Pion-nucleus scattering and absorption as a solution of the Boltzmann equation

J. Hüfner and M. Thies*
Institut für Theoretische Physik der Universität Heidelberg, Heidelberg, Federal Republic of Germany and Max-Planck-Institut für Kernphysik, Heidelberg, Federal Republic of Germany
(Received 27 November 1978)

We calculate inelastic pion-nucleus collisions and pion absorption from the Boltzmann equation. The pion-nucleus optical potential and the elementary pion-nucleon cross section are the dynamical input. We describe the construction of a scattering solution and propose an approximation scheme based on a straight line trajectory (allowing for backward scattering). The total reaction cross sections, the one for absorption and for double charge exchange, are calculated as a function of the pion energy (from 50 to 300 MeV). Doubly differential cross sections $d^3\sigma/d\Omega dE$ are computed for inelastic (π^+,π^+) and (π^+,π^0) reactions. Without adjusting any parameter, the absolute magnitude and the shape of the experimental data are fairly well reproduced. Pion absorption and single quasifree scattering are the two dominant channels in pion-nucleus interactions.

[NUCLEAR REACTIONS Inelastic pion-nucleus scattering, single and double charge exchange, reaction cross section, absorption cross section, calculated from Boltzmann equation. ^{4}He, ^{12}C, ^{16}O.]

I. THE AIM OF THIS WORK

How does a pion propagate into and inside a nucleus? Despite thirty years of research in pion-nucleus physics, this question has not yet been answered satisfactorily. Elastic pion-nucleus scattering, on which attention has been focused during recent years, cannot help for the following simple reason: Whenever a pion penetrates the nucleus it interacts inelastically. It knocks out one or several nucleons or it is absorbed. In both cases, the pion is removed from the elastic channel, hence it is unobservable in elastic scattering. Therefore, in order to learn how a pion behaves inside the nucleus, we have to focus attention on inelastic processes. The excitation of low-lying excited states carries only a little strength and is too similar to elastic scattering to serve our purpose. Instead, we shall concentrate on inclusive reactions of the form

$$\pi + {}^AZ \to \pi' + X. \tag{1.1}$$

A pion with energy ω interacts with a target ^AZ and an outcoming pion with energy ω' $(<\omega)$ is measured at an angle Θ while everything else (X) remains unobserved. The final pion π' need not carry the same charge as the initial one. Besides, we shall look at the total cross section for pion absorption,

$$\pi + {}^AZ \to \text{no } \pi + X. \tag{1.2}$$

When we attempt to describe reactions of the type of Eqs. (1.1) and (1.2) we take the point of view of multiple scattering: A pion which penetrates a nucleus is scattered elastically or inelastically by a few target nucleons before it either leaves the nucleus, Eq. (1.1), or is absorbed, Eq. (1.2). A calculation of the corresponding cross sections has to follow the pion through the nucleus. It must contain the sum over all unobserved information of the inclusive cross section. Here lies the problem: There are no well developed theoretical tools to describe inclusive pion-nucleus reactions. The Monte Carlo calculation or the intranuclear cascade calculation are the only methods available at present (Ginocchio[1]). It aims at a "brute force" computer simulation of complex processes and requires extensive numerical work, but only a minimum of ideas about what is going on. Hence the results of this type of calculation are sometimes as puzzling as the experimental data, even if they do agree.

In this paper we present an alternative approach to pion-nucleus inclusive scattering. The theoretical motivation has been discussed in several papers (Hüfner,[2] Thies,[3] and Remler[4]) and will not be repeated here. The relation of this approach to the pre-equilibrium calculations (Griffin,[5] Blann[6]) is not yet clear. Starting from quantal multiple scattering theory, the problem of inclusive nuclear scattering can be cast into the form of a transport equation for the Wigner transform[7] of the projectile density matrix. This equation reduces to the Boltzmann equation under

the following assumptions:

(a) There are no nucleon-nucleon correlations.
(b) The pion is on-shell between two scatterings.
(c) The wavelength of the pion is short as compared to its mean free path.

In this limit, the Wigner transform may be interpreted as the classical distribution function $f(\vec{x}, \vec{p}, t)$ in phase space. It evolves in time according to the details of the scattering process, while the number of pions (the "norm" of f)

$$N(t) = \int d\vec{x}\, d\vec{p}\, f(\vec{x}, \vec{p}, t) \tag{1.3}$$

decreases because of true absorption. $f(\vec{x}, \vec{p}, t)$ is the "optimum" quantity for our purpose in the sense that no redundant (unobserved) information about the complicated nuclear states is carried along. Yet we are aware that the Boltzmann equation provides a purely classical description, and hence the same reservation of principle applies as for the Monte Carlo method.

The Boltzmann equation has never been applied to pion-nucleus physics. We try to do it here. We construct an approximate solution, which allows for a lot of physical insight and for rapid computation. A variety of experimental data can be explained semiquantitatively and in a rather consistent way. The picture of how a pion propagates into and inside a nucleus turns out to be very simple: Pion absorption is a major if not the dominant reactive channel. An appreciable fraction of the pions are absorbed before they have a chance to knock out a nucleon. Thus there is virtually no inelastic cascade inside the nucleus. Those pions which are observed finally have merely "scratched" the nuclear surface, whereas the interior region is essentially unexplorable by means of the (π, π') reaction. These findings confirm the conclusions drawn by Lenz[8] from a qualitative analysis of the available experimental data and also support some of Ginocchio's results.[1]

II. THE BOLTZMANN EQUATION FOR A SCATTERING PROBLEM

The Boltzmann equation is well known in statistical physics, see e.g., Reif,[9] as a model for approaches towards equilibrium. In nuclear physics it has been used extensively for problems of neutron transport and moderation in chain reactors. Its application to an intranuclear cascade at intermediate energies seems rather novel. We only are aware of work in low-energy heavy ion reactions (Agassi *et al.*[10] and Bertsch[11]) where similar ideas are used. Our starting point is the equation[2,3]

$$\frac{\partial}{\partial t} f + \{H, f\} = 2 \operatorname{Im} U_{\text{opt}} f + Df\,. \tag{2.1}$$

It describes how the distribution function $f(\vec{x}, \vec{p}, t)$ for a pion in phase space changes with time. The classical Poisson bracket

$$\{H, f\} = \frac{\partial H}{\partial \vec{p}} \frac{\partial f}{\partial \vec{x}} - \frac{\partial H}{\partial \vec{x}} \frac{\partial f}{\partial \vec{p}} \tag{2.2}$$

provides the "drift terms" which account for a steady streaming of the particles in the average force field. We specialize to our problem by choosing the Hamiltonian *function* H for a relativistic pion moving in the real part of the optical potential,

$$H(\vec{x}, \vec{p}) = (\mu^2 + \vec{p}^{\,2})^{1/2} + \operatorname{Re} U_{\text{opt}}(\vec{x}, \vec{p})\,. \tag{2.3}$$

Here μ is the pion rest mass and $\hbar = c = 1$ by convention. The appearance of the optical potential means that the left-hand side (lhs) of Eq. (2.1) describes elastic scattering only. The right-hand side (rhs) or "collision terms"[9] arise from inelastic π-nucleus collisions and pion absorption. It is split into a "loss term" proportional to the imaginary part of the optical potential (note that $\operatorname{Im} U_{\text{opt}} < 0$) and into a "gain term" Df. D is a linear integral operator in phase space:

$$(Df)(\vec{x}', \vec{p}', t) = \int d\vec{x}\, d\vec{p}\; D(\vec{x}', \vec{p}' \leftarrow \vec{x}, \vec{p}) f(\vec{x}, \vec{p}, t)\,. \tag{2.4}$$

The physical picture underlying the Boltzmann equation implies that the nucleus is excited via quasi-free π-N collisions. Thus the basic ingredients in D are the elementary differential cross section and the distribution function $f_N(\vec{x}, \vec{p}, t)$ of the target nucleons. We take f_N to be appropriate to a local cold Fermi gas with Fermi momentum k_F:

$$f_N(\vec{x}, \vec{p}, t) = \frac{3}{4\pi k_F^{\,3}} \Theta(k_F - p) \rho(\vec{x})\,. \tag{2.5}$$

$\rho(\vec{x})$ is the nuclear density normalized to A. Then D assumes the form

$$\begin{aligned} D(\vec{x}', \vec{p}' \leftarrow \vec{x}, \vec{p}) = \delta(\vec{x} - \vec{x}') \int d\vec{k}'\, \Theta(k' - k_F) \\ \times \int d\vec{k} f_N(\vec{x}, \vec{k}, t) \delta(\vec{k}' + \vec{p}' - \vec{k} - \vec{p}) \\ \times \frac{\delta(p_r' - p_r)}{p_r^{\,2}} v_r \frac{d\sigma_{\pi N}}{d\Omega_{\text{c.m.}}}(\vec{p}_r', \vec{p}_r)\,. \end{aligned} \tag{2.6}$$

The notation is the following: The momenta of pion and nucleon are $\vec{p}$ and $\vec{k}$ before the collision and $\vec{p}'$ and $\vec{k}'$ afterwards. The center of mass

(c.m.) differential cross section depends on the relative momenta $\vec{p}_r$ before and $\vec{p}_r'$ after the collision. The δ functions ensure energy and momentum conservation and the relative velocity v_r corrects the flux factor in the definition of the cross section for a moving system.

