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Generalized spectator expansion for interacting composite systems

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Interactions of two composite clusters are treated in a multiple scattering framework whereby many-particle operators are decomposed into a systematic and finite series such that there is an ordered sequestering according to particle rank. Thus an N-body operator is written as the superposition of all distinct groupings of interactions that occur between pairs of particles, three particles, four particles, etc., such that all groupings contain at least one particle from each of the composite systems. It is shown how the transition operator, a reaction operator, and an optical potential may be described in this framework. The general nature of this decomposition is demonstrated and its connection to more standard multiple-scattering prescriptions delineated.

[NUCLEAR REACTIONS scattering theory, multiple scattering, nucleus-nucleus scattering, nuclear reactions, reaction operator expansion, transition operator expansion.]

I. INTRODUCTION

A substantial number of different approaches to multiple scattering have appeared in the literature, with wide variations in the physical systems to which they are addressed. The unique physical situation available for study in the collision of nuclei at intermediate and high energies (and perhaps also the collisions of isolated nucleons as well) prompts us to consider a subset of those approaches in which there is an ordered sequestering of the degrees of freedom into groups of constituents, which either participate in the reaction or observe (the spectators) the interaction region. In this paper we describe the interaction of two composite structures in a manner which allows for the presence of arbitrary N-body forces. Our work extends the development of Ernst, Londergan, Miller and Thaler (ELMT),[1] and of Siciliano and Thaler (ST).[2] We show schematically how the mean-field effects arising from many-body interactions may be accommodated. Furthermore, in outlining a reaction operator formalism we indicate how Pauli effects may be included in a natural manner. Thus, even in the particle-nucleus limit our approach constitutes an extension of the work of ST.

While multi-body interactions[3,4] (more than two or three body) are not normally addressed in nuclear applications, which is the area of our primary interest, the formal development allows for their presence and may be applied to a wide variety of reacting systems. In this spirit we attempt to limit the restrictive assumptions on the character (e.g., fermions or bosons) and dynamical framework (i.e., nature of the interactions) governing the behavior of the constituents in the composite systems. As an example of possible applications for multi-particle forces, this development may be used to describe nucleon-nucleon scattering as the interaction of two systems of quarks. Another area in which many-body forces are sometimes treated explicitly is that of chemical reactions.[5]

In Sec. II we deal with the definition of the Hamiltonian and introduce the connection with the transition operator T. Notational conventions and general definitions are described. We show how the Hamiltonian may be reexpressed in terms of mean fields and provide a methodology for embedding effective few-body operators in the full N-body problem. Section III is concerned with a generalization of the correlation expansion of ST to a cluster-cluster expansion. This correlation expansion, while reminiscent of the generalized cumulant expansion method,[6,7] is developed from algebraic identities for finite systems. In the limit of a structureless projectile this reduces to the results of ST. We call this prescription the generalized spectator expansion. General procedures for decomposing arbitrary N-body operators are demonstrated. In Sec. IV, we employ physical intuition to exploit the freedom inherent in such decompositions and thus delineate those quantities not formally specified by the Hamiltonian. In Sec. V we use projection operators to develop a many-body reaction operator K, whose two-body components resemble the Brueckner reaction matrix of nuclear structure calculations. In addition, an optical potential for interacting composite systems is presented. Finally, in Sec. VI we consider the construction of the matrix elements for two-body operators arising in the generalized spectator expansion (GSE).

II. HAMILTONIAN FRAMEWORK AND DEFINITIONS

This section is devoted to a concise description of a general N-body Hamiltonian, partitioned into two clusters A and B such that $A+B=N$. No restrictions are imposed upon the nature of the fundamental objects which comprise N. They are elementary in the sense that it is possible to write many-body potentials which describe their mutual interactions, but they need not all be identical.

We shall introduce auxiliary potentials, since in many applications important physical effects can be included in this way, even in low-order expansions. No requirement is made that such auxiliary potentials be used nor that they approximate a particular form. Restrictions, such as they must satisfy mean-field equations, may be imposed to facilitate rapid convergence in specific applications.

Consider an N-body system of "elementary" particles, subdivided into two clusters A and B. We write the Hamiltonian for the cluster A as

$$H^A = \sum_{i=1}^{A} (\hat{k}^i + \hat{w}^i) + \sum_{i<j}^{A} \hat{\mathfrak{v}}^{ij} + \sum_{i<j<k}^{A} \hat{\mathfrak{v}}^{ijk} + \ldots \hat{\mathfrak{v}}^{12\cdots A}, \qquad (2.1)$$

where the particles in cluster A are labeled sequentially as $1,2,3,\ldots,A$. Furthermore, $\hat{\mathfrak{v}}^{(\nu)}$ is defined as a multi-body interaction, where $(\nu) \subset (A)$ and $|\nu|$ is the number of particles in the subset (ν). Implicit is the restriction that if one of the particles in the subset is removed from the system, then $\hat{\mathfrak{v}}^{(\nu)} \to 0$. As an example, let $(\nu)=149$, then $|\nu|=3$ and $\hat{\mathfrak{v}}^{(\nu)}=\hat{\mathfrak{v}}^{149}$ is the three-body interaction between the objects in A labeled 1, 3, and 9.

We wish to consider the possibility of including auxiliary potentials in the formalism. To be specific, we introduce a set of auxiliary potentials that satisfy mean-field equations. However, such restrictions are not essential to the developments which follow.

We define a set of n-body mean fields due to the averaging of the sum of the m-body interactions for $m>n$. Thus, we write schematically in terms of the $\hat{\mathfrak{v}}^{(\nu)}$,

$$U^{(\nu)}(n) \equiv \sum_{(j<k<l\cdots)=(\mu)\subset(A)} (\hat{\mathfrak{v}}^{(\nu\mu)})_{(\mu)}, \qquad (2.2)$$

$$(\mu)\cap(\nu)\equiv 0, \quad |\nu|+|\mu|=n, \quad |\mu|\geq 1.$$

Here $U^{(\nu)}(n)$ is the $|\nu|$-body mean field experienced by the subset (ν) due to the n-body interactions averaged over the other $|\mu|$ distinct particles. The notation $(\hat{\mathfrak{v}}^{(\nu\mu)})_{(\mu)}$ means the averaging of the operator $\hat{\mathfrak{v}}^{(\nu\mu)}$ over a set of basis functions for the labeled particles contained in (μ). The selection of these basis functions may be dictated by the specific application. As an example of (2.2), consider the two-body mean-field interaction for $(\nu)=13$ when $A=5$; then

