_μ are the conjugate momenta. We also showed in II that, in spite of the fact that the first two terms of (1) are rather complicated, the behavior of the potential energy term $V(\beta_\mu)$ gives a good insight into what is the shape of the nucleus under consideration. In the sixth-order theory, we may write $V(\beta_\mu) = V(\beta, \gamma)$ as

$$V(\beta,\gamma) = c_2\beta^2 + c_3\beta^3\cos(3\gamma) + c_4\beta^4 + c_5\beta^5\cos(3\gamma) + c_6\beta^6 + c_7\beta^6\cos^2(3\gamma). \tag{2}$$

The same expression holds also for the fourth-order theory,[2] but with $c_5 = c_6 = c_7 = 0$.

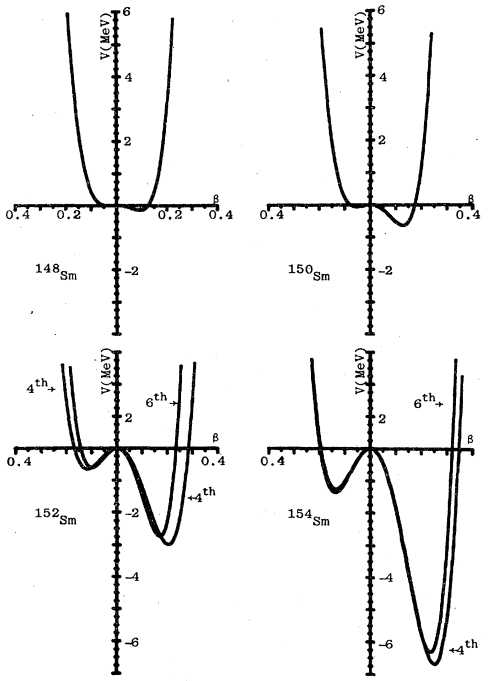

FIG. 2. Potential energy surface plotted against the deformation β for $\gamma = 0°$ (right half) and $\gamma = 60°$ (left half). Both the fourth- and sixth-order results are plotted for ^{152}Sm and ^{154}Sm, but only the sixth-order results for ^{148}Sm and ^{150}Sm, because the fourth- and sixth-order results are virtually identical in these lighter elements.

In order to understand further the features seen in Fig. 1, we plotted $V(\beta,\gamma)$ in Fig. 2 for the four Sm isotopes. For ^{148}Sm and ^{150}Sm, the fourth order $V(\beta,\gamma)$ were too close to the sixth order $V(\beta,\gamma)$ to be shown separately. However, the difference is sufficiently large for ^{152}Sm and ^{154}Sm, and we give both of them explicitly. This reflects the fact, seen in Fig. 1, that the fourth-order and sixth-order levels differ little in ^{148}Sm and ^{150}Sm, but significantly for ^{152}Sm and ^{154}Sm. In this figure, it is also clearly seen that the nuclear shape changes from spherical to prolately deformed, in going from ^{148}Sm to ^{154}Sm.

We note that the coefficient c_3 is negative for all these nuclei, reflecting the fact they are all prolate (although some of them are deformed very weakly). In the sixth-order calculation, c_5, c_6, and c_7 are all positive. Since c_3 remains about the same in going from fourth- to sixth-order calculations, the positive c_5 works to reduce the deformation, particularly for ^{152}Sm and ^{154}Sm, and the effect is clearly seen in Fig. 2. The positive c_6 and c_7 steepen the potential wall at large β. This, along with the factor $\cos^2(3\gamma)$ in the c_7 term, results, for large β, in the potential wall against deformation into axially asymmetric shapes ($\gamma \neq 0°$ or 60°) being less stiff than it is in the fourth-order potential. This explains why the γ-band head is brought down in the sixth-order calculation compared with the fourth-order calculation, as was seen in Fig. 1.

One might naively expect that the decreased value of β, corresponding to the minimum of the potential, results in a decreased nuclear deformation, i.e., in a decreased moment of inertia, and then in turn to increased spacings between levels in the ground and the β bands. The relative behavior of the fourth- and sixth-order level schemes shown in Fig. 1 is, however, opposite to this expectation. It should be noted, however, that in our calculations, we always demand that the energy of the 2_1^+ state agrees exactly with experiment, and thus, in this sense we always have a fixed value of the moment of inertia (for any given nucleus). Then a pushing of the outside potential wall to a smaller value of β (in going from the fourth- to sixth-order calculations for ^{152}Sm and ^{154}Sm), means that the nucleus is forced to behave more like a spherical than like a deformed nucleus, the spacings between the higher members of the band in turn being forced to become closer to that between the 2^+ and 0^+ states. This explains why the above, naive expectation is not realized in Fig. 1.

C. Comparison with experimental spectra

In Figs. 3–6, we compare the results of our sixth-order calculations with the experimental

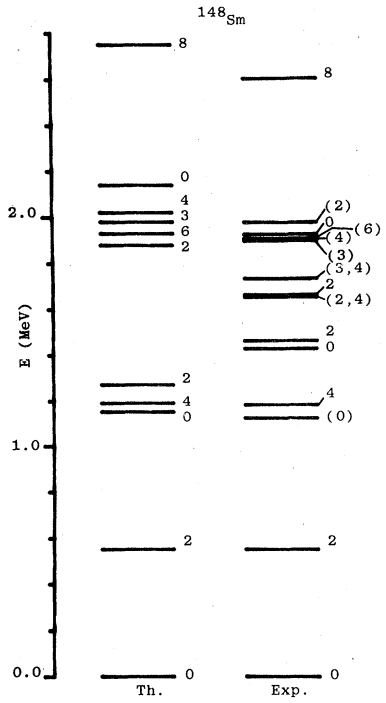

FIG. 3. Comparison of the sixth-order spectrum (Th) with experiment (Exp) for ^{148}Sm.

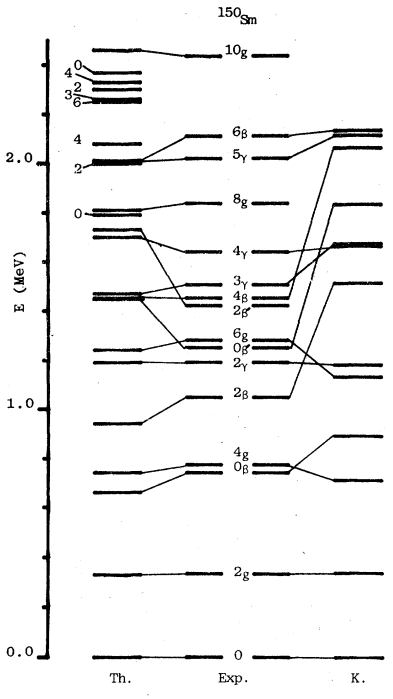

FIG. 4. Comparison of the sixth-order spectrum (Th) with experiment (Exp), as well as with the result of Kumar (K) for ^{150}Sm. Only the experimental levels are labeled and are connected to corresponding levels in theoretical spectra. Theoretical levels are also labeled when there is no counterpart in experiment.

spectra known for $^{148-154}$Sm. For $^{150-154}$Sm, there are available results of calculations[4] based on a theory of Kumar and Baranger,[13] and they are also compared with our results.