We have adopted semirelativistic kinematics for the pion, in which the nonrelativistic formulas are kept, but with the pion mass replaced by the total energy $\omega(\vec{p}) = (\mu^2 + \vec{p}^2)^{1/2}$. For instance, the relative momentum takes the form $\vec{p}_r = (M\vec{p} - \omega\vec{k})/(M+\omega)$ with the nucleon mass M.

There is an important unitary relation for the right-hand side of the Boltzmann equation which relates the imaginary part of the optical potential to an integral over the operator D. The norm $N(t)$ of $f(\vec{x}, \vec{p}, t)$ has been defined in Eq. (1.3). Integrating Eq. (2.1) over $\vec{x}$ and $\vec{p}$, one gets

$$\frac{dN(t)}{dt} = \int d\vec{x}\, d\vec{p}\{2\,\mathrm{Im}U_{\mathrm{opt}} f + Df\}. \tag{2.7}$$

If we switch off true absorption for the moment, the number of pions is a constant of motion and hence the right-hand side of Eq. (2.7) must vanish. An obvious choice for Im U_{opt} which identically satisfies this requirement is

$$2\,\mathrm{Im}U_{\mathrm{opt}}(\vec{x}, \vec{p}) = -\int d\vec{x}'\, d\vec{p}'\, D(\vec{x}', \vec{p}' - \vec{x}, \vec{p}). \tag{2.8}$$

Inserting this expression into the Boltzmann equation (2.1), we recover the more symmetric form of the collision gain and loss terms prevailing in the textbooks on statistical mechanics. On the other hand, Eqs. (2.6) and (2.8) yield at once a well-known result[12] for the first order optical potential,

$$2\,\mathrm{Im}U_{\mathrm{opt}} = -\rho(\vec{x})(v_r\sigma_{\mathrm{tot}})_{\mathrm{av}}, \tag{2.9}$$

where $(\ )_{\mathrm{av}}$ stands for Fermi averaging and Pauli blocking. Because of pion absorption, the argument leading to Eq. (2.9) breaks down. We shall see in Sec. IV that the optical potential can be split into a scattering and an absorptive part,

$$U_{\mathrm{opt}} = U_{\mathrm{sc}} + U_{\mathrm{abs}}. \tag{2.10}$$

We interpret $-2\,\mathrm{Im}U_{\mathrm{sc}}$ as the scattering rate and $-2\,\mathrm{Im}U_{\mathrm{abs}}$ as the absorption rate for pions. Then the unitary relation (2.8) holds for U_{sc}, so that the loss of pions into absorptive channels is described by the equation

$$\frac{dN(t)}{dt} = \int d\vec{x}\, d\vec{p}\, 2\,\mathrm{Im}U_{\mathrm{abs}} f. \tag{2.11}$$

A comment on the linearity of Eq. (2.1) may be appropriate: While the Boltzmann equation used in classical physics is quadratic in the phase space density, Eq. (2.1) is linear. What is the reason for this difference? In statistical mechanics one usually considers a gas of many particles which interact *among* themselves. Here we deal with *one* pion, which penetrates through nuclear matter and interacts with the nucleons. Thus, the equation must be linear in the pion distribution (one pion) but the collision term is bilinear in the pion and nucleon distribution functions. The nuclear distribution function Eq. (2.5) can be assumed time independent and appropriate to a cold Fermi gas as long as the pion moves much faster through the nucleus than the velocity of sound for nuclear matter ($v_N \approx k_F/M \approx 0.3$). Therefore $T_\pi \gg \mu v_N^2/2 \approx 10$ MeV.

A scattering solution of Eq. (2.1) is defined by the following initial condition: At very early times $t \to -\infty$ a pion approaches the target nucleus with a velocity $\vec{v}_0 = \vec{p}_0/\omega(p_0)$ on a trajectory which is characterized by the impact parameter $\vec{b}$ (from now on, we shall use $\vec{b}$ as a subscript for f and N):

$$f_{\vec{b}}(\vec{x}, \vec{p}, t) = \delta(\vec{p} - \vec{p}_0)\delta(\vec{x} - (\vec{b} + \vec{v}_0 t)). \tag{2.12}$$

Equation (2.1) has to be solved with this initial condition. However, as it stands, Eq. (2.1) still generates a piece in f coming from *elastic* scattering. Since elastic scattering is almost a pure wave phenomenon (diffraction) in the energy region of interest, our classical approach would fail badly. We subtract it out and content ourselves with the inclusive *inelastic* cross section.

This subtraction can easily be done: First solve the Boltzmann equation without a collision gain term for the elastic distribution function, f^{el}:

$$\frac{\partial}{\partial t} f^{\mathrm{el}} + \{H, f^{\mathrm{el}}\} - 2\,\mathrm{Im}U_{\mathrm{opt}} f^{\mathrm{el}} = 0. \tag{2.13}$$

Then solve the full Eq. (2.1) for f and define the inelastic distribution function as $f^{\mathrm{in}} = f - f^{\mathrm{el}}$. As shown in the Appendix, the differential cross section for inclusive inelastic scattering is simply given by the asymptotic momentum distribution integrated over all impact parameters,

$$\frac{d^3\sigma^{\mathrm{in}}}{d\vec{p}} = \lim_{t\to\infty} \int d\vec{b}\, d\vec{x} f^{\mathrm{in}}_{\vec{b}}(\vec{x}, \vec{p}, t). \tag{2.14}$$

Let us denote the norm of $f_{\vec{b}}$, $f^{\mathrm{el}}_{\vec{b}}$, and $f^{\mathrm{in}}_{\vec{b}}$ by $N_{\vec{b}}$, $N^{\mathrm{el}}_{\vec{b}}$ and $N^{\mathrm{in}}_{\vec{b}}$, respectively [cf. Eq. (1.3)]. Integrating Eq. (2.14) over all final pion momenta, we get for the total nuclear excitation cross section

$$\sigma_{n\mathrm{exc}} = \lim_{t\to\infty} \int d\vec{b}\, N^{\mathrm{in}}_{\vec{b}}(t). \tag{2.15}$$

$N_{\vec{b}}(t)$ is the probability that the pion with initial

impact parameter $\vec{b}$ does still exist at time t. Therefore, we can immediately write down the total absorption cross section,

$$\sigma_{abs} = \lim_{t\to\infty} \int d\vec{b}\,[1 - N_{\vec{b}}(t)]. \tag{2.16}$$

Adding Eqs. (2.15) and (2.16) we obtain for the total reaction cross section

$$\sigma_{re} = \sigma_{nexc} + \sigma_{abs} = \lim_{t\to\infty} \int d\vec{b}\,[1 - N^{el}_{\vec{b}}(t)]. \tag{2.17}$$

This equation resembles the corresponding quantum mechanical one: We relate the impact parameter $\vec{b}$ to the initial pion-nucleus relative angular momentum l and introduce the S matrix S_l for pion-nucleus elastic scattering. Then

$$\sigma_{re} = \frac{\pi}{k^2} \sum_l (2l+1)(1 - |S_l|^2). \tag{2.18}$$

III. A SCHEMATIC SOLUTION OF THE BOLTZMANN EQUATION

The Boltzmann equation (2.1) is an integro-differential equation in seven variables $(\vec{x}, \vec{p}, t)$. To solve it exactly represents an enormous task. One possibility is to convert it into an integral equation which can then be solved by Monte Carlo methods. While this still seems to be the only practical option if one is interested in a complete description of complex final states, it is highly desirable to develop simpler and more transparent techniques for the inclusive reaction. Since there exists little experience on solving Eq. (2.1) for pions and a finite geometry, our approximation scheme is largely based on physical intuition. It comes closest to a technique used by Adler *et al.*[13] and Sternheim,[14] who follow a suggestion by Fermi, and solve a one-dimensional transport equation. Adler *et al.*[13] have applied their approach to an exactly soluble model and found it to be an excellent approximation. Therefore, we feel rather confident about the reliability of our solution.