$$U^{13}(3) = (\hat{\mathfrak{v}}^{123})_2 + (\hat{\mathfrak{v}}^{134})_4 + (\hat{\mathfrak{v}}^{135})_5,$$

$$U^{13}(4) = (\hat{\mathfrak{v}}^{1234})_{24} + (\hat{\mathfrak{v}}^{1235})_{25} + (\hat{\mathfrak{v}}^{1345})_{45},$$

$$U^{13}(5) = (\hat{\mathfrak{v}}^{12345})_{245}.$$

Alternatively, it is possible to define the auxiliary potentials $U^{(\nu)}(n)$ as any "convenient" or physically motivated set of functions, rather than obtaining them from the "fundamental" interactions. In either choice, the total auxiliary potential of the $|\nu|$th rank becomes

$$U^{(\nu)} = \sum_{n=|\nu|+1}^{A} U^{(\nu)}(n). \qquad (2.3)$$

We now rewrite (2.1) in terms of the quantities

$$h^i = \hat{k}^i + \hat{w}^i + U^i, \qquad (2.4)$$

$$\sum_{i<j} \mathfrak{v}^{ij} = \sum_{i<j} (\hat{\mathfrak{v}}^{ij} + U^{ij}) - \sum_i U^i(2), \qquad (2.5a)$$

$$\sum_{i<j<k} \mathfrak{v}^{ijk} = \sum_{i<j<k} (\hat{\mathfrak{v}}^{ijk} + U^{ijk}) - \sum_i U^i(3) - \sum_{i<j} U^{ij}(3), \qquad (2.5b)$$

and generally

$$\sum_{(\nu)\subset(A)} \mathfrak{v}^{(\nu)} = \sum_{(\nu)\subset(A)} (\hat{\mathfrak{v}}^{(\nu)} + U^{(\nu)}) - \sum_{m=1}^{|\nu|-1} \sum_{\substack{(\mu_m)\subset(A)\\ |\mu_m|=m}} U^{(\mu_m)}(|\nu|). \qquad (2.5c)$$

Equations (2.5a)–(2.5c) imply that the $\mathfrak{v}^{(\nu)}$ may be defined as

$$\mathfrak{v}^{ij} \equiv \hat{\mathfrak{v}}^{ij} + U^{ij} - \left(\frac{1}{A-1}\right)[U^i(2) + U^j(2)], \qquad (2.5a')$$

$$\mathfrak{v}^{ijk} \equiv \hat{\mathfrak{v}}^{ijk} + U^{ijk} - \frac{2}{(A-1)(A-2)}[U^i(3)+U^j(3)+U^k(3)] - \left(\frac{1}{A-2}\right)[U^{ij}(3)+U^{ik}(3)+U^{jk}(3)], \qquad (2.5b')$$

and generally

$$\mathfrak{v}^{(\nu)} \equiv \hat{\mathfrak{v}}^{(\nu)} + U^{(\nu)} - \sum_{m=1}^{|\nu|-1} \left[\binom{A-m}{|\nu|-m}\right]^{-1} \left[\sum_{\substack{(\mu_m)\subset(\nu)\\ |\mu_m|=m}} U^{(\mu_m)}(|\nu|)\right]. \qquad (2.5c')$$

With these definitions we may now rewrite the Hamiltonian H^A as

$$H^A = \sum_{i=1}^{A} h^i + \sum_{i<j}^{A} v^{ij} + \dots v^{12\cdots A}, \qquad (2.6)$$

which retains the same form as the initial Hamiltonian but allows the use of auxiliary potentials.

We now introduce the subcluster interaction $V^{(\nu)}$, defined by the expression

$$V^{(\nu)} \equiv \sum_{i<j\subset(\nu)} v^{ij} + \sum_{i<j<k\subset(\nu)} v^{ijk} + \dots v^{(\nu)}. \qquad (2.7)$$

As an example, we let $A=9$, $(\nu)=127$, then $V^{127} = v^{12}+v^{17}+v^{27}+v^{127}$. It is apparent that for $(\nu) = (A)$, $V^{(A)}$ is the total interaction for the system A, i.e., $H^A = \sum h^i + V^{(A)}$.

An analogous set of equations may be written for the system of particles B using Greek subscripts $(\alpha,\beta,\gamma,\dots)$ in place of the Latin superscripts $(i,j,k,\dots)$. Equations (2.1)–(2.7) could be rewritten in this fashion, for example, (2.6) becomes

$$H_B = \sum_{\alpha=1}^{B} h_\alpha + \sum_{\alpha<\beta}^{B} v_{\alpha\beta} + \sum_{\alpha<\beta<\gamma}^{B} v_{\alpha\beta\gamma} + \dots v_{12\cdots B}. \qquad (2.6')$$

And we can write $H_B = \sum h_\alpha + V_{(B)}$. The remainder of the equations are apparent.

Consistent with the notation used in the preceding discussion, we define the interaction existing between clusters A and B, using Latin (Greek) superscripts (subscripts) to refer to elements of cluster A (B):

$$V_{(B)}^{(A)} \equiv \sum_{i,\alpha}^{A,B} \hat{v}_\alpha^i + \sum_{i<j,\alpha}^{A,B} \hat{v}_\alpha^{ij} + \sum_{i,\alpha<\beta}^{A,B} \hat{v}_{\alpha\beta}^i + \dots \hat{v}_{(B)}^{(A)}. \qquad (2.8)$$

Here $\hat{v}_{(\mu)}^{(\nu)}$ is the real $(|\nu|+|\mu|)$-body interaction, occurring between the particles (ν) of A and (μ) of B. If any one particle is removed from (ν) or (μ) then $\hat{v}_{(\mu)}^{(\nu)} \to 0$. In illustration, $\hat{v}_{14}^2$ is the three-body potential between particle 2 of A and the particles 1 and 4 of B.

One could invoke a mean-field treatment of the many-body interactions between elements of A and B in a fashion similar to that describing the mean fields for system A or B. Such a treatment could be used as a formal basis for two-center shell model reaction calculations, since the mean fields would reflect the mutual interactions of A and B. Both nuclear and atomic applications of such calculations are common.[3]

It should be clear that, having incorporated whatever mean-field effects that are desired, we could recover the form of Eq. (2.8). To signify this possibility we eliminate the carets in (2.8).