1. ^{148}Sm

In Fig. 3, the sixth-order energy spectrum for ^{148}Sm is compared with that of experiment.[14-16] Since ^{148}Sm is basically a spherical nucleus, we expect to have a vibrational pattern for its spectrum. In other words, we expect two-phonon triplet states to appear at twice the energy of the first 2^+ (2_1^+) state, and three-phonon quintet states at thrice the 2_1^+ energy. Experimentally the quintet states appear at about 1.9 MeV, and our calculation predicts them at the correct energy, but with somewhat too large splitting and with incorrect ordering. To our knowledge, however, experimental confirmation of the three-phonon nature of this quintet of states is not definite. It should also be remarked that there are observed experimentally a number of states below 2 MeV, which we believe are of quasiparticle nature. It thus seems premature to make a detailed comparison between theory and experiment for states above 1.4 MeV. A further experimental investigation is needed. We also would need[2] to extend the calculation so as to treat *explicitly* the excitation of some of the quasiparti-

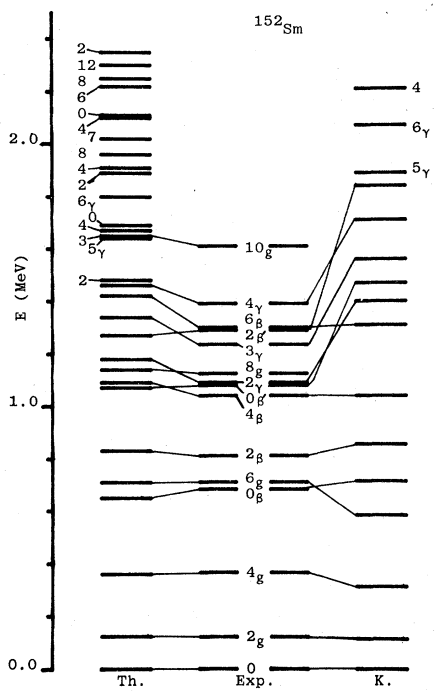

FIG. 5. Same as in Fig. 4 except that this is for ^{152}Sm.

cle modes.

As for the lower states, the presence of the 1.12 MeV (0^+) state might be questioned; the presence of this state was reported in Ref. 16, but not in Ref. 15. Suppose that this state is indeed absent. We can then show that, if we took $f_2 = 0.92$ and $g_2 = 0.76$, the resultant theoretical spectrum fits the lowest five states (0^+_g, 2^+_1, 4^+_1, 0^+_2, and 2^+_2) almost perfectly. This calculation, though looking better as regards a fit to the energy spectrum, results in a potential energy surface which is slightly more deformed than that shown in Fig. 2 for ^{150}Sm, violating a smooth transition from spherical to deformed shape as the mass number is increased. In addition, the e_{eff} needed to fit the experimental $B(E2; 2^+_1 \to 0^+_g)$ becomes unreasonably small, compared with what was obtained for the other isotopes. It should also be remarked that the above choice of $f_2 = 0.92$ and $g_2 = 0.76$ is inconsistent with values for the other isotopes. We thus tend to believe that the 0^+ state at 1.12 MeV does exist. This then leaves the problem that our theory predicts the 2^+_2 state a little too low. However, because of the reason we stated at the end of the last paragraph, we shall not go further into this problem at present.

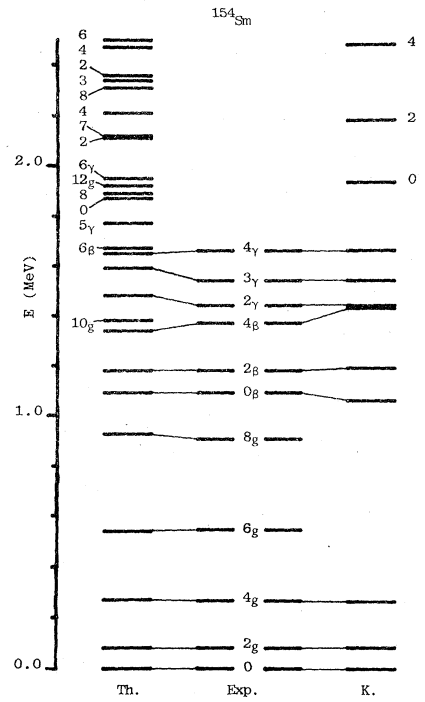

FIG. 6. Same as in Fig. 4 except that this is for ^{154}Sm.

2. ^{150}Sm

This is a rather typical transitional nucleus. As is seen in Fig. 4, the experimental spectrum[17,18] has fairly well developed ground and β bands. It also has a γ band, but there the presence of γ instability is evident in that the 3^+_γ and 4^+_γ states appear rather close together. The labeling of states as β- and γ-band states should not be taken too seriously; it was done to a large extent only for convenience. The nucleus is not very well deformed.

It is seen that the sixth-order calculation fits the data very well. Out of the 13 states belonging to ground, β, and γ bands, only three (0^+_β, 2^+_β, and 4^+_γ) are slightly off the experimental positions, but by amounts that do not exceed 80 keV. Noticeable deviations are nevertheless seen in the energies of states which we tentatively denoted as 0^+_β, and 2^+_β, states. It is very likely, however, that a consideration of *explicit* coupling with quasiparticle excitation modes would push down these states, thus removing the difficulty.

As is well known, and as was discussed in some detail in II, the theory of Kumar and Baranger[13] uses an adiabatic assumption very seriously. It is thus believed that this theory is good for well deformed nuclei, but gets poorer for weakly deformed nuclei. Kumar's result[4] reproduced in Fig. 4, seems to reflect this fact. The ground-band states are too narrowly, and the β-band states are too widely, spaced compared with experiment. Also the 3^+_γ state appears too high, although Kumar's 4^+_γ state appears much closer to experiment than does ours.

3. ^{152}Sm

This nucleus is rather well deformed, particularly in the sense that, as is seen in Fig. 5, the experimental energies[19,20] of the ground-band states obey rather closely the $I(I+1)$ law. These ground-band states are fit perfectly by our sixth-order results. Our theory also predicts very well the spacings between γ-band states, although the bandhead was predicted slightly too high. A somewhat more serious problem is seen in the β band, in that the predicted spacings are slightly too large, compared with experiment. It is a little surprising to find that the energies of the 0^+_β, and 2^+_β, states are now predicted in good agreement with experiment.

In spite of the slight discrepancy we mentioned above concerning the β and the γ bands, the improvement achieved by going from fourth-order to sixth-order calculation is obvious from Fig. 1. In the fourth-order calculations, all the higher spin members of these two bands had appeared too high. The sixth-order calculations bring them down to energies not very far away from experimental values.

Kumar's calculation[4] is expected to work better here than in ^{150}Sm. Nevertheless, it is seen in Fig. 5, that a difficulty similar to that which we pointed out concerning the ^{150}Sm calculation is again seen here. Overall, our results fit the experimental ^{152}Sm spectrum much better than do those of Kumar.

4. ^{154}Sm

This nucleus is known to be well deformed, and if the boson expansion method, which is based on a complete set prepared for a spherically symmetric system, fits the spectrum of this nucleus, it can be considered a significant achievement.

The sixth-order calculation given, in Fig. 6, reproduces the experimental collective levels[21] very well. The same may be said for Kumar's calculation.[4] As mentioned, we included three more proton single-particle levels in this calculation, and the effect was to lower the γ bandhead energy by about 160 keV, with little or no effect on the ground and β bands. Thus it helped to soften the potential wall in the axially asymmetric directions.

D. Electromagnetic transition probabilities and the static quadrupole moments

There is a limited amount of experimental data concerning the electromagnetic properties of ^{148}Sm. Comparison of the theoretical predictions with the experimental data is as follows: $B(E2; 2_1^+ \to 0_g^+) = 0.141 e^2b^2$ (exp. $= 0.141 \pm 0.005$),[22] $Q(2_1^+) = -0.58eb$ (exp. $= -0.54 \pm 0.2$, and $= -0.97 \pm 0.27$),[22,23] and $B(E2; 4_1^+ \to 2_1^+) = 0.255 e^2b^2$ (exp. $= 0.25 \pm 0.07$).[24] It is seen that good agreement has been obtained.

The $B(E2)$ values and the static quadrupole moments $Q(2_1^+)$ for $^{150-154}$Sm are summarized in Table I, which contains three columns, one for experimental values, the other for sixth-order results (followed by fourth-order results in parentheses) of the present work and finally for Kumar's results.[4] Each column has three entries, presenting values for ^{150}Sm, ^{152}Sm, and ^{154}Sm in that order.

Most of the experimental values shown in the table are those given by experimentalists, as such. We derived a few of the other entries, however, by using experimentally known $B(E2)$'s and related branching ratios which are also known experimentally. The derived values are underlined.