A pion which propagates through a nucleus collides a few times before it either escapes or is absorbed. "Few" means "once" or "twice" on the average. This behavior has been deduced by Lenz[8] on the basis of various experiments. It also results from our calculation since the mean free paths for πN scattering and pion absorption are comparable for pion energies between 100 and 300 MeV. The small number of πN collisions inside a nucleus suggests the following expansion scheme. The distribution function $f(\vec{x}, \vec{p}, t)$ is written as a sum of terms each of which describes a definite number n of inelastic collisions:

$$f(\vec{x}, \vec{p}, t) = \sum_{n=0}^{\infty} f^{(n)}(\vec{x}, \vec{p}, t). \tag{3.1}$$

The term $f^{(0)}$ corresponds to the elastic channel, and $f^{(1)}$ corresponds to the fraction with one (and only one) quasi-free collision, etc. Each $f^{(n)}$ satisfies an equation similar to Eq. (2.1), except that the collision gain term now becomes a source term involving $f^{(n-1)}$. D increases the number of inelastic collisions by one:

$$\frac{\partial}{\partial t} f^{(n)} + \{H, f^{(n)}\} = 2\,\mathrm{Im}\,U_{opt} f^{(n)} + D f^{(n-1)}. \tag{3.2}$$

Equation (3.2) holds true for $n \geq 0$ with the convention $f^{(-1)} \equiv 0$. At $t \to -\infty$, only $f^{(0)}$ is populated.

The "inclusive multiple scattering expansion," Eq. (3.1), does not yet simplify the original Boltzmann equation. In order to arrive at a more tractable set of equations, we have to separate space and momentum variables as far as possible, at the price of losing some of the correlations between them. The mathematical realization of our approximation goes as follows: We postulate that $f^{(n)}_{\vec{b}}(\vec{x}, \vec{p}, t)$ (where $\vec{b}$ is the initial impact parameter) can be factorized as

$$f^{(n)}_{\vec{b}}(\vec{x}, \vec{p}, t) = \xi^{(n)}_{\vec{b}}(\vec{x}, t)\Pi^{(n)}(\vec{p}),$$
$$\int d\vec{p}\,\Pi^{(n)}(\vec{p}) = 1. \tag{3.3}$$

Here $\Pi^{(n)}(\vec{p})$ is the momentum distribution after n collisions, normalized to one. It becomes time independent once we neglect the small changes in pion momentum due to the force, keeping only the effect of the "violent" collisions. $\xi^{(n)}_{\vec{b}}(\vec{x}, t)$ carries the information about the space-time dependence and the norm of the distribution function. The only correlation to $\Pi^{(n)}$ which we allow is a discrete one: Whenever the pion energy appears in the evaluation of $\xi^{(n)}_{\vec{b}}(\vec{x}, t)$, we replace it by the *average* energy after n collisions as determined by the second moment of $\Pi^{(n)}$. This information is now contained in the (discrete) index (n). Similarly, we replace the momenta p in the Poisson bracket and the optical potential by the average root mean square momentum after n collisions and indicate this choice by the label (n), e.g.,

$$\mathrm{Im}\,U_{opt}(\vec{x}, \vec{p})\, f^{(n)}_{\vec{b}}(\vec{x}, \vec{p}, t) \simeq \mathrm{Im}\,U^{(n)}_{opt}(\vec{x})\, \xi^{(n)}_{\vec{b}}(\vec{x}, t)\, \Pi^{(n)}(\vec{p}). \tag{3.4}$$

Finally, in the collision gain term we pull out a factor $v_r \sigma^{tot}_{\pi N}$ [at the average energy after $(n-1)$ collisions] out of the integral, Eq. (2.6):

$$\int d\vec{x}'\, d\vec{p}'\, D(\vec{x}, \vec{p} \to \vec{x}', \vec{p}') f_{\vec{b}}^{(n-1)}(\vec{x}', \vec{p}', t) \simeq -2\,\mathrm{Im}\, U_{sc}^{(n-1)}(\vec{x})\, \xi_{\vec{b}}^{(n-1)}(\vec{x}, t) \int d\vec{p}'\, W(\vec{p} - \vec{p}')\Pi^{(n-1)}(\vec{p}'), \tag{3.5}$$

where

$$W(\vec{p} - \vec{p}') = C \int d\vec{k}'\, d\vec{k}\, \Theta(k_F - k')\Theta(k - k_F)\delta(\vec{k} + \vec{p} - \vec{k}' - \vec{p}')\, \frac{\delta(p_r' - p_r)}{p_r^2} \left(\frac{v_r[\vec{k}']}{v_r[0]}\right) \frac{1}{\sigma_{\pi N}^{tot}} \frac{d\sigma_{\pi N}}{d\Omega_{c.m.}}(\vec{p}_r', \vec{p}_r) \tag{3.6}$$

and the normalization constant C follows from

$$\int d\vec{p}\, W(\vec{p} - \vec{p}') = 1. \tag{3.7}$$

This approximation amounts to neglecting the effect of Fermi averaging in the optical potential, without violating the unitarity condition discussed in Sec. II, owing to Eq. (3.7). The reduction in $\sigma_{\pi N}^{tot}$ due to the Pauli principle, i.e., the restriction of phase space for the scattered nucleons, is taken into account when evaluating $\mathrm{Im}U_{sc}^{(n-1)}$.

Collecting Eqs. (3.2) to (3.7), the Boltzmann equation decays into the following set of equations:

$$\Pi^{(n)}(\vec{p}) = \int d\vec{p}'\, W(\vec{p} - \vec{p}')\Pi^{(n-1)}(\vec{p}), \tag{3.8}$$

$$\left(\frac{\partial}{\partial t} + v^{(n)} \frac{\partial}{\partial z}\right) \xi_{\vec{b}}^{(n)}(\vec{x}, t) = 2\,\mathrm{Im}\, U_{opt}^{(n)}(\vec{x})\, \xi_{\vec{b}}^{(n)}(\vec{x}, t) - 2\,\mathrm{Im}\, U_{sc}^{(n-1)}(\vec{x})\, \xi_{\vec{b}}^{(n-1)}(\vec{x}, t). \tag{3.9}$$

The momentum distribution arises by successively folding the transition probability $W(\vec{p} - \vec{p}')$ for a single quasi-free scattering event [cf. Eq. (3.6)]. Equations (3.9) are partial differential equations for the $\xi_{\vec{b}}^{(n)}(\vec{x}, t)$. The coupling between Eqs. (3.8) and (3.9) is only a trivial one: The velocity $v^{(n)}$ and the optical potentials have to be evaluated at average pion energies as determined from the $\Pi^{(n)}$. To evaluate the cross sections, one needs to know the functions $\Pi^{(n)}$ explicitly, but only the asymptotic *norm* of $\xi_{\vec{b}}^{(n)}(\vec{x}, t)$ (cf. Sec. II). Therefore, we solve Eqs. (3.9) via a moment expansion. The moments to which we limit ourselves are the norm and the mean position, i.e.,

$$N_{\vec{b}}^{(n)}(t) = \int d\vec{x}\, \xi_{\vec{b}}^{(n)}(\vec{x}, t), \tag{3.10}$$

$$\langle z\rangle_{\vec{b}}^{(n)}(t) = \int d\vec{x}\, z\, \xi_{\vec{b}}^{(n)}(\vec{x}, t). \tag{3.11}$$