In a manner reminiscent of (2.6) we define a subcluster-subcluster interaction by the expression

$$V_{(\mu)}^{(\nu)} \equiv \sum_{\substack{i\subset(\nu)\\ \alpha\subset(\mu)}} v_\alpha^i + \sum_{\substack{i<j\subset(\nu)\\ \alpha\subset(\mu)}} v_\alpha^{ij} + \sum_{\substack{i\subset(\nu)\\ \alpha<\beta\subset(\mu)}} v_{\alpha\beta}^i + \dots v_{(\mu)}^{(\nu)}, \qquad (2.9)$$

where $(\nu)\subset(A)$ and $(\mu)\subset(B)$.

In illustration of (2.9), let $(\nu)=17$ and $(\mu)=59$, then

$$V_{59}^{17} = v_5^1 + v_9^1 + v_5^7 + v_9^7 + v_5^{17} + v_9^{17} + v_{59}^1 + v_{59}^7 + v_{59}^{17}.$$

Physically, this includes all two-body, three-body, and four-body interactions which occur between the elements 1 and 7 of A and the elements 5 and 9 of B. $V_{(\mu)}^{(\nu)}$ includes all two-body through $(|\nu|+|\mu|)$-body interactions occurring between the sets (ν) and (μ), which are not purely system A or system B interactions. Specifically, $V_{(\mu)}^{(\nu)} \cap V^{(A)} \equiv 0$ and $V_{(\mu)}^{(\nu)} \cap V_{(B)} \equiv 0$.

We recognize that for $(\nu)\equiv(A)$ and $(\mu)\equiv(B)$, we recover the full cluster-cluster interaction $V_{(B)}^{(A)}$.

The total Hamiltonian may be written as

$$H_0 = H^A + H_B, \qquad (2.10a)$$

$$H = H_0 + V_{(B)}^{(A)} = H_0 + V. \qquad (2.10b)$$

The transition operator T is defined by

$$T = V + VGV, \qquad (2.11a)$$

or equivalently by

$$T = V + VG_0T = V + TG_0V, \qquad (2.11b, c)$$

where G_0 and G are the respective propagators associated with the Hamiltonians H_0 and H. They satisfy the resolvent identities

$$G = G_0 + G_0VG \qquad (2.12a)$$

and

$$G = G_0 + G_0TG_0. \qquad (2.12b)$$

In this paper we take the viewpoint that formal developments cast in the framework of expansions for T are desirable and present a systematic set of simplifications, which offer flexibility and convenience for obtaining the transition matrix element T. It is our contention that this framework is not only computationally feasible, but also provides insight into some phenomenological treatments.

III. CORRELATION EXPANSION OF T

Section II introduced the Hamiltonians, the notation, and general definitions. The transition operator T was shown to be expressible in terms of the full N-body Green's function G, or in terms of G_0, the channel Green's function. A practical approach to the N-body scattering problem may require some approximations. We develop expansions of T which are based upon the assumption that the scattering is dominated by a simple superposition of two-body interactions, followed by a superposition of effective three-body interactions, and so forth. Thus, in leading order, we allow pairwise encounters of the participant particles with the remaining $(N-2)$ particles behaving as spectators. The passive particles may, for example, be responsible for defining mean fields in which the other particles interact. In a corrective sense, the next order includes the effects of a pair of particles from one cluster and one particle from the other cluster participating with $(N-3)$ "passive" spectators, and so forth. We show how a systematic progression of such terms is related to the exact T.

We introduce the algebraic identity of lemma A, given in Appendix A. This lemma is an extension of the one proved by Siciliano and Thaler[2] (ST). In summary, it proves that for any many-body operator, Eq. (3.1) is an identity:

$$\begin{aligned}\theta = &\sum_{i=1}^{A}\sum_{\alpha=1}^{B}\phi_\alpha^i + \sum_{i<j}^{A}\sum_{\alpha=1}^{B}(\phi_\alpha^{ij}-\phi_\alpha^i-\phi_\alpha^j) + \sum_{i=1}^{A}\sum_{\alpha<\beta}^{B}(\phi_{\alpha\beta}^i-\phi_\alpha^i-\phi_\beta^i)\\ &+\sum_{i<j}^{A}\sum_{\alpha<\beta}^{B}(\phi_{\alpha\beta}^{ij}-\phi_\alpha^{ij}-\phi_\beta^{ij}-\phi_{\alpha\beta}^i-\phi_{\alpha\beta}^j+\phi_\alpha^i+\phi_\alpha^j+\phi_\beta^i+\phi_\beta^j)\\ &+\sum_{i<j<k}^{A}\sum_{\alpha=1}^{B}(\phi_\alpha^{ijk}-\phi_\alpha^{ij}-\phi_\alpha^{ik}-\phi_\alpha^{jk}+\phi_\alpha^i+\phi_\alpha^j+\phi_\alpha^k)\\ &+\sum_{i=1}^{A}\sum_{\alpha<\beta<\gamma}^{B}(\phi_{\alpha\beta\gamma}^i-\phi_{\alpha\beta}^i-\phi_{\alpha\gamma}^i-\phi_{\beta\gamma}^i+\phi_\alpha^i+\phi_\beta^i+\phi_\gamma^i)\\ &+\sum_{i<j<k}^{A}\sum_{\alpha<\beta}^{B}(\phi_{\alpha\beta}^{ijk}-\phi_{\alpha\beta}^{ij}-\phi_{\alpha\beta}^{ik}-\phi_{\alpha\beta}^{jk}-\phi_\alpha^{ijk}-\phi_\beta^{ijk}+\phi_\alpha^{ij}+\phi_\alpha^{ik}+\phi_\alpha^{jk}+\phi_\beta^{ij}+\phi_\beta^{ik}+\phi_\beta^{jk}\\ &\qquad -\phi_\alpha^i-\phi_\beta^i-\phi_\alpha^j-\phi_\beta^j-\phi_\alpha^k-\phi_\beta^k)\\ &+\dots\\ &+(\phi_{12\cdots B}^{12\cdots A}-\dots)\,,\end{aligned}\tag{3.1}$$

for arbitrary $\phi_{(\mu)}^{(\nu)}$, provided only that Eq. (3.2) holds,

$$\theta=\phi_{(B)}^{(A)}\,. \tag{3.2}$$

For notational convenience we introduce the quantities $\theta_{(\mu)}^{(\nu)}$ defined by

$$\theta_\alpha^i=\phi_\alpha^i\,, \tag{3.3a}$$

$$\theta_\alpha^{ij}=(\phi_\alpha^{ij}-\phi_\alpha^i-\phi_\alpha^j)\,, \tag{3.3b}$$

$$\theta_{\alpha\beta}^i=(\phi_{\alpha\beta}^i-\phi_\alpha^i-\phi_\beta^i)\,, \tag{3.3c}$$

$$\begin{aligned}\theta_{\alpha\beta}^{ij}=&(\phi_{\alpha\beta}^{ij}-\phi_\alpha^{ij}-\phi_\beta^{ij}-\phi_{\alpha\beta}^i-\phi_{\alpha\beta}^j\\ &+\phi_\alpha^i+\phi_\alpha^j+\phi_\beta^i+\phi_\beta^j)\,,\end{aligned} \tag{3.3d}$$

$$\vdots$$

$$\theta_{(B)}^{(A)}=(\phi_{(B)}^{(A)}-\dots)\,. \tag{3.3e}$$

Using the summation convention introduced in Appendix A, we rewrite Eq. (3.1) as

$$\theta=\sum\theta_\alpha^i+\sum\theta_\alpha^{ij}+\sum\theta_{\alpha\beta}^i+\dots\theta_{(B)}^{(A)}\,. \tag{3.4}$$