We shall discuss the sixth-order results first. It is seen in Table I, that $Q(2_1)$ and the $B(E2)$'s we obtained for the transitions between ground-band states agree almost perfectly with experiment. The same is also the case for Kumar's results, although the number of quantities he calculated is somewhat less than what we did.

The agreement of the calculated $B(E2)$'s with experiment, for transitions from β- and γ-band states to the ground-band states are also rather good, though not as good as they are for transitions within the ground-band states. We underlined theoretical values (ours and Kumar's) whenever they deviate from experiment by more than a factor of 2. It is seen that about one third of our values are underlined, and the same is also true with Kumar's values. Even when this sort of deviation takes place, however, it is recognized that the trend, i.e., the mass number dependence of the triad of quantities, often agrees with that of experiment, a feature which is to be desired at the least. In Table I, we attached an asterisk to a triad whose trend disagrees with that of experiment. It is seen that there is only one such case in our calculations. With Kumar's calculations, there appear four asterisks.

It is gratifying to find that our calculations reproduce the experimental electromagnetic properties as well as in Table I. It is well known that the comparison of the theoretical and experimental electromagnetic properties presents a much more severe test of a theory, than does the comparison of the energy levels.

When the electromagnetic transitions get weaker, the pertinent experimental data are normally supplied in the form of branching ratios. Therefore, the comparison of the theoretical branching ratios with experiment offers a still more stringent test of the theory. It begins to test the validity of the prediction of rather small components of the wave functions.

The calculated branching ratios are compared with experiment in Table II. We underlined theoretical values whenever they deviate from experiment by more than a factor of 5 (rather than of 2 as was the case in Table I). It is seen that only a few of the entries are underlined; in particular only one is underlined for ^{152}Sm. We may thus say that, overall, the agreement of the calculated branching ratios with experiment is rather good.

Nevertheless, one may be embarrassed to find a case in which the theory deviates by more than a factor of 100. An example is $B(E2; 2_\gamma^+ \to 2_g^+)/B(E2; 2_\gamma^+ \to 4_g^+) = 3916$ for ^{154}Sm, which is to be compared with the experimental value of 25. If one looks at Table I, it is easy to see that the very small value of $B(E2; 2_\gamma^+ \to 4_g^+) = 0.000\,012$ is largely responsible for the very large value of the above theoretical branching ratio. The corresponding experimental value $B(E2; 2_\gamma^+ \to 4_g^+) = 0.0008$ is, however, also very small. This means that the comparison of theoretical and experimental branching ratios often forces us to compare extremely small quantities. In other words, an occasional appearance of a large discrepancy in branching ratios

TABLE I. $B(E2)$ values (e^2b^2) and static quadrupole moments (eb) for ^{150,152,154}Sm. Experimental data were taken for ^{150}Sm from Refs. 22, 23, 35, and 36; for ^{152}Sm from Refs. 22, 36, 37, 38, 39, and 40; and for ^{154}Sm from Refs. 41, 42, and 43. The values which were derived from some other known $B(E2)$ values and related branching ratios are underlined. Theoretical values which differ from experiment by more than a factor of two are also underlined. An asterisk is attached to a triad of theoretical entries whenever the A dependence of this triad differed from that of experiment. Fourth-order results are enclosed in parentheses next to the sixth-order results.

Transition	Experiment			Theory					
				Present			Kumar		
$g \rightarrow g$									
$2 \rightarrow 0$	0.274(6)	0.670(15) 0.666(14)	0.922(40) 0.843(21)	0.275(0.275)	0.673(0.673)	0.881(0.909)	0.233	0.648	0.94
$2 \rightarrow 2$	−1.31(19)	−1.73(20)		−1.01(−1.00)	−1.67(−1.68)	−1.91(−1.92)	−0.95	−1.64	−1.97
$4 \rightarrow 2$	0.53(6)	1.017(14) 0.989(35)	1.186(39)	0.51(0.52)	0.98(0.99)	1.25(1.26)	0.431	0.993	1.40
$6 \rightarrow 4$		1.179(33) 1.20(6)	1.374(47)	0.64(0.67)	1.09(1.12)	1.35(1.37)	0.561	1.19	
$8 \rightarrow 6$		1.39(14) 1.45(22)	1.49(15)	0.73(0.74)	1.11(1.17)	1.38(1.41)			
$\beta \rightarrow g$									
$0 \rightarrow 2$	0.26(3)	0.176(11) 0.159(10)		0.422(0.460)	0.120(0.156)	0.054(0.054)	0.325	0.20	0.235
$2 \rightarrow 0$	0.0036(14)	0.00456(34)	0.0060(14)	0.020(0.026)	0.007(0.005)	0.001(0.008)*	0.003	0.003	0.013
$2 \rightarrow 2$	0.043(20)	0.0258(26)	0.012	0.181(0.210)	0.025(0.031)	0.010(0.014)	0.002	0.028	0.054*
$2 \rightarrow 4$	0.166(98)	0.091(11)	0.024	0.077(0.083)	0.070(0.050)	0.008(0.033)	0.090	0.137	0.129*
$4 \rightarrow 2$		0.0053(35)		0.007(0.009)	0.005(0.004)	0.018(0.011)	0.001	0.0002	0.012
$4 \rightarrow 4$		0.037(23)		0.105(0.118)	0.016(0.026)	0.003(0.010)	0.004	0.027	0.057
$4 \rightarrow 6$		0.100(57)		0.063(0.073)	0.047(0.070)	0.014(0.029)	0.071	0.139	
$\gamma \rightarrow g$									
$2 \rightarrow 0$	0.0088(20)	0.0163(11)	0.013(3)	0.020(0.019)	0.050(0.049)	0.021(0.027)	0.01	0.022	0.033*
$2 \rightarrow 2$	0.0387(141)	0.0417(42)	0.02	0.024(0.021)	0.053(0.050)	0.047(0.037)	0.125	0.051	0.047
$2 \rightarrow 4$	0.0194(100)	0.00416(32)	0.0008	0.087(0.110)	0.006(0.007)	10^{-5}(0.001)	0.034	0.003	0.010*
$4 \rightarrow 2$		0.00350(13)		0.022(0.024)	0.026(0.034)	0.021(0.020)	0.002	0.008	0.016
$4 \rightarrow 4$		0.0370(13)		0.032(0.029)	0.076(0.068)	0.040(0.043)	0.084	0.049	0.048
$4 \rightarrow 6$		0.0002		0.052(0.057)	0.002(0.008)	0.002(0.002)	0.027	0.007	
$\beta \rightarrow \beta$									
$2 \rightarrow 0$				0.19(0.18)	0.45(0.45)	0.62(0.68)	0.14	0.74	0.84
$2 \rightarrow 2$				0.18(0.25)	−1.30(−1.27)	−1.63(−1.70)	−0.50	−1.75	−1.80
$4 \rightarrow 2$				0.35(0.35)	0.63(0.72)	0.86(0.96)	0.30	1.11	1.30
$\gamma \rightarrow \gamma$									
$2 \rightarrow 2$				−0.43(−0.55)	1.21(1.16)	1.64(1.71)	0.38	1.39	1.83
$3 \rightarrow 2$				0.30(0.29)	0.76(0.77)	1.28(1.30)	0.23	0.90	
$4 \rightarrow 3$				0.12(0.12)	0.63(0.61)	0.98(0.98)	0.27	0.36	
$5 \rightarrow 4$				0.23(0.24)	0.47(0.50)	0.62(0.69)	0.16	0.72	

TABLE II. Branching ratios in ^{150,152,154}Sm. See the caption of Table I for experimental references. Theoretical ratios which differ from experiment by more than a factor of 5 are underlined. Fourth-order results are enclosed in parentheses next to the sixth-order results.