In order that the equations for the moments close, we introduce a "smoothness assumption": We assume the spatial distribution of the pion to be well localized to volumes in which the potential does not change significantly. For instance, we set

$$\int d\vec{x}\, U_{opt}^{(n)}(\vec{x})\, \xi_{\vec{b}}^{(n)}(\vec{x}, t) \simeq U_{opt}^{(n)}(\vec{b}, \langle z\rangle_{\vec{b}}^{(n)}) N_{\vec{b}}^{(n)}(t). \tag{3.12}$$

Then, by taking mean values of Eq. (3.9), a set of coupled ordinary differential equations is obtained:

$$\frac{d}{dt} N_{\vec{b}}^{(n)} = 2\,\mathrm{Im}\, U_{opt}^{(n)}(\vec{b}, \langle z\rangle_{\vec{b}}^{(n)}) N_{\vec{b}}^{(n)} - 2\,\mathrm{Im}\, U_{sc}^{(n-1)}(\vec{b}, \langle z\rangle_{\vec{b}}^{(n-1)}) N_{\vec{b}}^{(n-1)}, \tag{3.13}$$

$$\frac{d}{dt}\langle z\rangle_{\vec{b}}^{(n)} = v^{(n)} + 2\,\mathrm{Im}\, U_{sc}^{(n-1)}(\vec{b}, \langle z\rangle_{\vec{b}}^{(n-1)}) \times \frac{N_{\vec{b}}^{(n-1)}}{N_{\vec{b}}^{(n)}} \{\langle z\rangle_{\vec{b}}^{(n)} - \langle z\rangle_{\vec{b}}^{(n-1)}\}. \tag{3.14}$$

Equations (3.13) and (3.14) are now in a form in which they can readily be solved on a computer, for each impact parameter $\vec{b}$. One first evaluates the $\Pi^{(n)}$ from the recursion relation (3.8). For $n = 1$ we have done the integration exactly, so that our single scattering term is proportional to a plane wave impulse approximation appropriate to a Fermi gas. For $n > 1$, neglecting the slowly varying flux factor, we derive from Eq. (3.8) analytic recursion relations for the first and second moments of $\Pi^{(n)}$. The same formulas have previously been used in a different context (cf. the Appendix of Ref. 15) and need not be repeated here. Then we reconstruct the functions $\Pi^{(n)}$ for $n > 1$ by assuming a normal distribution. The results of our calculation justify *a posteriori* the somewhat crude treatment of multinucleon knockout processes, since they represent only a tiny fraction of the reactive cross section.

In the actual calculation we had to go one step beyond the scheme described so far. Because of the p-wave dominated π-N interaction, large angle scattering occurs frequently, thus invalidating the assumption that the distribution function remains well localized in space. Also, the mean energy averaged over all the angles seems too "global" a quantity to characterize the energy dependent aspects of the scattering process even

qualitatively. Guided by the success of the Adler model,[13] we have attempted to improve on the preceding approximations by choosing finer subdivisions for the distribution function f. We split the collision gain term into a sum of two pieces, one corresponding to a "forward" scattering event ($\Theta < 90°$, label "+"), the other to a "backward" scattering event ($\Theta > 90°$, label "−"). Then the distribution function $f^{(n)}$ for an n-fold collision can further be decomposed into 2^n terms according to the possible sequences of forward/backward scatterings, e.g., $f_+^{(1)}$, $f_-^{(1)}$ for single, $f_{++}^{(2)}$, $f_{+-}^{(2)}$, $f_{-+}^{(2)}$, and $f_{--}^{(2)}$ for double scattering. The only modification in the equations discussed before is that $f^{(n)}$ and $U_{\rm opt}^{(n)}$ get an additional label for the "history" of the collision process, and that the velocity $v^{(n)}$ in Eq. (3.9) may have either sign, depending on the total number of large angle scatterings. Since the rapid convergence of the multiple scattering expansion allows us to truncate the series after the triple scattering term, the extra amount of computation is still reasonable.

IV. IN-MEDIUM PION-NUCLEON SCATTERING

The elementary πN scattering information enters the Boltzmann equation in the optical potential and in the gain part of the collision term. In lowest order the optical potential is linear in the πN scattering amplitude,[12]

$$U_{\rm opt} = -\frac{4\pi}{2\omega} f_{\pi N}(\Theta = 0)\rho(\vec{x}) . \tag{4.1}$$

In free space the πN scattering amplitude is given by the experimental phase shifts. Inside the nucleus the pion can also be absorbed. Absorption may be incorporated by constructing an effective or in-medium πN amplitude. We follow the arguments of Hirata *et al.*[16] and Oset *et al.*[17]: In the resonant (3, 3) channel, pion and nucleon form a Δ, which, while propagating through the nucleus, can collide with another nucleon

$$\Delta + N \rightarrow N + N \tag{4.2}$$

and convert its mass excess into kinetic energy of the two nucleons. According to Refs. 16 and 17, this mechanism accounts for most of the pion absorption phenomena. Therefore, we construct our in-medium πN amplitude in the following way: Experimental πN phase shifts are used in all partial waves, except the (3, 3) channel. The resonant part of the (3, 3) amplitude, which has the form

$$f_{3,3}^{\rm free}(\omega) = \frac{-\Gamma(\omega)/2p}{\omega - \omega_R + \frac{1}{2} i\,\Gamma(\omega)} , \tag{4.3}$$

is modified in the medium to

$$f_{3,3}(\omega) = \frac{-\Gamma(\omega)/2p}{\omega - \omega_R + \frac{1}{2} i[P(\omega)\Gamma(\omega) + \Gamma_{\rm abs}\ \rho(\vec{x})/\rho_0]} . \tag{4.4}$$

The two modifications in the denominator of Eq. (4.4) have the following origin: (a) The Pauli-quenching factor $P(\omega)$ (≤ 1) reduces the decay width of the delta in nuclear matter, because not all the states of the final nucleon are available. For example, $P(\omega) = 0.7$ at the resonance for a Fermi momentum $k_F = 190$ MeV/c. (b) The absorption width $\Gamma_{\rm abs}$ accounts for absorption via Eq. (4.2). The factor $\rho(\vec{x})/\rho_0$ (ρ_0 = density of nuclear matter) takes into account that a second nucleon must be present. We assume $\Gamma_{\rm abs}$ to be independent of energy ω since no detailed information is available. The magnitude can be estimated from several sources. The experimental value $\mathrm{Im}\,C_0$ in the p-wave part of the optical potential for pionic atoms[12] is related via

$$\Gamma_{\rm abs} \approx 2\omega_R \cdot \frac{\mathrm{Im}\,C_0 \rho_0}{\mathrm{Re}\,c_0} , \tag{4.5}$$

where C_0 governs the magnitude of the real part of the optical potential. Depending on whether we trust the experimental values $\mathrm{Im}\,C_0 = 0.04\ \mu^{-4}$ or $0.08\ \mu^{-4}$, we find $\Gamma_{\rm abs} = 30$ or 60 MeV. Hirata *et al.*[16] deduced $\Gamma_{\rm abs}$ by fitting the total pion-nucleus cross section. Their value of $\Gamma_{\rm abs}$ lies between 100 and 150 MeV. We follow them and take $\Gamma_{\rm abs} = 120$ MeV, but find the calculation, even of the absorption cross section, very insensitive to the precise value of $\Gamma_{\rm abs}$ (see Sec. VI).

The pion-nucleus optical potential also contains an absorptive part of the form $\Delta U_{\rm opt}^{\rm abs} = 4\pi i\,\mathrm{Im}B_0\ \rho^2(\vec{x})/2\omega$, which, for instance, accounts for the s-wave pion absorption in pionic atoms.[12] We include such a term also in our calculation. We take $\mathrm{Im}B_0 = 0.04\ \mu^{-4}$.