As pointed out by ST the arbitrariness in the identity expansion gives us a great deal of flexibility for decomposing many-body operators. We exploit this freedom in developing our expansions. The "arbitrary" quantities will be chosen on physically motivated bases to exhibit a systematic progression in the decomposition of the N-body operator into effective $2,3,\dots,(N-1)$-body operators. Furthermore, we will utilize this flexibility to demonstrate connections between different multiple-scattering formalisms.

We expand the Green's functions G and G_0 via the identity (3.1). Thus we obtain for G the expression

$$G=\sum g_\alpha^i+\sum (g_\alpha^{ij}-g_\alpha^i-g_\alpha^j)$$
$$+\sum (g_{\alpha\beta}^i-g_\alpha^i-g_\beta^i)$$
$$+\sum (g_{\alpha\beta}^{ij}-g_\alpha^{ij}-g_\beta^{ij}-g_{\alpha\beta}^i-g_{\alpha\beta}^j+g_\alpha^i+g_\alpha^j+g_\beta^i+g_\beta^j)$$
$$+\dots, \tag{3.5a}$$
$$=\sum \gamma_\alpha^i+\sum \gamma_\alpha^{ij}+\sum \gamma_{\alpha\beta}^i+\sum \gamma_{\alpha\beta}^{ij}+\dots, \tag{3.5b}$$

where the definitions of the $\gamma_{(\mu)}^{(\nu)}$'s in terms of the $g_{(\mu)}^{(\nu)}$'s follow from Eqs. (3.3). For G_0, we obtain the identical form, except we now use $\tilde{g}_{(\mu)}^{(\nu)}$ instead of $g_{(\mu)}^{(\nu)}$:

$$G_0=\sum \tilde{g}_\alpha^i+\sum (\tilde{g}_\alpha^{ij}-\tilde{g}_\alpha^i-\tilde{g}_\alpha^j)+\dots, \tag{3.6a}$$
$$=\sum \tilde{\gamma}_\alpha^i+\sum \tilde{\gamma}_\alpha^{ij}+\dots. \tag{3.6b}$$

The only restrictions imposed on the $\{g_{(\mu)}^{(\nu)}\}$ and the $\{\tilde{g}_{(\mu)}^{(\nu)}\}$ are that $g_{(B)}^{(A)}\equiv G$ and $\tilde{g}_{(B)}^{(A)}\equiv G_0$. These ensure that lemma A holds.

The cluster-cluster interaction $V_{(B)}^{(A)}$ may be expanded in terms of (3.1) also; however, here we can immediately select the physically relevant choice for the expansion set. The subcluster-subcluster potentials $V_{(\mu)}^{(\nu)}$ defined by (2.9) satisfy the conditions of lemma A, and thus we write

$$V_{(B)}^{(A)}=\sum V_\alpha^i+\sum (V_\alpha^{ij}-V_\alpha^i-V_\alpha^j)$$
$$+\sum (V_{\alpha\beta}^i-V_\beta^i-V_\alpha^i)$$
$$+\sum (V_{\alpha\beta}^{ij}-V_\alpha^{ij}-V_\beta^{ij}-V_{\alpha\beta}^i-V_{\alpha\beta}^j+V_\alpha^i+V_\beta^i+V_\alpha^j+V_\beta^j)$$
$$+\dots, \tag{3.7}$$
$$=\sum \upsilon_\alpha^i+\sum \upsilon_\alpha^{ij}+\sum \upsilon_{\alpha\beta}^i+\sum \upsilon_{\alpha\beta}^{ij}+\dots. \tag{3.8a}$$

It is physically apparent what each expression in (3.7) represents, when we recall the meaning of the $\upsilon_{(\mu)}^{(\nu)}$. The element $\upsilon_{(\mu)}^{(\nu)}$ is the "real" $|\nu|+|\mu|$th-body interaction existing between the (ν) subcluster of A and the (μ) subcluster of B. In the event that a particle is removed from either (ν) or (μ), this potential vanishes.

Appendix A includes two useful corollaries which simplify the construction of expansions for operators defined by algebraic functions of other operators. Corollary I (addition corollary) states that for any operator $Z=X+Y$, if the sets $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$ satisfy the conditions of lemma A for X and Y respectively, then the set defined by $Z_{(\mu)}^{(\nu)}\equiv X_{(\mu)}^{(\nu)}+Y_{(\mu)}^{(\nu)}$ does so for Z. Similarly, corollary II (multiplication corollary) specifies that for $Z=XY$, the set $Z_{(\mu)}^{(\nu)}\equiv X_{(\mu)}^{(\nu)}Y_{(\mu)}^{(\nu)}$ is a legitimate expansion set of Z.

Lemma A allows us to write the expansion of the cluster-cluster transition operator T:

$$T=\sum t_\alpha^i+\sum (t_\alpha^{ij}-t_\alpha^i-t_\alpha^j)+\sum (t_{\alpha\beta}^i-t_\alpha^i-t_\beta^i)$$
$$+\sum (t_{\alpha\beta}^{ij}-t_\alpha^{ij}-t_\beta^{ij}-t_{\alpha\beta}^i-t_{\alpha\beta}^j+t_\alpha^i+t_\alpha^j+t_\beta^i+t_\beta^j)$$
$$+\dots, \tag{3.8b}$$
$$=\sum \tau_\alpha^i+\sum \tau_\alpha^{ij}+\sum \tau_{\alpha\beta}^i+\sum \tau_{\alpha\beta}^{ij}+\dots\tau_{(B)}^{(A)}, \tag{3.8c}$$

which we will refer to as the generalized spectator expansion for the transition operator T.