		^{150}Sm			^{152}Sm			^{154}Sm		
			Theory			Theory			Theory	
Initial	Final	Exp.	Present	Kumar[4]	Exp.	Present	Kumar[4]	Exp.	Present	Kumar[4]
2_β	$4_g/2_g$	3.87(48)	0.43(0.39)	43.7	3.18(68)	2.80(1.61)	4.9	2.07	0.86(2.36)	2.4
	$0_g/2_g$	0.084(10)	0.11(0.12)	1.47	0.176(16)	0.26(0.16)	0.11	0.49	0.10(0.61)	0.24
	$0_\beta/0_g$	27.6(39)	9.50(7.10)	45.7		69.2(90.8)	236		616.0(82.4)	
4_β	$2_g/4_g$	0.0024(5)	0.056(0.081)	0.27	0.10(3)	0.32(0.17)	0.01	0.93	6.48(1.10)	0.21
	$2_\beta/4_g$	5.93(85)	3.36(2.99)	72.7	41(18)	39.4(27.7)	41.3		317.8(99.0)	
2_γ	$2_g/0_g$	4.4(6)	1.18(1.11)	12.6	2.44(13)	1.06(1.02)	2.33	1.56	2.23(1.37)	1.42
	$2_g/4_g$	2.0(7)	0.27(0.20)	3.75	11.9(13)	9.14(7.57)	19.7	25	3916(37.0)	8
	$0_\beta/0_g$	5.2(7)	4.85(6.16)	9.24		0.42(0.37)	0.036		0.80(0.63)	
	$2_\beta/2_g$	12(6)	12.2(13.3)	1.82	1.2(1)	2.64(3.70)	0.96		0.70(0.70)	
	$0_\beta/2_\beta$	0.10(5)	0.33(0.42)	0.40		0.15(0.10)	0.016		0.51(0.63)	
3_γ	$2_g/4_g$	0.29(6)	1.09(1.14)	0.52	0.95(7)	2.68(2.83)	1.42	2.5	3.45(3.57)	1.35
	$2_\beta/2_g$	11(3)	1.80(1.97)	8.17	$\leq$0.06	0.40(0.33)	0.046		0.63(0.54)	
	$2_\gamma/2_g$	24(5)	4.34(3.89)	10.2		8.91(9.06)	22.7		33.7(26.0)	
4_γ	$2_g/4_g$	0.050(7)	0.69(0.83)	0.028	0.088(13)	0.34(0.50)	0.16	0.055	0.51(0.47)	0.32
	$2_\beta/2_g$	2.95(59)	0.08(0.06)	15.5	0.8(4)	1.65(0.31)	0.029		2.77(0.99)	
	$4_\beta/2_\beta$	70(21)	60.2(71.3)	1.07		1.58(0.91)	140.1		0.17(1.26)	
	$4_\beta/4_g$	10.3(27)	3.10(3.69)	0.46		0.89(1.40)	0.65		0.24(0.56)	
	$3_\gamma/2_\gamma$	3.7(13)	0.53(0.48)	0.58		2.60(2.35)	1.98		2.63(2.33)	

does not necessarily mean that the theory is poor to such an extent.

We shall finally discuss briefly the results of the fourth-order calculations. First of all, it should be noted in Tables I and II, that the fourth- and sixth-order results are close to one another. Also the quadrupole operators resulting from the fourth- and sixth-order calculations were nearly identical. The lowest-order term dominates in both cases and the same effective charges can be used. These facts clearly support our previous conclusion, made with regard to the energy spectra, that our calculation is indeed converging. Of course, there are found a few cases in which noticeable differences are encountered. However, in all these cases, the $B(E2)$ values involved are small. Since, as we emphasized above, small $B(E2)$'s reflect small components of the wave functions, a difference of this extent is not unexpected.

The results for ^{154}Sm are of particular interest. As we remarked above, there were sixth-order $B(E2)$'s for ^{154}Sm which disagreed with experiment, and thus were underlined in Table I. As seen in this table, however, none of the fourth-order $B(E2)$'s are underlined. In other words, the fourth-order predictions fit all the known $B(E2)$'s, even the worst deviation being within a factor of 2. These results reflect themselves into that of the branching ratios in Table II. In particular, the trouble, pertaining to the $2_\gamma^+ \to 4_g^+$ transition discussed above, has completely disappeared.

It is likely that the excellent $B(E2)$ predictions of the fourth-order calculation in ^{154}Sm are somewhat accidental, since the energies of the levels are not predicted as well as in the sixth-order calculation. We may also remark that it should be possible to make the sixth-order ^{154}Sm $B(E2)$ results as good as the fourth-order results. Since only subtle features of the wave functions are at issue, small modifications of some input quantities might achieve this. Reasonable possibilities are to make the energies of the single-particle orbits dependent (apart from the $41/A^{1/3}$ scale factor) on the mass number[31] or to increase the number of single-particle orbits used in the calculation.

E. Isomer shifts

The isomer shift $\delta\langle r_p^2\rangle$, which is the proton mean-square radius in the 2_1^+ state minus that in the ground state, was calculated by using formulas given in Appendix B, and the results are summarized in Table III. This table also contains experimental results[25,26] and other theoretical predictions.[4,27,28] As is seen, our value agrees with experiment in trend, but is about a factor of 1.6 too large for ^{150}Sm, and is grossly too large for ^{154}Sm. The agreement is perfect for ^{152}Sm.

Marshalek[27] calculated many isomer shifts, by using the pairing-plus-quadrupole interaction (PPQ), as we do here, and by solving the self-consistent cranking equation to third order. The results thus obtained were too large by about a factor of 3, due to an overestimate of the centrifugal stretching. Meyer and Speth[28] then suspected that this overestimate was due to the use of PPQ, and showed that better agreement with experiment was obtained, if an effective interaction of the Migdal type was used. The fact that we obtained a good fit to data, at least for ^{152}Sm, however, seems to indicate that the use of PPQ does not always result in disagreement with experiment. It should be emphasized that our calculation of the isotope shift was made as part of a larger calculation which obtained energies, $B(E2)$'s and so forth pertaining to a number of levels, while Meyer and Speth considered only the 2_1^+ and 0_g^+ states.

Concerning the too large isomer shifts we encountered in ^{154}Sm, it should be remarked that a sufficiently small value can be obtained only through very delicate cancellations of contributions from a number of single-particle states; cf. Eq. (B3) in Appendix B. Normally an orbit above (below) the Fermi energy gives a positive (negative) contribution to the isomer shift. Thus, had we suppressed the contribution of the two highest orbits ($2f_{7/2}$ and $1h_{9/2}$) in our ^{154}Sm calculation, we would have obtained a theoretical value for the isomer shift, which agreed very well with experiment. The same may be achieved by adding contributions of orbits that lie below the $Z=28$ shell. In

TABLE III. Isomer shifts in units of 10^{-3} fm^2, for ^{150,152,154}Sm. Experimental values were taken from Ref. 26 for ^{150,152}Sm and Ref. 25 for ^{154}Sm.

		Theory			
	Exp.	Present	Kumar[4]	Meyer-Speth[28]	Marshalek[27]
^{150}Sm	49.6 ± 2.6	73.3	56.		
^{152}Sm	18 ± 4	18.4	44.	14.6	49.3
^{154}Sm	1.2 ± 0.8	12.9		5.55	5.4
	0.75 ± 0.5				

this way, it appears that the calculation of the isomer shifts requires far more delicate arguments of how to treat the contributions of far-off levels, than is needed in discussing the other properties of a nucleus. (We suspect that this delicate problem is also embedded in the work of Meyer and Speth.[28]) Because of this, we shall be content with results obtained so far in the present calculations.

F. (t, p) cross sections

By using the ground-state wave functions obtained in the course of the above calculations for the four Sm isotopes, we calculated the (relative) cross sections for (t, p) reactions between ground states of the consecutive even-even nuclei. First the spectroscopic amplitudes are calculated, by using the formula given in Appendix C. They are then multiplied with the pair of radial wave functions, and the results were summed over these pairs of states. The resultant sum was then transformed to a sum of products of relative and center-of-mass (c.m.) parts of the wave functions, and the term corresponding to the Os relative state has been retained. Its coefficient is nothing but the c.m. wave function to be used as the DWBA form factor for the corresponding (t, p) reaction.