According to the calculations, the most important physics happens in the nuclear surface. It is necessary to choose the correct local Fermi momentum. We do not follow the prescription of a local Fermi gas, which predicts $k_F(\vec{x}) \propto \rho^{1/3}(\vec{x})$, but rather we follow the suggestion by Campi *et al.*[18] For a nuclear wave function built of Gaussians, the local Fermi momentum is practically independent of $\vec{x}$. For an exponential tail, $k_F(\vec{x})$ decreases with density, but much more slowly than predicted by the naive local Fermi gas. We choose $k_F = 190$ MeV/c, independent of $\vec{x}$, which seems appropriate for that part of the density to which we are sensitive.

There is no reason to believe that the *real* part of the energy denominator in Eq. (4.4) is not af-

fected by the medium. A difference in the potentials felt by the nucleon and the delta or dispersive effects of pion absorption could induce large shifts. We ignore this kind of effect since it is not yet well enough understood at present. However, some of the discrepancies between calculated and measured spectra in the (π, π') reaction seem to call for a sizable shift in the position of the Δ resonance.

Finally we decompose the optical potential into a scattering part U_{sc} and an absorption part U_{abs}, cf. Eq. (2.10). Inserting expression (4.4) into Eq. (4.1) and taking the imaginary part of (which only enters our calculation), we identify

$$\begin{aligned} \mathrm{Im}\, U_{sc} &= \frac{P(\omega)\Gamma(\omega)}{\Gamma_{tot}} \mathrm{Im}\, U_{opt}, \\ \mathrm{Im}\, U_{abs} &= \frac{\Gamma_{abs}\rho(x)/\rho_0}{\Gamma_{tot}} \mathrm{Im}\, U_{opt}, \end{aligned} \tag{4.6}$$

with the total in-medium width

$$\Gamma_{tot} = P(\omega)\Gamma(\omega) + \Gamma_{abs} \cdot \frac{\rho(\vec{x})}{\rho_0}. \tag{4.7}$$

Note that U_{sc} is also modified by absorption.

V. CALCULATED CROSS SECTIONS AND COMPARISON WITH EXPERIMENT

We have approximately solved the Boltzmann equation as explained in Sec. III, with the model for the πN scattering amplitude described in Sec. IV. There is no adjustable parameter. All cross sections which are shown in the following are calculated in shape and absolute magnitude and no "cosmetics" were attempted. We have tried to cover a variety of data, many of which have been discussed by Lenz.[8,19] We have ordered the presentation according to decreasing inclusiveness: First we present the integrated cross sections, then the single differential cross sections, and finally the double differential ones.

Figure 1 shows the reaction and the absorption cross sections for pions on ^{4}He, ^{12}C, and ^{16}O as a function of the pion energy. (No attempt was made to evaluate the integrated elastic cross section, since it derives a large part from quantum effects absent in our approach). The reaction cross sections show a pronounced peak in the resonance region, the shape and absolute magnitude of which are approximately reproduced by the calculation. However, there seems to be a systematic discrepancy on the low-energy side. The calculated absorption cross sections are rather flat. They seem to reflect some kind of saturation phenomenon. There is no reminiscence of the (3, 3) resonance at higher energies, presumably because decreasing *direct* absorption (out of the elastic channel) is compensated by a rising probability for absorption after one or several quasi-free collisions. The present accuracy of the data does not yet allow a detailed comparison. The difference between σ_{re} and σ_{abs} is due to inelastic scattering. As indicated already and as will be discussed in more detail in Sec. VI, it is dominated by single nucleon knockout.

The cleanest experiment which studies single inelastic scattering to date seems to be the re-

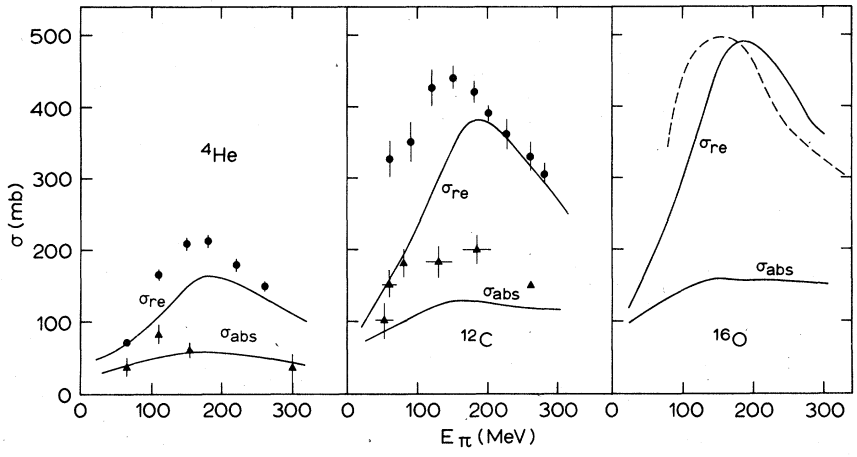

FIG. 1. The reaction cross section σ_{re} and the total absorption cross section σ_{abs} as a function of pion laboratory energy and for three target nuclei. The experimental data (triangles) for the absorption cross sections are taken from Refs. 22–24 for ^{4}He and compiled by Ginocchio (Ref. 1) for ^{12}C. The reaction cross sections (points) are given in Refs. 25–27 for ^{4}He and in Refs. 27 and 28 for ^{12}C. For ^{16}O, the dotted line stems from a phase shift analysis of recent SIN data Ref. 29.

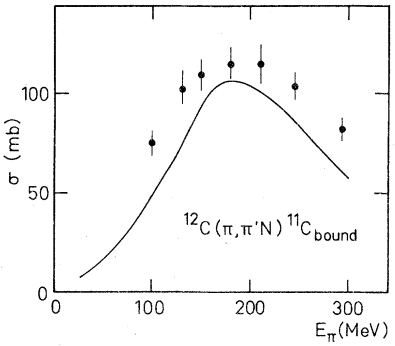

FIG. 2. The integrated cross section for one-nucleon knockout as a function of the pion laboratory energy. The experimental data (Ref. 30) correspond to the sum

$$\sigma(^{12}C(\pi^+,\pi N)^{11}C)+\sigma(^{12}C(\pi^-,\pi^- N)^{11}C).$$

The solid line is the calculated cross section for single scattering multiplied by 0.5, as explained in the text.

action $^{12}C(\pi,\pi'N)^{11}C$, in which a bound final state of ^{11}C is measured radioactively. This reaction differs from the inclusive one, and one now has to think about the final state interaction of the knocked-on nucleon. According to the estimate of Ref. 20, about half of the outgoing nucleons excite the ^{11}C nucleus above the particle threshold. Therefore, we scale the calculated inclusive single scattering cross section down by a factor of $\frac{1}{2}$ before comparing it to the measured excitation function for $^{12}C(\pi,\pi'N)^{11}C$, Fig. 2.

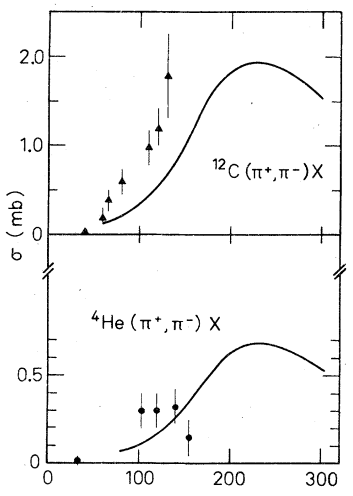

FIG. 3. The total cross section for double charge exchange on ^4He and ^{12}C. The experimental data are taken from Refs. 31 to 33. The triangles correspond to the (π^+,π^-) reaction in emulsion and are given to show the energy dependence.

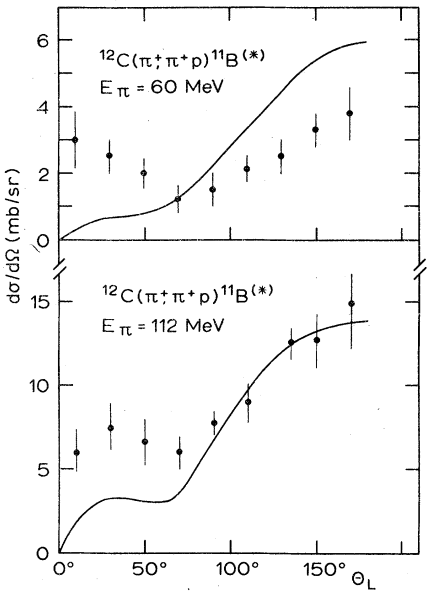

FIG. 4. The angular distribution of the pions in a one-proton knockout reaction for two different initial pion energies. Experimental data are from Refs. 34–36.