Making use of corollaries I and II of Appendix A, the definition (2.11a) for T, Eq. (3.5a) for G, and Eq. (3.7) for V, we obtain defining relations for the $t_{(\mu)}^{(\nu)}$:

$$t_\alpha^i=\upsilon_\alpha^i+\upsilon_\alpha^i g_\alpha^i \upsilon_\alpha^i, \tag{3.9a}$$
$$t_\alpha^{ij}=(\upsilon_\alpha^{ij}+\upsilon_\alpha^i+\upsilon_\alpha^j)+(\upsilon_\alpha^{ij}+\upsilon_\alpha^i+\upsilon_\alpha^j)g_\alpha^{ij}(\upsilon_\alpha^{ij}+\upsilon_\alpha^i+\upsilon_\alpha^j) \tag{3.9b}$$
$$=V_\alpha^{ij}+V_\alpha^{ij}g_\alpha^{ij}V_\alpha^{ij}, \tag{3.9b$'$}$$
$$t_{\alpha\beta}^i=V_{\alpha\beta}^i+V_{\alpha\beta}^i g_{\alpha\beta}^i V_{\alpha\beta}^i, \tag{3.9c}$$
$$t_{\alpha\beta}^{ij}=V_{\alpha\beta}^{ij}+V_{\alpha\beta}^{ij}g_{\alpha\beta}^{ij}V_{\alpha\beta}^{ij}, \tag{3.9d}$$

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$$t_{(\mu)}^{(\nu)}=V_{(\mu)}^{(\nu)}+V_{(\mu)}^{(\nu)}g_{(\mu)}^{(\nu)}V_{(\mu)}^{(\nu)}, \tag{3.9e}$$

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$$t_{(B)}^{(A)}=V_{(B)}^{(A)}+V_{(B)}^{(A)}g_{(B)}^{(A)}V_{(B)}^{(A)}\equiv T. \tag{3.9f}$$

From the form of Eqs. (3.9), it is clear that for the appropriate definitions of the $g_{(\mu)}^{(\nu)}$, the $t_{(\mu)}^{(\nu)}$ become purely $(|\nu|+|\mu|)$-body operators. Specifically, only the labeled particles (ν) from cluster A and (μ) from cluster B may participate actively in $t_{(\mu)}^{(\nu)}$. The remaining particles play the role of passive spectators. Inspection of the terms in (3.8c) displays the symmetry under interchange of clusters A and B. In τ_α^i we note that the labeling expressly separates the ith and αth particles from the $(A-i)+(B-\alpha)$ remaining particles and treats them as "effective" two-body operators. The so-called passive spectators may participate to the extent of defining background mean fields.

If, instead of using (3.5a) for G, we use (3.6a) for G_0 and the iterative definition (2.11b) for T, the resulting expansion (written with $\tilde{t}^{(\nu)}_{(\mu)}$ and $\tilde{\tau}^{(\nu)}_{(\mu)}$) of (3.8b) and (3.8c) has the following definitions for the $\tilde{t}^{(\nu)}_{(\mu)}$:

$$\tilde{t}^{\,i}_{\alpha}=\upsilon^i_\alpha+\upsilon^i_\alpha\tilde{g}^i_\alpha\tilde{t}^{\,i}_\alpha\,, \tag{3.10a}$$

$$\tilde{t}^{\,ij}_{\alpha}=V^{ij}_\alpha+V^{ij}_\alpha\tilde{g}^{ij}_\alpha\tilde{t}^{\,ij}_\alpha\,, \tag{3.10b}$$

$$\tilde{t}^{\,i}_{\alpha\beta}=V^{i}_{\alpha\beta}+V^{i}_{\alpha\beta}\tilde{g}^{i}_{\alpha\beta}\tilde{t}^{\,i}_{\alpha\beta}\,, \tag{3.10c}$$

$$\vdots$$

$$\tilde{t}^{(\nu)}_{(\mu)}=V^{(\nu)}_{(\mu)}+V^{(\nu)}_{(\mu)}\tilde{g}^{(\nu)}_{(\mu)}\tilde{t}^{(\nu)}_{(\mu)}\,, \tag{3.10d}$$

$$\vdots$$

$$\tilde{t}^{(A)}_{(B)}=V^{(A)}_{(B)}+V^{(A)}_{(B)}\tilde{g}^{(A)}_{(B)}\tilde{t}^{(A)}_{(B)}\equiv T\,. \tag{3.10e}$$

In principle we need not require that $\tilde{t}^{(\nu)}_{(\mu)}\equiv t^{(\nu)}_{(\mu)}$, for general (ν) and (μ). If we impose this restriction then we can show that $\tilde{g}^{(\nu)}_{(\mu)}$ and $g^{(\nu)}_{(\mu)}$ are related by the resolvent expressions

$$g^{(\nu)}_{(\mu)}=\tilde{g}^{(\nu)}_{(\mu)}+\tilde{g}^{(\nu)}_{(\mu)}V^{(\nu)}_{(\mu)}g^{(\nu)}_{(\mu)}\,, \tag{3.11a}$$

$$=\tilde{g}^{(\nu)}_{(\mu)}+\tilde{g}^{(\nu)}_{(\mu)}t^{(\nu)}_{(\mu)}\tilde{g}^{(\nu)}_{(\mu)}\,. \tag{3.11b}$$

As we have seen, the identity lemma and corollaries I and II have provided us with considerable simplicity in constructing operator expansions. This has facilitated establishing decomposition sets for operators defined in terms of other operators with previously defined expansion sets. In particular, we have exploited the lemma to generate freedom in the choice of the subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$ or $g^{(\nu)}_{(\mu)}$ without introducing any approximations in T. A primary emphasis of this work is to introduce certain physically motivated choices for these subcluster propagators (Sec. IV) and discuss their significance.