Instead of calculating the (t, p) cross section, we took the square of the form factor, at its peak at the nuclear surface, and assumed it to be simply proportional to the cross section. By carrying out this calculation for $^{A}\mathrm{Sm}(t, p)^{A+2}\mathrm{Sm}$ transitions with $A = 148$, 150, and 152, we were thus able to obtain the relative cross sections for these three reactions. Normalized so that the $^{150}\mathrm{Sm} \rightarrow {}^{152}\mathrm{Sm}$ cross section agrees with experiment, the results are given in Table IV, and are compared with experiment.[29] As is seen, the $152 \rightarrow 154$ result agrees very well with experiment, but the $148 \rightarrow 150$ result was too small by about a factor of 2. The latter discrepancy may again suggest the need of adding the noncollective modes explicitly in our calculation.

TABLE IV. Peak differential cross sections in mb/sr, for (t,p) reactions on Sm targets. The theoretical values were normalized so that the result agrees with experiment for the $^{150}\mathrm{Sm} \rightarrow {}^{152}\mathrm{Sm}$ transition. See text for more detail.

$A \rightarrow (A+2)$	$148 \rightarrow 150$	$150 \rightarrow 152$	$152 \rightarrow 154$
Exp.	0.57	0.19	0.30
Theory	0.27	0.19	0.27

IV. SUMMARY AND DISCUSSIONS

We have compared our theoretical results with a large number of quantities known experimentally for $^{148-154}\mathrm{Sm}$, and the overall agreement obtained is seen to be very good. There nevertheless remain discrepancies with regard to quantities, like (some of the) branching ratios and isomer shifts, which are considered to depend sensitively on the fine details of the wave functions.

We remarked earlier that the choice of single-particle energies, with which all our calculations begin, plays a very crucial role. We found that the use of either Nilsson or Klinkenberg level schemes resulted in noticeable changes in the obtained spectra. It is thus possible that a further search for a better set of single-particle energies would help to decrease the still remaining discrepancies. In the above calculations, we used a fixed set of single-particle energies for all the four isotopes, which might have happened to be the most appropriate for $^{152}\mathrm{Sm}$. Recall that the best overall fit to experiment was found for this nucleus.

In spite of the fact that we considered here isotopes whose neutron numbers differ only by six, there is reason to believe that different choices of single-particle energies are well justified for different isotopes. Let us note that Kisslinger and Sorensen[30] and Hammamoto[31] pointed out the importance of considering the short-range residual interactions between protons and neutrons that are in orbits with the same quantum numbers. In our mass region, the $2f_{7/2}$ and $1h_{9/2}$ neutron orbits lie very close to the Fermi energy, and thus their amplitudes of occupation, i.e., the u and v amplitudes of the BCS theory vary rather significantly as the neutron number is increased. The short-range proton-neutron interaction then affects the position of these orbits, as well as those of their proton counterpart. The $2f_{7/2}$ and $1h_{9/2}$ proton orbits lie rather high above the Fermi energy. Nevertheless, their precise positions can affect our results, because these orbits give significant contributions to our Hamiltonian through the quadrupole-pairing interaction. By allowing for the mass number dependence of the single-particle energies in this way, we may expect to improve further the results obtained so far. This may be particularly the case with regard to the branching ratios and isomer shifts.

A number of other theories which introduced bosons in one way or another have been recently utilized. The work of Lie and Holzworth[9] demonstrated that a nearly perfect fit can be obtained when the coefficients in the boson Hamiltonian were allowed to be free parameters. Their applications were, however, limited to nuclei that are basically

spherical. Also the number of free parameters was as large as eight. As we stated in the introduction, when a microscopic version of their boson theory was used, they encountered the trouble of having too large level spacings (by about a factor of 1.5). A similar difficulty was experienced by Sorensen,[3] who used a boson expansion theory very close to ours, but without collective-noncollective interactions.

Janssen *et al.*[5] started with a set of quadrupole operators and their conjugates, and then formed their commutators. There are thus altogether 35 operators which were then recognized to form generators of an SU(6) group. They wrote down a Hamiltonian with these SU(6) operators, and applied it to nuclei in the $A = 150$ region, in particular ^{150}Sm and ^{152}Sm. They introduced a cutoff parameter N and parametrized their Hamiltonian, keeping only second- and third-order boson terms. N can be related to the number of pairs of valence nucleons, and thus they were left with three free parameters. Their fits to the experimental energy levels were not as good as we presented in Figs. 4 and 5. They also had difficulties in fitting some of the branching ratios, to an extent similar to what we experienced.

Another parametrized boson theory, the interacting boson approximation (IBA),[32] which also uses SU(6), but in a way very different from that of Janssen *et al.*,[5] was applied by Sujkowski *et al.*[18] to ^{150}Sm. The fit obtained, with four free parameters, was somewhat poorer than our result, if only low-lying states in Fig. 4 are concerned. IBA fit, however, the energies of high-spin ($I \leq 14$) states of the ground band very well.

We have also calculated the energies of these high-spin states for ^{150}Sm, and found that the obtained values became progressively too high as I increased. The experimental energies for $I = 10$, 12, and 14 states are 2.433, 3.048, and 3.676 MeV, respectively. Our results were higher than these values by 26, 141, and 325 keV, respectively. One may suspect that this discrepancy is caused either because we truncated our Hamiltonian, cf. Sec. II, or because the calculation has not converged even with sixth order, when applied to such high-spin states. We have confirmed, however, that neither of these suspicions hold, particularly by noticing that the fourth- and sixth-order calculations gave the same energies for these states.

A possible way to improve the situation may be the following. Let us take two neutrons in high-spin orbits, e.g., an $(i_{13/2})^2$ pair, and allow them to be coupled to various angular momenta. These newly coupled modes will then be represented by what we may call (active) noncollective bosons and will be treated on an equal basis with our current collective quadrupole mode. To work fully in this extended space, in the same way as has been done with a pure collective quadrupole boson, will certainly make the calculation impractical. Actually, however, such a large scale calculation can be avoided. We have already solved the problem of purely quadrupole bosons. We shall then just retain the wave functions for the ground-band and other low-lying states, and couple them to the additional mode of noncollective nature. The Hamiltonian to be diagonalized in this new space is the interaction, between this new mode and the collective quadrupole mode, which has been ignored in the calculations made so far.

It should be recognized that the idea behind this proposed approach is very closely related to that of Stephens and Simon,[33] who pointed out the importance of considering the coupling of the decoupled pairs to the nuclear rotation, in explaining the back-bending phenomena found in a number of deformed nuclei. In other words, we are proposing here a possible new approach to the problem of back bending, and such work is under preparation. It is hoped that the problem of high-spin states in ^{150}Sm will also be solved in the course of this type of calculation.

We recall that we have compared our calculations to those of Kumar and have been successful in achieving a great deal of improvement. Recently, Kumar and collaborators[34] have introduced an improved version of his earlier work, but this new theory has not yet been applied to samarium.

We may summarize the following achievements in the present work. (i) Transition from spherical to deformed nucleus, in going from ^{148}Sm to ^{154}Sm is well explained, by reproducing energies, $B(E2)$'s and other quantities of these nuclei in a very consistent way. (ii) Convergence of the calculations at the fourth order has been confirmed for two lighter elements, ^{148}Sm and ^{150}Sm. The rather small, though significant, difference between the fourth- and sixth-order calculations experienced in heavier elements ^{152}Sm and ^{154}Sm leads us to be convinced that the calculation has converged at the sixth order. (iii) It was found that good choice of single-particle energies is sometimes very crucial in getting a good fit to data.