Double charge exchange requires at least two πN collisions, and has only a very small cross section (Fig. 3). In addition to a factor of ~5 which one loses when going from single to double scattering, it is also cut down by a trivial isospin factor (starting with a π^+, only 1 out of 36 double scatterings lead to a π^-, near resonance). The order of magnitude seems to be all right, but the scarcity of data does not permit a stringent test

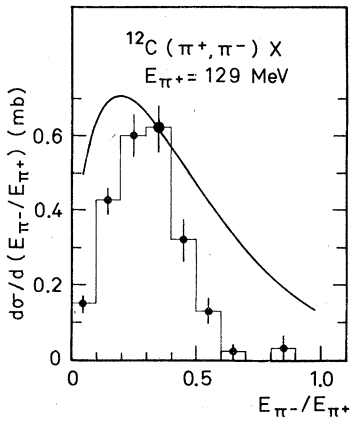

FIG. 5. The energy distribution of the outgoing π^- (integrated over angles) for the double charge exchange reaction. Since the experimental data (Ref. 32) are in relative units, we have normalized them to the calculation at the maximum of the experimental distribution.

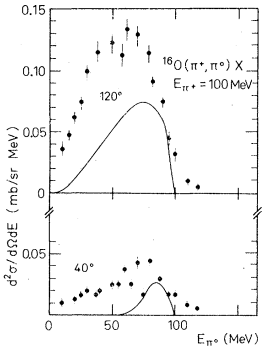

FIG. 6. The double differential cross section for the single charge exchange reaction on ^{16}O. The experimental data (of which not all points are drawn) are from Ref. 21.

of the model. For instance, a verification of the predicted energy dependence (with a maximum well above resonance) would be highly welcome.

Figures 4 and 5 show two examples of single differential cross sections. In Fig. 4 the angular distribution has been obtained by integrating $d^2\sigma/d\Omega dE$ over the final energy. In Fig. 5 we have integrated over the angles to get the energy spectrum in double charge exchange. The shape and normalization (where measured) agree fairly well. At low energies, inelastic pion-nucleus scattering favors large angles. This is a consequence of the p-wave dominance together with the Pauli principle, which suppresses the forward part. Whereas Fig. 4 is almost exclusively due to single scattering, Fig. 5 shows that the features of the two-step process (π^+, π^-) can also be understood in our model.

Doubly differential cross sections are displayed in Figs. 6 and 7. The qualitative features of the data are reproduced, but significant discrepancies remain. For the single charge-exchange reaction at 100 MeV, large angle (120°) scattering is three times more likely than small angle (90°) scattering. Our calculation overestimates the ratio, as does a Monte Carlo calculation.[21] It is impossible to understand the low-energy tail at 40° in terms of one quasi-free collision, for simple kinematical reasons. But double scattering seems also much too small to account for the discrepancy, so that one has to think of other explanations. The fact that the peaks in Figs. 6 and 7 are shifted with respect to the data is probably due to our neglect of the nucleon binding energies.

VI. THE ANATOMY OF INCLUSIVE PION-NUCLEUS REACTIONS

Pion absorption is an extremely important channel in pion-nucleus interactions. It screens the interior region to a large extent. The relative importance of quasi-free scattering versus absorption can be inferred already from a comparison between the corresponding mean free paths, Fig. 8, where

$$\frac{1}{\lambda_{sc}} = -\frac{2}{v}\,\mathrm{Im}\,U_{sc}, \quad \frac{1}{\lambda_{abs}} = -\frac{2}{v}\,\mathrm{Im}\,U_{abs}. \tag{6.1}$$

For nuclear matter density and on top of the (3, 3) resonance, λ_{sc} and λ_{abs} are both of the order of 1 fm. It is instructive to examine the sensitivity

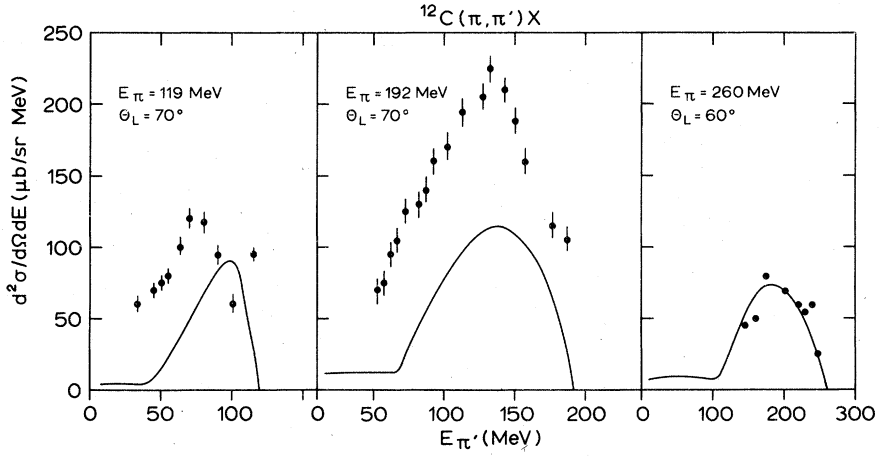

FIG. 7. The double differential cross section for the (π^+, π^+) reaction on ^{12}C for several energies and angles. Experimental points are from Refs. 37 and 38.

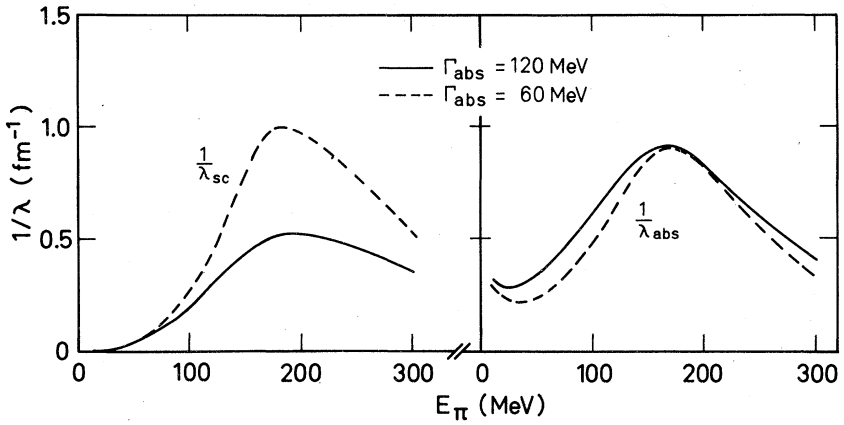

FIG. 8. The mean free paths λ_{abs} for pion absorption and λ_{sc} for pion inelastic scattering as a function of the pion energy. Nuclear matter density is assumed.

of λ_{sc} and λ_{abs} with respect to variations of Γ_{abs}. Right at the resonance we have the function dependence

$$\frac{1}{\lambda_{sc}} \propto \frac{1}{(1+x)^2}, \quad \frac{1}{\lambda_{abs}} \propto \frac{x}{(1+x)^2}, \qquad x = \frac{\Gamma_{abs}}{P(\omega)\Gamma(\omega)} . \tag{6.2}$$

The function $x/(1+x)^2$ has a very flat maximum at $x=1$ and stays practically constant in the relevant region $\frac{1}{2} \leq x \leq 2$. Hence, the pion absorption rate is insensitive to the precise value of the absorption width. λ_{sc}, on the other hand, increases by a factor of 4 in the same interval. We conclude that in order to pin down the value of Γ_{abs}, it may be more profitable to focus on the inelastically scattered pions, rather than on the absorption cross section itself. The value $\Gamma_{abs} = 120$ MeV used here yields a reduction factor greater than 5 per increase of the number of collisions by one (Fig. 9). Had we neglected Γ_{abs} altogether, the double scattering term would go up by a factor of 3, the triple scattering term by a factor of 10 around 300 MeV, and we would get poor convergence.