We may associate the expressions of (3.10) with a grouping in terms of the particle rank or total number of participant particles:

$$T(2)=\sum \tau^i_\alpha\,, \tag{3.12a}$$

$$T(3)=\sum \tau^{ij}_\alpha+\sum \tau^i_{\alpha\beta}\,, \tag{3.12b}$$

$$T(4)=\sum \tau^{ijk}_\alpha+\sum \tau^{ij}_{\alpha\beta}+\sum \tau^i_{\alpha\beta\gamma}\,, \tag{3.12c}$$

$$\vdots$$

$$T(A+B)=\tau^{(A)}_{(B)}\,, \tag{3.12d}$$

where

$$T=\sum_{n=2}^{(A+B)} T(n)\,. \tag{3.13}$$

The physical content of the $T(n)$'s is clear. $T(n)$ is the contribution in which n particles participate in the interaction. Thus $T(2)$ is the superposition of all pairwise scatterings of one particle from A with one particle from B. $T(3)$ contains a pair from $A(B)$ encountering one particle from $B(A)$. Similarly, for $T(4)$ we have the sum of contributions in which one particle from $A(B)$ interacts with a triplet cluster from $B(A)$, and those in which a correlated pair from A interacts with a correlated pair from B. The extension to arbitrary $T(n)$ is apparent. For particular choices of $\tilde{g}^{(\nu)}_{(\mu)}$, the $\tau^{(\nu)}_{(\mu)}$ can be shown to be completely connected[7-9] with respect to the labeled particles and completely disconnected from the spectator particles. This fact allows us to view the $T(n)$'s as a perturbative sequence, in which we hope to obtain a good approximation to T by the use of the term $T(2)$ with $T(3)$, $T(4)$, etc. providing the successive corrections.

The scope of the identity expansion (3.8b) of T is evident through its reduction to the generalization of the Watson multiple scattering series for cluster-cluster scattering. Let all $\tilde{g}^{(\nu)}_{(\mu)}\equiv G_0$ in Eqs. (3.10) and allow only two-body interactions. As the results in Appendix A show, the arbitrariness in $\tilde{g}^{(\nu)}_{(\mu)}$ permits this choice, Then we obtain

$$\bar{t}^{\,i}_\alpha=\upsilon^i_\alpha+\upsilon^i_\alpha G_0\bar{t}^{\,i}_\alpha\,. \tag{3.14}$$

A careful examination of the expressions for $\tau^{(\nu)}_{(\mu)}$ in (3.8b) yields

$$\begin{aligned} T=&\sum \bar{t}^{\,i}_\alpha+\sum(\bar{t}^{\,i}_\alpha G_0\bar{t}^{\,j}_\alpha+\bar{t}^{\,j}_\alpha G_0\bar{t}^{\,i}_\alpha+\bar{t}^{\,i}_\alpha G_0\bar{t}^{\,j}_\alpha G_0\bar{t}^{\,i}_\alpha+t^j_\alpha G_0 t^i_\alpha G_0 t^j_\alpha\cdots)\\ &+\sum(\bar{t}^{\,i}_\alpha G_0\bar{t}^{\,i}_\beta+\bar{t}^{\,i}_\beta G_0\bar{t}^{\,i}_\alpha+\bar{t}^{\,i}_\alpha G_0\bar{t}^{\,i}_\beta G_0\bar{t}^{\,i}_\alpha+\bar{t}^{\,i}_\beta G_0\bar{t}^{\,i}_\alpha G_0\bar{t}^{\,i}_\beta+\cdots)+\sum(\bar{t}^{\,i}_\alpha G_0\bar{t}^{\,j}_\alpha G_0\bar{t}^{\,k}_\alpha+\cdots)\\ &+\cdots\,. \end{aligned} \tag{3.15}$$

Equation (3.15) may be reorganized to give

$$T+\sum \bar{t}^{\,i}_{\alpha}+\sum_{i\neq j}\bar{t}^{\,i}_{\alpha}G_0\bar{t}^{\,j}_{\alpha}+\sum_{\alpha\neq\beta}\bar{t}^{\,i}_{\alpha}G_0\bar{t}^{\,i}_{\beta}+\sum_{\substack{i\neq j\\ \alpha\neq\beta}}\bar{t}^{\,i}_{\alpha}G_0\bar{t}^{\,j}_{\beta}+\sum_{\substack{i\neq j\\ j\neq k}}\bar{t}^{\,i}_{\alpha}G_0\bar{t}^{\,j}_{\alpha}G_0\bar{t}^{\,k}_{\alpha}+\sum_{\substack{\alpha\neq\beta\\ \beta\neq\gamma}}\bar{t}^{\,i}_{\alpha}G_0\bar{t}^{\,i}_{\beta}G_0\bar{t}^{\,i}_{\gamma}+\sum_{\substack{i\neq j,\, j\neq k\\ \alpha\neq\beta,\,\beta\neq\gamma}}\bar{t}^{\,i}_{\alpha}G_0\bar{g}^{\,j}_{\beta}G_0\bar{t}^{\,k}_{\gamma}+\cdots . \tag{3.16}$$

Here Eq. (3.16) contains an infinite set of terms and for $\beta=1$ reduces to

$$T(B=1)=\sum \bar{t}^{\,i}_1+\sum_{i\neq j}\bar{t}^{\,i}_1G_0\bar{t}^{\,j}_1 +\sum_{\substack{i\neq j\\ j\neq k}}\bar{t}^{\,i}_1G_0\bar{t}^{\,j}_1G_0\bar{t}^{\,k}_1+\cdots , \tag{3.17}$$

which is the Watson multiple scattering series.[10] Thus, beginning with the expansion (3.8b) for T we have obtained in a straightforward fashion the generalization (3.16) of the Watson multiple scattering formalism (3.17). We observe that the content of T in terms of the identity expansion readily yields the connection with conventional multiple-scattering series. The flexibility inherent in our ability to choose the $\tilde{g}^{(\nu)}_{(\mu)}$ permits us to construct different expansions on the basis of utility and convenience. In brief, we have a method for examining cluster-cluster interactions in a systematic fashion, which readily displays the connection to the exact result stemming from different leading-order approximations.

IV. SELECTION OF SUBCLUSTER PROPAGATORS

In Sec. III we decomposed the cluster-cluster T operator into a finite series of participant-spectator operators. While this expansion is not a "perturbation" series in the standard sense of order-by-order in some coupling parameter, we hope that the selection of the subcluster propagators can be made so as to maximize the content of $T(2)$ which is readily calculable and minimize the corrections due to $T(3)$, $T(4)$, etc. The correlation decomposition resembles the hole-line expansion[11] of nuclear-structure calculations which is effectively an expansion in the density times a correlation volume.