Because of the confirmation of the convergence we have shown, our theory may be called a sixth-order boson theory, as compared with some other theories which appear to have claimed to achieve a good fit to data by using a fourth-order boson Hamiltonian[5,32,44]; and one would certainly prefer to have a lower-order theory, since it is simpler. We note, however, that for nuclei which are basically spherical, like ^{148}Sm, our theory is in fact

a fourth-order theory. We further note that our computer program has been written in such a way that the sixth-order calculation takes only slightly more computational time than does the fourth-order calculation.[45]

The work of Janssen *et al.*[5] is, as we stressed in Sec. III A, a purely parametrized theory, and the truncation of the Hamiltonian to the SU(6) form was made as an ansatz. More recently, Holzwarth, Janssen, and Jolos[46] revisited the work of Ref. 5, based on an extended form of the formalism of Ref. 9, and indicated that the conditions to justify the SU(6) truncation are not necessarily satisfied. They further showed, probably more importantly, that the fifth-order terms are by no means small, even when applied to spherical nuclei. The work of Refs. 9 and 46 is microscopic, as is ours, though the methods used are fairly different. As far as the convergence argument is concerned, however, both microscopic theories point to the use of a sixth-order theory,[45] at the same time showing that the expansion can in fact be terminated there. The comparison of the work of Refs. 5 and 46 may further be considered as a warning against taking too seriously the predictions of a parametrized theory.[47]

The interacting boson approximation (IBA)[32,44] is likewise a fully parametrized theory (as it has been used to fit data so far), and the significance of the obtained fit might again be questioned. As for Sm isotopes, comparison with experimental spectra was made in Ref. 44, where the theoretical spectra were obtained by freely varying four parameters in the Hamiltonian; ϵ, κ, κ' and κ'' in Eq. (7.1) there. In spite of the use of the complete freedom thus allowed, the obtained fit is by no means better, if not worse, than what we obtained above. [This can be confirmed by plotting the theoretical levels in Fig. 25 of Ref. 44 in our Figs. 3–6. It is not difficult to see that a similar conclusion can be drawn with regard to $B(E2)$ values as well.] It is also important to note (in Table II of Ref. 48) that, not only does κ increase by an order of magnitude, but also κ' and κ'' change sign in going from $A=148$ to 154. When such a drastic variation is made to *free* parameters, justifications must be made of their use. Otherwise the obtained fit remains rather arbitrary.

As in Ref. 5, the IBA Hamiltonian has been chosen to be of fourth order, again as an *ansatz*. Only very recently, Otsuka *et al.*[48] attempted to justify this Hamiltonian microscopically. Their procedure may briefly be summarized as follows: (i) An even number of fermions are taken assuming that they occupy a single shell with a large j value. (ii) A set of fermion wave functions are then constructed, so that each member of this set behaves as close as possible to a corresponding member of the boson wave functions. A large number of the other unwanted fermion states are just thrown away. (iii) The parameters in the IBA Hamiltonian are fixed by equating the expectation values of this Hamiltonian, taken for one- and two-boson states, with those of the fermion Hamiltonian taken for states with j^2 and j^4 configurations. (iv) It is then checked whether the expectation values and other matrix elements of the thus fixed IBA Hamiltonian taken for states with higher N, the boson number, agree with those taken for corresponding fermion states with $v=2N$, v being the seniority. If the equality of the corresponding matrix elements holds, the use of the (fourth-order) IBA Hamiltonian is considered justified.

Let us consider first the step (iv). As is seen in several tables given in Ref. 48, the expected equality holds rather well for $N=3$. However, for $N=4$, noticeable violation of the equality begins to take place; in many cases it is by 20 to 30%, and in a few cases it is by as much as 50 to 60%. Since it is expected that the discrepancy increases as N increases, a serious doubt is cast as to whether the justification of the fourth-order IBA Hamiltonian was in fact achieved. As Otsuka *et al.* themselves alluded to, the difficulty will certainly be eased, if one extends the IBA Hamiltonian to sixth, or still higher orders. If this is done however, the claimed simplicity of IBA is lost. In passing, it may be worthwhile to note that, in the matrix elements of our sixth-order Hamiltonian taken for states with $N\leq 4$, the contributions from the fifth- and sixth-order terms are less than 3%. In other words, within states with $N\leq 4$, our sixth-order Hamiltonian is in effect a fourth-order Hamiltonian with a confidence level of 97%.

The step (ii), mentioned above, may also be questioned, because it appears to be a procedure which maps the boson space into the fermion space. A correct procedure would have to be the opposite. The step (i) is also highly questionable, because a realistic microscopic theory must be based on a set of nondegenerate subshells. Because of these reasons, it may not be unfair to conclude that the task of justifying the IBA Hamiltonian is far from being accomplished by the work of Ref. 48. Note that, with our method, none of these difficulties is encountered.

At this stage, it may be worthwhile to note further that the physical understanding of the origin of the nuclear collective motions, taken as the basis of IBA is quite different from that of all the rest of the microscopic boson theories, including ours. In the latter, the particle-hole (ph) mode is considered to be responsible for the collective motion, while in the former it is the particle-particle

(pp) mode. It is our understanding that the ph modes successfully explain the giant resonances ($E0$, $E1$, and $E2$), while there does not exist a counterpart with the pp description. We may also consider the collective 3^- state in ^{208}Pb, which is known to be easily accounted for in terms of the ph mode. If the pp picture is taken literally, however, there is no way to explain this state. Therefore, if one sticks to the pp picture in understanding the (nongiant) quadrupole collective mode, one is forced to conclude that the physical origins of the quadrupole and octupole collective motions are entirely different from one to the other. This conclusion contradicts the basic idea of the original Bohr-Mottelson model.

Returning to the question of the convergence, we note that, to our knowledge, there are two known methods which make the boson expansion exactly finite. The one is that of Schwinger (referred to as S henceforth),[49] and the other is that of Dyson.[50] The former was explored in great detail recently by Blaizot and Marshalek (BM),[51] who compared it with the method of Holstein and Primakoff (HP),[52] which is an infinite boson expansion. What BM noted was that these two methods are partially equivalent, in that their bosons can be transformed mutually from one to the other, by a *one-sided* unitary operator. Had this operator been two sided, the S and HP bosons are completely (rather than partially) equivalent, and there would be no doubt that the use of the S method is preferred. However, the reason why the above unitary operator is one-sided is that the S method includes unphysical components in its Hilbert space (contrary to HP which does not) and thus introduces a number of spurious states. This is the penalty one has to pay for making the expansion finite. BP[51] showed that there is a way to remove the spurious states by using a perturbation method. Because of the use of this perturbation, however, the S expansion becomes in practice an infinite expansion.[51]

Contrary to Schwinger's, the method of Dyson[50] seems to be free from the spurious states, and the use of this method would be worthwhile to explore further. One apparent disadvantage is the fact that Dyson's Hamiltonian is non-Hermitian. However, as has been discussed, e.g., by Ring and Schuck,[53] ways are known to diagonalize such a Hamiltonian, obtaining real eigenvalues. Thus the non-Hermitian Hamiltonian does not cause any practical difficulty. It may nevertheless be noted that Dyson's Hamiltonian is of sixth order (had the BCS transformation been made in the original fermion Hamiltonian), rather than fourth, and thus is as complicated as is (the sixth-order version of) our Hamiltonian. In other words, the confirmation that we can terminate our expansion at the sixth order can be regarded as a confirmation that our seemingly more complicated infinite boson Hamiltonian is not in practice more complicated than is the exactly finite boson expansion of Dyson. This may be considered a significant achievement made in the present work. To be more precise, however, the reason why we were able to terminate at the sixth order was that we have so far been interested only in states with compartively low energies. If we want to extend our description to still higher states, we may find that the use of Dyson's method is more preferable.

Staying with lower-energy states, it is our belief that extension to higher orders is now unnecessary. More important than increasing the vertical expansion would be the inclusion of more explicitly the excitation of noncollective modes in our calculations.[2] For this purpose, the use of our method, rather than Dyson's would be more preferred, because our Hamiltonian is Hermitian, making more transparent the interpretation of the behavior of the Hamiltonian and of the resultant wave functions.