The central region of the nucleus is essentially unreachable for pions, due to absorption. Figures 10 and 11 illustrate that everything happens in the surface. We look first at the impact parameter dependence, i.e., the profile functions of the principal contributions to the reactive cross

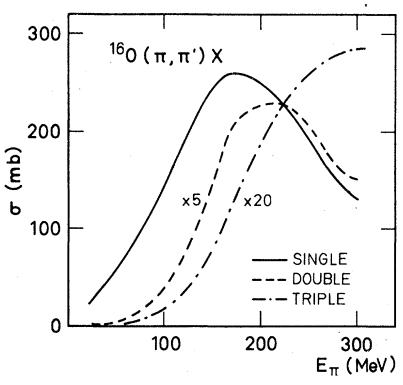

FIG. 9. The cross sections for single, double, and triple inelastic scattering as a function of the pion energy. Note the shifted peak positions.

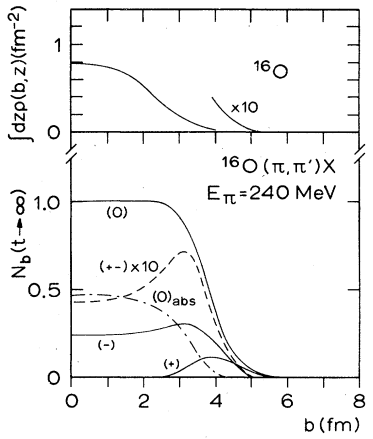

FIG. 10. The contributions of the various scattering and absorption processes as a function of impact parameter $\vec{b}$. Above is the nuclear thickness function $\int dz\, \rho(b,z)$ and below are the norms $N_{\vec{b}}^{\alpha}(t)$ for $t \to \infty$ as a function of $\vec{b}$, where (0): The pion has not scattered inelastically (here $1 - N_{\vec{b}}^{el}$ is drawn). $(0)_{abs}$: The pion is absorbed without an inelastic collision. (+): One forward scattering on the trajectory. (−): One backward scattering on the trajectory. (+−): First a forward, then a backward inelastic collision.

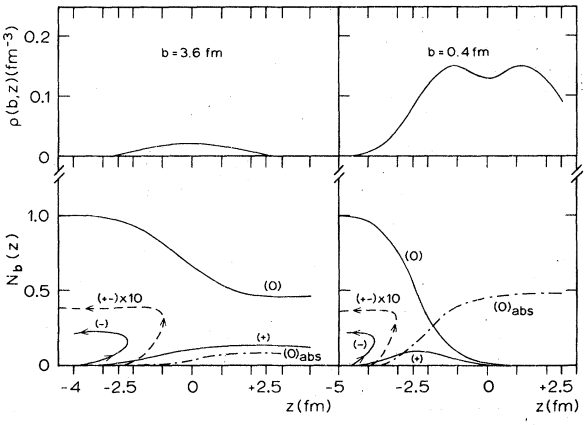

FIG. 11. The development of the norm $N_b(z)$ as the pion moves along the trajectory in a z direction, for two values of the impact parameter. Above are the corresponding density profiles. Below is the development of the norm. The pion approaches from the left; it either goes through (to the right) or returns (to the left) in the case of a backward scattering event. The notation is the same as in Fig. 10.

section (Fig. 10). At small b no flux gets through in the forward direction, but backward scattering ($-$ or $+-$) is still possible. Figure 11 displays the motion of the pion in the z direction for two representative impact parameters. For large b the forward scattered pion has a good chance of escaping, in contrast to what happens in a central collision. The backscattered pion reverses its direction of motion rather early and abruptly, and does not enter the dense regions at all; therefore, it cannot be absorbed. The dominant absorption mode is direct absorption out of the elastic channels $(0)_{abs}$. It sets in later but still reaches its peak probability at roughly central density.

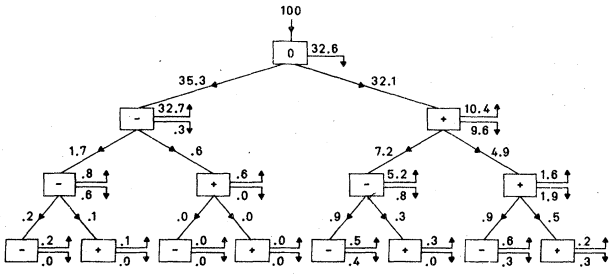

FIG. 12. An illustration of all possible branchings in an inelastic $\pi-{}^{16}O$ collision at 240 MeV according to our calculation. The numbers are in percent of the reactive cross section. Each box with a + (−) corresponds to a small (large) angle scattering. At the right-hand side of the boxes the percentages for escape (↑) and absorption (↓) are given. The cascade has been truncated after three collisions.

We wind up this Section with a survey of the total probabilities (i.e., integrated over db) for the various inelastic pion-nucleus interactions according to our model (Fig. 12). We choose 240 MeV pions on ^{16}O and ignore the pion charge states to keep the number of branches small. Following a quasi-free collision, the pion has four options: To escape (↑), to be absorbed (↓), or to rescatter, either by a small angle (+) collision or a large angle (−) collision. At the beginning of the cascade (0), the pion has about an equal chance of scattering forward or backward or being absorbed. However, its fate is rather different, depending on whether the first collision is forward or backward. After the − collision, it survives in 97% of all cases, escaping most likely right away. In the + branch, the prognosis is less favorable: This pion is absorbed with a probability of 41%, due to its longer path through nuclear matter.

VII. CONCLUSIONS

We have described a new way of computing inclusive inelastic pion-nucleus reactions. The basic idea is to consider the pion-nucleus collision as a *transport* phenomenon. The pion penetrates into the target nucleus, loses energy, and is eventually absorbed or reemitted, in much the same way as a neutron interacts with a piece of ordinary matter. Thus it is plausible that similar tools (e.g., the Boltzmann equation) should be applicable. As far as the neglect of quantum corrections is concerned, we are unfortunately in a much less favorable regime here than in the neutron case: The de Broglie wavelength and the mean free path of the pion are quite similar. It came as a surprise to us that a purely classical picture can account at least semiquantitatively for the data down to energies below 100 MeV. The reason for our success may lie in the special nature of pion-nucleus interaction. The inelastic cross section is dominated by single collisions. For this case the solution of the Boltzmann equation is nearly equivalent to the eikonal distorted wave impulse approximation. The assumptions on classical propagation between scatterings enter decisively only in double and higher scatterings.

Of course, many discrepancies remain, which may become even more significant with the advent of better and more systematic data from the meson factories in the near future. We are not sure to what extent they originate in the basic approach, in the numerical approximations, or in some missing physical effect (e.g., real part of Δ-nucleus potential, residual interactions, more exotic mechanisms). We find—confirming earlier

suspicions[19]—that quasi-free one nucleon knockout essentially exhausts the nuclear excitation cross section—that some of these ambiguities might be resolved by a more accurate evaluation of the distorted wave impulse approximation.

We have enjoyed numerous discussions which helped to clarify our ideas. We are especially grateful to T. Fujita, A. Klar, F. Lenz, C. Mahaux, H. C. Pauli, and H. J. Pirner. This work was supported by a grant from the Federal Ministry of Research and Technology (BMFT).