This section is concerned with developing a systematic set of definitions for the subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$ and $g^{(\nu)}_{(\mu)}$. We desire a set of $\tilde{g}^{(\nu)}_{(\mu)}$ for which the $T(n)$ will satisfy one of the conditions of (4.1):

$$\langle T\rangle \cong \langle T(2)\rangle \gg \sum_{n=3}^{N} T(n) \ , \tag{4.1a}$$

$$\langle T\rangle \cong \sum_{n=2}^{m} T(n) \gg \sum_{l=m+1}^{N} T(l) \ , \tag{4.1b}$$

$$\langle T(2)\rangle \gg \langle T(3)\rangle > \langle T(4)\rangle > \langle T(5)\rangle > \cdots > \langle T(N)\rangle . \tag{4.1c}$$

In particular applications it is possible that a specific subcluster-subcluster interaction provides the dominant character of the series, such as in collective scattering processes. Thus, another convenient condition might be

$$\langle T\rangle \cong \langle T(m)\rangle \gg \sum_{n\neq m}^{N} T(n) \ . \tag{4.1d}$$

An example of such an application might be in treating nucleon-nucleon scattering as a collection of quarks. Thus, for nucleons described as bound states of three quarks, the scattering operator T may be dominated naturally by $T(6)$. Another example might be the collisions of even-even $N=Z$ nuclei. It may well be that the total amplitude is dominated by the alpha-alpha terms which are contained in $T(8)$.

There is not an *a priori* guarantee that a set of subcluster propagators exists for which any of the conditions of (4.1) hold. We shall assume, for practical purposes, that we can find a set for which (4.1a) or (4.1b), with m small, is satisfied. Much work has been done in which one or more of these conditions appears to have been fulfilled with satisfactory results. To cite just one example, consider the work of Fujita and Hüfner[12] in which they develop a phenomenological treatment of back-angle proton scattering from nuclei. They find that the differential cross section appears to be the superposition of scatterings from single target nucleons, correlated pairs, correlated triplets, and so forth. Such a description demonstrates an application where condition (4.1b) appears to be satisfied.

In what follows we hope to motivate choices of the propagators which reduce the computational magnitude of the "lower-order" $T(n)$ by ensuring that the respective $t^{(\nu)}_{(\mu)}$'s are effective $n(=|\nu|+|\mu|)$-body operators rather than N-body operators. Furthermore, we believe such choices should possess transparent physical interpretations.

For example, in $T(2)$ we would like a choice of the subcluster propagators $\tilde{g}^i_\alpha$ which reduces the t^i_α to effective two-body operators rather than being a fully coupled N-body operator. Similarly in $T(3)$, we desire fully coupled three-body operators, "disconnected" from the remaining $N-3$ objects. Or generally, $T(n)$ is to be an n-body operator which is diagonal in the space of the remaining N-n particles.

By using lemma A we can expand H_0, the noninteracting cluster-cluster Hamiltonian, in terms of arbitrary partial Hamiltonians $h^{(\nu)}_{(\mu)}$ as

$$H_0=\sum h^i_\alpha+\sum (h^{ij}_\alpha-h^i_\alpha-h^j_\alpha)$$

$$+\sum (h^i_{\alpha\beta}-h^i_\alpha-h^i_\beta)+\cdots, \tag{4.2a}$$

$$=\sum H^i_\alpha+\sum H^{ij}_\alpha+\sum H^i_{\alpha\beta}+\cdots, \tag{4.2b}$$

with the restriction that

$$h^{(A)}_{(B)}\equiv H_0 . \tag{4.3}$$

It is then natural to use the $h^{(\nu)}_{(\mu)}$ selected for a given application to define the hitherto arbitrary $\tilde{g}^{(\nu)}_{(\mu)}$ by

$$\tilde{g}^{(\nu)}_{(\mu)}\equiv(Z-h^{(\nu)}_{(\mu)})^{-1} . \tag{4.4a}$$

Here Z represents the complex parametric energy which includes the appropriate boundary conditions. From (4.4a) and the resolvent Eqs. (3.11), we obtain for the interacting subcluster-subcluster propagators $g^{(\nu)}_{(\mu)}$ the expression

$$g^{(\nu)}_{(\mu)}\equiv(Z-h^{(\nu)}_{(\mu)}-V^{(\nu)}_{(\mu)})^{-1} . \tag{4.4b}$$

Note that in terms of the $h^{(\nu)}_{(\mu)}$, $g^{(A)}_{(B)}=(Z-h^{(A)}_{(B)}-V^{(A)}_{(B)})^{-1}=G$, and that $\tilde{g}^{(A)}_{(B)}=(Z-h^{(A)}_{(B)})^{-1}=G_0$, which are the proper limits of $g^{(\nu)}_{(\mu)}$ and $\tilde{g}^{(\nu)}_{(\mu)}$ respectively.

In order for the $t^{(\nu)}_{(\mu)}$ to operate upon the participants (i.e., the labeled particles) the $h^{(\nu)}_{(\mu)}$ must be defined to contain the $(\nu)+(\mu)$ particles. The remaining particles may appear provided no interactions are permitted which link the two groups. We make the following definitions:

$$\bar{h}^{(\nu)}\equiv\sum_{i\subset(\nu)} h^i+V^{(\nu)}, \tag{4.5a}$$

$$\bar{h}_{(\mu)}\equiv\sum_{\alpha\subset(\mu)} h_\alpha+V_{(\mu)}, \tag{4.5b}$$

$$\bar{h}^{(\nu)}_{(\mu)}=\bar{h}^{(\nu)}+\bar{h}_{(\mu)}. \tag{4.6}$$

The $\bar{h}^{(\nu)}$ and $\bar{h}_{(\mu)}$ are the subcluster Hamiltonians of the (ν) and (μ) subclusters of A and B respectively. We can write residual subcluster Hamiltonians as $\bar{h}^{(A-\nu)}$ and $\bar{h}_{(B-\mu)}$. Equation (4.6) defines the noninteracting subcluster-subcluster Hamiltonian, $\bar{h}^{(\nu)}_{(\mu)}$.

We are free to define the partial Hamiltonian $h^{(\nu)}_{(\mu)}$ by the following expression:

$$h^{(\nu)}_{(\mu)}\equiv\bar{h}^{(\nu)}_{(\mu)}+\bar{h}^{(A-\nu)}_{(B-\mu)}, \tag{4.7}$$

which obeys the restriction of lemma A, that $h^{(A)}_{(B)}\equiv H_0$, as can be seen by examining the definitions (4.5) and (4.6). It is important to emphasize that the definition (4.7) specifically leaves out interactions between elements of (ν) and $(A-\nu)$ as well as those between (μ) and $(B-\mu)$. In this way our definitions ensure the spectator separation of our expansions.

It is possible to introduce into these definitions auxiliary potentials. These may arise from mean-field definitions or may merely be physically or computationally desirable. The only restriction is that such fields must become zero for $(\nu)=(A)$ and $(\mu)=(B)$.