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APPENDIX A: SUMMARY OF THE SIXTH-ORDER HAMILTONIAN

There were errors and misprints for the sixth-order Hamiltonians given in I. We first give here their corrected versions. The equation numbers are those of I.

$$H_{BB}^{(6,1)}(42) = -2xQ^2\Big\{6iK_1(K_2^2 - K_2) + v[3K_1I_2 + 6I_1(K_2 - 1)] + w\sum_{abcdef}[6Y(\bar{a}bge)Y(cfgd) + 3Y(c\bar{a}gd)Y(fbge)]A_a^\dagger A_b^\dagger A_c^\dagger A_d^\dagger A_e A_f\Big\}, \tag{5.2a}$$

$$H_{BB}{}^{(6,2)}(33) = -Q^2\left[4r^2K_2(K_2{}^2 - 3K_2 + 2) + 4rsI_2(K_2 - 2) + s^2\sum_{abcdefg} Y(abcd)Y(aefg)A_b^\dagger A_c^\dagger A_g^\dagger A_d A_e A_f\right], \tag{5.20b}$$

$$\begin{aligned} H_{BB}{}^{(6,2)}(31) + H_{BB}{}^{(6,2)}(22) + H_{BB}{}^{(6,2)}(11) \\ = -Q^2\{[8r^2 + (12F^+ + 20F^-)rs/7 + s^2(14G_{00}^{+-} + 20G_{22}^{+-} + 36G_{44}^{+-})/35](K_1K_2 + \text{H.c.}) + 4r^2(K_2{}^2 - K_2) \\ + 4rsI_2 + s^2[(21G_{00}^{++} - 30G_{22}^{++} + 9G_{44}^{++})K_1K_1^\dagger + (60G_{22}^{++} + 45G_{44}^{++})(K_2{}^2 - K_2) + 15(G_{44}^{++} - G_{22}^{++})(J^2 - 6K_2)]/105 \\ + (32r^2 + 8rsF^+)(K_2{}^2 - K_2) + 8rsI_2 + 4s^2[(2G^- - G^+ - 2G_1^-)K_1K_1^\dagger + (3G^+ - G^- + 2G_1^-)(K_2{}^2 - K_2) \\ - (G^+ - G^- + 4G_1^-)(J^2 - 6K_2)/3]/7 \\ + (24r^2 + 48s^2 + 8rsF^+ - 4s^2F^+)K_2\}, \end{aligned} \tag{5.21}$$

$$\bar{F}^{(\pm)} = \sum_k (\pm)^k(2k+1)f_k{}^2/5, \tag{5.21a}$$

$$\bar{O}_4^{42} = 5[[A_2^\dagger A_2^\dagger]_2 A_2]_0[[A_2^\dagger A_2^\dagger]_2 A_2]_0 - 4K_1K_2/7, \tag{5.26}$$

$$E_1^{[W3]} = 8(G_\delta^{(+,-)} + G^{(-)}) + 2G_\delta^{(-,-)}, \quad E_2^{[W3]} = 6G_\delta^{(+,+)} + 4(G_\delta^{(+,-)} + G^{(-)}) + 3G^{(+)} + G_\delta^{(-,-)},$$

$$E_3^{[W3]} = \tfrac{9}{2}G_{2,2,s}^{(+,+)} + 6G_{2,2,s}^{(+,-)} + 6G_{2,2,s}^{(-,-)} + \tfrac{15}{2}\sum_g \delta_{\lambda_g,0}[f_2(g)]^2,$$

$$E_4^{[W3]} = 8(G_{2,2,s}^{(+,-)} + G_{2,2,s}^{(-,-)}) + 10\sum_g \delta_{\lambda_g,0}[f_2(g)]^2,$$

$$\begin{aligned} E_5^{[W3]} = 2E_5^{[W3]}\ \text{(of Ref. 1)} + \tfrac{12}{5}(6G_{4,4}^{(+,-)} - G_{0,0}^{(+,-)} + 3G_{4,4}^{(+,+)}) - 6G_{2,2}^{(+,-)} - 3G_{2,2}^{(+,+)} \\ - \tfrac{6}{5}G_{0,0}^{(+,+)} - \tfrac{6}{5}\sum_{k=0}^{4}(2k+1)G_{1,k,\delta}^{(-,-)}, \end{aligned} \tag{5.33}$$

$$E_1^{[W4]} = 9G_{2,2,s}^{(+,+)},$$

$$E_2^{[W4]} = 20(G_{2,2,s}^{(+,-)} + G_{2,2,s}^{(-,-)}) + 25\sum_g \delta_{\lambda_g,0}[f_2(g)]^2, \tag{5.42}$$

$$\begin{aligned} E_1^{[S2]} = 9F_2^{(+)}F_2^{(+)}, \\ E_2^{[S2]} = 20F_2^{(-)}(F_2^{(+)} + F_2^{(-)}), \end{aligned} \tag{5.44}$$

$$\begin{aligned} H_{BC}^5 = -4\sqrt{5}rQ\Lambda_2[[A_2^\dagger A_2^\dagger]_2 A_{\tilde{2}}]_0K_2 - \sum_\epsilon \sqrt{5}sQ\Lambda_2(\epsilon)[2F^-(\epsilon) + F_2^+(\epsilon)][[A_2^\dagger A_2^\dagger]_2 A_{\tilde{2}}]_0 \\ - 2\sqrt{5}sQ\Lambda_2\sum_{kk'}\sum kk'F_k^\dagger W(22kk';22)[[[A_2^\dagger A_2^\dagger]_k A_2^\dagger]_{k'}[A_2A_2]_{\tilde{k}'}]_0 + \text{H.c.} \end{aligned} \tag{5.49}$$

Finally the second line of (5.51) should read $(2F_2^{(+)} + 48F_2^{(-)} + 34F_0^{(+)} - 20F_0^{(-)})\bar{O}_2^{[32]}$.

In terms of the basic quantities defined in I, with the corrections shown above, we shall now summarize the collective part of the fifth- and sixth-order Hamiltonian. They are given as

$$H_{BC}^{(5)} = -(sQ_2/99)\sum_{i=1}^{3} h_{32i}(\bar{O}_i^{[32]} + \text{H.c.}), \tag{A1}$$

$$\begin{aligned} H_{BB}^{(6)} = H_{BB}^{(6,1)}[42] + H_{BB}^{(6,2)}[33] \\ = -Q_1\sum_{i=1}^{5}[h_{42i}(\bar{O}_i^{[42]} + \text{H.c.}) + h_{33i}\bar{O}_i^{[33]}]. \end{aligned} \tag{A2}$$

The operators $\bar{O}^{[32]}$, $\bar{O}^{[33]}$, and $\bar{O}^{[42]}$ were defined in Eqs. (5.50), (5.40), and (5.26), respectively, of Ref. 1, while the coefficients are defined as

$$\begin{aligned} h_{321} = \sqrt{5}(41F_2^{(+)} - 6F_2^{(-)} + 4F_0^{(+)} - 14F_0^{(-)}), \\ h_{322} = \sqrt{5}(2F_2^{(+)} + 48F_2^{(-)} + 34F_0^{(+)} - 20F_0^{(-)}), \\ h_{323} = -7F_2^{(+)} + 30F_2^{(-)} - 20F_0^{(+)} + 70F_0^{(-)}, \end{aligned} \tag{A3}$$

and

$$\begin{aligned} h_{42i} = s^2D1(i) + xwD3(i), \\ h_{33i} = s^2D2(i) + xwD4(i), \end{aligned} \tag{A4}$$

with

$$D1(i)=\sum_{j=1}^{5} c_{ij}^{[42]}E_j^{[S1]},$$
$$D2(i)=2\sum_{j=1}^{5} c_{ij}^{[33]}E_j^{[S2]},$$
$$D3(i)=6\sum_{j=1}^{5} c_{ij}^{[42]}E_j^{[W3]},$$
$$D4(i)=36\sum_{j=1}^{5} c_{ij}^{[33]}E_j^{[W4]}. \quad \text{(A5)}$$

The constants $c_{ij}^{[42]}$ and $c_{ij}^{[33]}$ were defined in Tables 1 and 2 of Ref. 1. Finally,

$$Q_1=\chi_2 q_0^2/2, \text{ and } Q=\chi_2 q_0 r_{00}/2, \quad \text{(A6)}$$

with q_0 and r_{00} defined by Eq. (2.5) of Ref. 2.