APPENDIX: DERIVATION OF THE INCLUSIVE DIFFERENTIAL CROSS SECTION, Eq. (2.14)

Let $f^{\rm in}_{\vec{b}}(\vec{x},\vec{p},t)$ be a solution of the Boltzmann equation (2.1) with the initial condition (2.12), where the elastic contribution has been eliminated as discussed in Sec. II. Suppose that at time t_0 the scattering process is over, so that $f^{\rm in}_{\vec{b}}(\vec{x},\vec{p},t)$ is well separated from the target (radius R_0). From then on the pion propagates freely, i.e.,

$$f^{\rm in}_{\vec{b}}(\vec{x},\vec{p},t)_{t>t_0} = \int d\vec{x}'\,\delta(\vec{x}'-\vec{x}+(\vec{p}/\omega)(t-t_0)) \times f^{\rm in}_{\vec{b}}(\vec{x}',\vec{p},t_0) \tag{A1}$$

$$\underset{|\vec{x}|\gg R_0}{\simeq} \delta(\vec{x}-(p/\omega)(t-t_0)) \times \int d\vec{x}'\, f^{\rm in}_{\vec{b}}(\vec{x}',\vec{p},t_0)\,. \tag{A2}$$

A stationary distribution function $\hat{f}^{\rm in}(\vec{x},\vec{p})$ corresponding to a steady stream of incoming particles uniformly distributed over all impact parameters can be constructed by integrating Eq. (A2) over time and $\vec{b}$,

$$\hat{f}^{\rm in}(\vec{x},\vec{p}) = \frac{\omega}{p}\,\frac{\delta(\Omega_{\vec{x}}-\Omega_{\vec{p}})}{\vec{x}^2}\int d\vec{b}\,d\vec{x}'\,f^{\rm in}_{\vec{b}}(\vec{x}',\vec{p},t_0)\,. \tag{A3}$$

[It is possible to extend the time integration below t_0 because $f^{\rm in}_{\vec{b}}(\vec{x},\vec{p},t)$ vanishes for $t<t_0$, $|\vec{x}|\gg R_0$.] The number of particles scattered through a surface element $d\vec{F}=\vec{x}^2\hat{x}\,d\Omega_{\hat{x}}$ at a large distance x from the target is

$$dN=\hat{f}^{\rm in}(\vec{x},\vec{p})\,d\vec{x}\,d\vec{p}=\hat{f}^{\rm in}(\vec{x},\vec{p})\vec{v}\cdot d\vec{F}\,d\vec{p}\,dt\,. \tag{A4}$$

Inserting Eq. (A3) and performing one angular integration, we get at once the rate at which particles are scattered into the momentum interval $d\vec{p}$ at $\vec{p}$. Since the absolute magnitude of the total incoming flux calculated from Eq. (2.12) is unity, the differential cross section becomes simply

$$\frac{d^3\sigma}{d\vec{p}} = \int d\vec{b}\,d\vec{x}'\,f^{\rm in}_{\vec{b}}(\vec{x}',\vec{p},t_0)\,. \tag{A5}$$

We arrive at the desired final result, Eq. (2.14), by noting that t_0 is arbitrary except for a lower bound.

[1] J. N. Ginocchio, Phys. Rev. C 17, 195 (1978).
[2] J. Hüfner, Ann. Phys. (N.Y.) 115, 43 (1978).
[3] M. Thies (unpublished).
[4] E. A. Remler, Ann. Phys. (N.Y.) (unpublished).
[5] J. J. Griffin, Phys. Rev. Lett. 17, 478 (1966).
[6] M. Blann, Nucl. Phys. A213, 570 (1973).
[7] E. Wigner, Phys. Rev. 40, 749 (1932).
[8] F. Lenz, in *Proceedings of the Seventh International Conference on High Energy Physics and Nuclear Structure, Zürich*, 1977, edited by M. P. Locher (Birkhäuser, Basel, 1977), p. 175.
[9] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).
[10] D. Agassi, C. M. Ko, and H. A. Weidenmüller, Ann. Phys. (N.Y.) 107, 140 (1977).
[11] G. F. Bertsch, in *Nuclear Physics with Heavy Ions and Mesons*, Les Houches Lectures, 1977, edited by R. Balian *et al.* (North-Holland, Amsterdam, 1978), pp. 178.
[12] J. Hüfner, Phys. Lett. 21C, 1 (1975).
[13] S. L. Adler, S. Nussinov, and E. A. Paschos, Phys. Rev. D 9, 2125 (1974).
[14] M. M. Sternheim and R. R. Silbar, Phys. Rev. D 6, 3117 (1972).
[15] J. Hüfner and J. Knoll, Nucl. Phys. A290, 460 (1977).
[16] M. Hirata, F. Lenz, and K. Yazaki, Ann. Phys. (N.Y.) 108, 116 (1977); M. Hirata, J. H. Koch, F. Lenz, and E. J. Moniz, Phys. Lett. 70B, 281 (1977).
[17] E. Oset and W. Weise, Phys. Lett. 77B, 159 (1978).
[18] X. Campi and A. Bouyssy, Phys. Lett. B73, 263 (1978).
[19] F. Lenz, Lectures at the Zuoz Winter School, 1976, Report No. PR-76-005 (unpublished).
[20] J. Hüfner, H. J. Pirner, and M. Thies, Phys. Lett. 59B, 215 (1975).
[21] T. Bowles, D. F. Geesaman, R. J. Holt, H. E. Jackson, R. M. Laszewski, J. R. Specht, L. L. Rutledge, R. E. Segel, R. P. Redwine, and M. A. Yates-Williams, Phys. Rev. Lett. 40, 97 (1978).
[22] E. C. Fowler, W. B. Fowler, R. P. Shutt, A. M. Thorndike, and W. L. Whittemore, Phys. Rev. 91, 135 (1953).
[23] M. S. Kozodgev, M. M. Kulyukin, R. M. Suljgev, A. I. Filopov, and Y. A. Shcherbakov, Zh. Eksp. Teor. Fiz. 38, 409 (1960) [Sov. Phys.—JETP 11, 300 (1960)].
[24] Y. A. Budagov, P. F. Ermolov, E. A. Kushivenko, and V. I. Moskalev, Zh. Eksp. Teor. Fiz. 42, 1191 (1961) [Sov. Phys.—JETP 15, 824 (1962)].
[25] C. Wilkin, C. R. Cox, J. J. Domingo, K. Gabathuler, E. Pedroni, J. Rohlin, P. Schwaller, and N. W. Tanner, Nucl. Phys. B62, 61 (1973).
[26] K. M. Crowe, A. Fainberg, J. Miller, and A. S. L.

Parsons, Phys. Rev. 180, 1349 (1969).
[27]F. Binon, P. Duteil, M. Gauanère, L. Hugon, J. Jansen, J. P. Lagnaux, H. Palevsky, J. P. Peigneux, M. Spighel, and J. P. Stroot, Phys. Rev. Lett. 35, 145 (1975).
[28]F. Binon, P. Duteil, J. P. Garron, J. Görres, L. Hugon, J. P. Peigneux, C. Schmit, M. Spighel, and J. P. Stroot, Nucl. Phys. B17, 168 (1970).
[29]F. Lenz, private communication.
[30]L. H. Batist, V. D. Vitman, V. P. Koptev, M. M. Makarov, A. A. Naberezhnov, V. V. Nelyubin, G. Z. Obrant, V. V. Sarantsev, and G. V. Scherbakov, Nucl. Phys. A254, 480 (1975).
[31]Yu. A. Scherbakov, in Meson-Nuclear Physics—1976, proceedings of the International Topical Conference, Pittsburgh, edited by P. D. Barnes, R. A. Eisenstein, and L. S. Kisslinger (AIP, New York, 1976), p. 365.
[32]Yu. A. Batusov, S. A. Bunyatov, V. M. Sidorov, and V. A. Yarba, Yad. Fiz. 6, 998 (1967) [Sov. J. Nucl. Phys. 6, 727 (1968)].
[33]Yu. A. Batusov, S. A. Bunyatov, V. M. Sidorov, and V. A. Yarba, Yad. Fiz. 3, 309 (1966) [Sov. J. Nucl. Phys. 3, 223 (1966)].
[34]Yu. R. Gismatullin and V. I. Ostroumov, Yad. Fiz. 11, 285 (1970) [Sov. J. Nucl. Phys. 11, 159 (1970)].
[35]Yu. R. Gismatullin, I. A. Lantsev, V. I. Ostroumov, and A. Ya. Smelyanskii, Yad. Fiz. 19, 45 (1974) [Sov. J. Nucl. Phys. 19, 22 (1974)].
[36]Yu. R. Gismatullin, I. A. Lantsev, and V. I. Ostroumov, Yad. Fiz. 21, 950 (1975) [Sov. J. Nucl. Phys. 21, 488 (1975)].
[37]F. Binon, P. Duteil, J. B. Garrou, J. Görres, L. Hugon, M. Krell, R. Meunier, J. P. Peigneux, C. Schmit, M. Spighel, and J. P. Stroot, CERN Report No. PH III-69/19 (unpublished).
[38]J. P. Albanes, J. Arvieux, E. T. Boschnitz, C. H. Q. Ingram, L. Pflug, and J. Zichy, quoted by Lenz, Ref. 1.