In the solution of the boson expanded commutation relation equations [see Eqs. (3.5) of Ref. 1], we truncated to the collective branch as was done for the Hamiltonian. The resulting solution is then $r=u=v=0$, $s=-1/(4x)$, and $w=-s^2/(18x)$, where x, r, s, u, v, w, were introduced in Ref. 1. As in Ref. 2, x is fixed to a value slightly greater than unity in order to normalize boson expansion of the pair operator to unity. The transformation of the Hamiltonian, from the above form in the $A^\dagger$ representation to that in the $\alpha^\dagger$ representation, is straightforward, if use is made of the formulas given in Ref. 2.

Addition of the noncollective coupling, as described in Ref. 2, was also extended to fifth and sixth order. Using similar notation as in Eq. (4.9) of Ref. 2, we define

$$g_{ij}=\frac{1}{2}\sum_{\nu} f_i^\nu f_j^\nu\left(\frac{1}{n_i\hbar\omega-W_\nu}+\frac{1}{n_j\hbar\omega-W_\nu}\right) \quad (i,j=1,13), \quad \text{(A7)}$$

with

$$n_1=n_{12}=n_8=n_{10}=-1,\quad n_2=n_7=n_{11}=n_{13}=1,$$
$$n_5=0,\quad n_4=2,\quad n_3=-2,\quad n_6=-3,\quad n_9=3.$$

The coefficients f_i^ν $(i=1,5)$ were given in Ref. 2 where $f_i^\nu=0$ $(i>5)$. Let $f_i^\nu=f_i^\nu$ (Ref. 2) $+p_i^\nu$.

$$p_i^\nu=-\sum_\alpha \phi_\alpha^\nu Y(\alpha)b(i) \quad (i=1,5),$$
$$p_i^\nu=-\sum_\alpha \phi_\alpha^\nu Z(\alpha)b(i) \quad (i=6,13), \quad \text{(A8)}$$

with

$$b(1,2,\ldots,13)=(\phi,\psi,0,0,0,\psi\phi^2,\psi^3,\phi^3,\times\psi^2\phi,2\psi^2\phi,2\psi\phi^2,-2.5\phi,-2.5\psi),$$
$$Y(\alpha)=\chi_2\left(\sum_\beta r_{\alpha\beta}r_{0\beta}/2-3r_{00}r_{0\alpha}/7\right),$$
$$Z(\alpha)=2\chi_2 r_{00}r_{0\alpha}/7. \quad \text{(A9)}$$

We may write, as in Eq. (4.10) of Ref. 2,

$$H'_{PP}=\sum_{ii'} g_{ii'}(O_{2\mu}^{(i')\dagger}O_{2\mu}^{(i)}+\text{H.c.}),$$

where the operators O_2^i $(i=1,5)$ were defined in Eq. (3.13) of Ref. 2. The new operators are given as

$$O_{2\mu}^6=\alpha_\mu^\dagger K_1^\dagger,\quad O_{2\mu}^7=\alpha_\mu^\dagger K_1,\quad O_{2\mu}^8=\alpha_\mu K_1^\dagger,$$
$$O_{2\mu}^9=\alpha_\mu K_1,\quad O_{2\mu}^{10}=\alpha_\mu^\dagger K_2,\quad O_{2\mu}^{11}=\alpha_\mu K_2,$$
$$O_{2\mu}^{12}=[\alpha^\dagger[\alpha^\dagger\alpha]_1]_{2\mu},\quad O_{2\mu}^{13}=[\alpha[\alpha^\dagger\alpha]_1]_{2\mu}, \quad \text{(A10)}$$

with the operators K_1 and K_2 defined in Ref. 2.

The above Hamiltonian H'_{PP} can easily be brought into the form

$$H'_{PP}=\sum_{[mn]i} h'_{[mn]i}\,\overline{O}_i^{[mn]} \quad \text{(A11)}$$

to obtain the contribution of collective-noncollective coupling to the fifth- and sixth-order Hamiltonian in the α representation. See Ref. 2 for details.

APPENDIX B: FORMULATION OF THE CALCULATION OF ISOMER SHIFTS

The isomer shift we intend to calculate is that of the 2_1^+ state. It is defined as the difference of the mean-square (ms) radius of protons in this state with that of the ground state. More precisely, it may be given by the following formula:

$$\delta\langle r_p^2\rangle=\langle r_p^2\rangle_{2^+}-\langle r_p^2\rangle_{0^+}-\frac{d}{dA}(r_0^2A^{2/3})(N_{2^+}-N_{0^+}). \quad \text{(B1)}$$

The inclusion of the last term in (B1) is made, because we use the BCS theory which does not conserve the nucleon number. In this term, N_{2^+} means the expectation value of the proton number in the 2^+ state, and N_{0^+} is defined similarly. The effect of this term was to decrease the isomer shift, given by the first two terms of (B1) by up to 25%.

In the first two terms of (B1), we may set

$$\langle r_p^2\rangle_{I^+}=Z^{-1}\langle I^+|M_0|I^+\rangle, \quad \text{(B2)}$$

the monopole operator being given by

$$M_0=\sum_{j(\text{protons})}(2j+1)^{1/2}\langle j||r^2||j\rangle\{(u_j^2-v_j^2)[(1-x^2)/2+P_{jj}(\psi\phi(K_1+\text{H.c.})+(\psi^2+\phi^2)\hat{N}+5\phi^2)]+v_j^2$$
$$-su_jv_jW_{jj}[(\psi^2\phi+\phi^2\psi)(K_{30}+\text{H.c.})+(\psi^3+\phi^3+2\psi\phi(\psi+\phi))(K_{21}+\text{H.c.})]\}. \quad \text{(B3)}$$

In (B3) the quantities x, s, ψ, and ϕ are those that have been defined in II, while $\hat{N}(=K_2)$, K_1, K_{21}, and K_{30} are boson operators defined in I. The quantities u_j and v_j are the usual BCS amplitudes. Further

$$P_{jj} = (2j+1)^{-1} \sum_{j'} \Lambda_{j'j} \Lambda_{j'j} ,$$
$$W_{jj} = 10(2j+1)^{-1} \sum_{j'j''} \Lambda_{j'j} \Lambda_{j''j} \Lambda_{j'j''} W(jj'22; 2j'') . \tag{B4}$$

Finally $\Lambda_{jj'}$ is the quantity called $D_{jj'}\psi^{(0)}_{jj'2}$ in I. See Eq. (2.7) of I for a precise definition of this quantity.

APPENDIX C: (t, p) SPECTROSCOPIC AMPLITUDE

The spectroscopic amplitude for the $O_g^+ \to O_g^+$ transition of a (t, p) reaction can be obtained by taking the matrix element of the following operator between the ground states of the parent and the daughter nuclei:

$$\begin{aligned} T_{jj'} = {} & [(2_j+1)/(1+\delta_{nn'})]^{1/2} \\ & \times \{ u_j v_j \delta_{nn'} (2 - x^2 + 10\phi^2 P_{jj'}) + (u_j v_{j'} + u_{j'} v_j) P_{jj'} [\psi\phi(K_1 + \text{H.c.}) + (\psi^2 + \phi^2)\hat{N}] \\ & + s W_{jj'} [(\phi^3 + 2\psi^2\phi)(u_j u_{j'} K_{21} - v_j v_{j'} K_{21}^\dagger) + (\psi^3 + 2\psi\phi^2)(u_j u_{j'} K_{21}^\dagger - v_j v_{j'} K_{21}) \\ & + \phi^2\psi(u_j u_{j'} K_{30} - v_j v_{j'} K_{30}^\dagger) + \psi^2\phi(u_j u_{j'} K_{30}^\dagger - v_j v_{j'} K_{30})] \} . \end{aligned} \tag{C1}$$

Notation is the same as that in Appendix B. The Kronecker delta $\delta_{nn'}$ signifies that a term multiplied with it contributes only when $j=j'$ and $n=n'$, where n and n' are the radial quantum numbers. In other terms $j=j'$ is still required, but n can be different from n'. The quantities $P_{jj'}$ and $W_{jj'}$ can still be defined by (B4), if one of the Λ factors is interpreted to correspond to an orbit having a spin j but a radial quantum number equal to n', rather than to n.

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