An alternative definition for the partial Hamiltonian $h^{(\nu)}_{(\mu)}$, which also maintains the separate character of the interacting particles and spectators, is apparent from the definition of the subcluster-subcluster Hamiltonian $\bar{h}^{(\nu)}_{(\mu)}$. We may choose

$$h^{(\nu)}_{(\mu)}\equiv\bar{h}^{(\nu)}_{(\mu)}+\Delta\epsilon^{(\nu)}_{(\mu)}, \tag{4.8}$$

where

$$\Delta\epsilon^{(A)}_{(B)}\equiv 0 . \tag{4.9}$$

The $\Delta\epsilon^{(\nu)}_{(\mu)}$ may be chosen to represent the energy shift due to the residual particles. It is readily verified that (4.8) obeys the restriction of (4.3).

The definitions (4.7) of the partial Hamiltonians, in conjunction with the definitions (4.4) of the propagators $\tilde{g}^{(\nu)}_{(\mu)}$ and Eqs. (3.10) for the partial transition operators $t^{(\nu)}_{(\mu)}$, have the virtue that the $t^{(\nu)}_{(\mu)}$ become effective $n(=|\nu|+|\mu|)$-body operators which are diagonal with respect to the residual Hamiltonian particles (spectators). Similarly, definition (4.8) yields $t^{(\nu)}_{(\mu)}$ which are true n-body operators. Other definitions for the partial Hamiltonians $h^{(\nu)}_{(\mu)}$ are possible.

It is of some interest to note that the partial transition operators $t^{(\nu)}_{(\mu)}$ corresponding to the partial Hamiltonians $h^{(\nu)}_{(\mu)}$ of Eq. (4.8) with the choice $\Delta\epsilon^{(\nu)}_{(\mu)}\equiv 0$ and in the absence of mean fields (see Sec. II) are the free (ν)-cluster (μ)-cluster transition operators. For example, t^i_α is then the free nucleon-nucleon transition operator, if i and α are nucleons. Thus, for $\Delta\epsilon^i_\alpha\neq 0$, t^i_α is an energy-shifted two-body t operator; we may say that $\Delta\epsilon^i_\alpha$ accounts in some average fashion for the off-shell nature of t^i_α when i and α are embedded in larger groups of particles. Including mean fields in $h^{(\nu)}_{(\mu)}$ permits a more complete treatment of the off-shell dynamics and renders t^i_α an off-shell two-nucleon t matrix.

The set of definitions (4.8) for the partial Hamiltonian yields the generalization of the propagators defined by ST[2]; this selection also corresponds in lowest order to the closure approximation[13] of Ernst, Londergan, Miller, and Thaler (ELMT).[1] In the work of ELMT, the T operator is approximated by replacing all of the $\tilde{g}^{(\nu)}_{(\mu)}$'s by the choice for $\tilde{g}^{i}_{\alpha}$ and retaining only two-body interactions. To obtain some systematic order-by-order estimate of the discrepancy between T and T_{ELMT}, we would look at $[T(n) - T(n)_{\mathrm{ELMT}}]$. For $n=2$, $T(2) = T(2)_{\mathrm{ELMT}}$.

We may give a spectral representation for the partial propagator $\tilde{g}^{(\nu)}_{(\mu)}$ if we first construct the scattering solutions to the subcluster Hamiltonians:

$$[\epsilon^{(\nu)}_{(\mu)}(a) - \bar{h}^{(\nu)}_{(\mu)}]\,\phi^{(\nu)}_{(\mu)}(a) = 0\,, \tag{4.10a}$$

$$[\epsilon^{(\nu)}_{(\mu)}(a) - \bar{h}^{(\nu)}_{(\mu)} - V^{(\nu)}_{(\mu)}]\,\Psi^{(\pm)(\nu)}_{(\mu)}(a) = 0\,, \tag{4.10b}$$

$$[E^{(\nu)}(b) - \bar{h}^{(A-\nu)}]\,\chi^{(A-\nu)}(b) = 0\,, \tag{4.11a}$$

$$[E_{(\mu)}(c) - \bar{h}_{(B-\mu)}]\,\chi_{(B-\mu)}(c) = 0\,. \tag{4.11b}$$

Thus, for the definition (4.8) of the partial Hamiltonian $h^{(\nu)}_{(\mu)}$, the spectral decompositions of the subcluster-subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$ and $g^{(\nu)}_{(\mu)}$ are

$$\tilde{g}^{(\nu)}_{(\mu)} = \sum_{a} \frac{|\phi^{(\nu)}_{(\mu)}(a)\rangle\langle\phi^{(\nu)}_{(\mu)}(a)|}{Z - \epsilon^{(\nu)}_{(\mu)}(a) - \Delta\epsilon^{(\nu)}_{(\mu)}} \tag{4.12}$$

and

$$g^{(\nu)}_{(\mu)} = \sum_{a} \frac{|\Psi^{(\pm)(\nu)}_{(\mu)}(a)\rangle\langle\Psi^{(\pm)(\nu)}_{(\mu)}(a)|}{Z - \epsilon^{(\nu)}_{(\mu)}(a) - \Delta\epsilon^{(\nu)}_{(\mu)}}\,. \tag{4.13}$$

We may also use definition (4.7), in which case we obtain the expression

$$\tilde{g}^{(\nu)\prime}_{(\mu)} = \sum_{a,b,c} \frac{S\,|\phi^{(\nu)}_{(\mu)}(a)\chi^{(A-\nu)}(b)\chi_{(B-\mu)}(c)\rangle\langle\phi^{(\nu)}_{(\mu)}(a)\chi^{(A-\nu)}(b)\chi_{(B-\mu)}(c)|}{Z - \epsilon^{(\nu)}_{(\mu)}(a) - E^{(\nu)}(b) - E_{(\mu)}(c)}\,, \tag{4.14}$$

with an analogous expression for $g^{(\nu)\prime}_{(\mu)}$. Here S is the appropriate symmetrization operator. For example, if all N particles are nucleons, then S antisymmetrizes the particles between the state vectors $\phi^{(\nu)}_{(\mu)}$, $\chi^{(A-\nu)}$, and $\chi_{(B-\mu)}$, unless the proper symmetrization has already been included in the intercluster potentials $v^{(\nu)}_{(\mu)}$.[4]

V. K MATRIX AND OPTICAL POTENTIAL APPLICATIONS

In reducing the N-body problem to systems of fewer than N bodies, we have not addressed all the physical and practical aspects that arise. For example, in applying the GSE to nuclear systems we have not considered specifically the effect of the Pauli principle. We now focus our attention on the nuclear applications and address the questions of the Pauli principle, the strong repulsive core of the nucleon-nucleon interaction (V_{NN}) and the reduction to calculable spaces. For bound states of finite nuclei the effective two-nucleon interaction has been derived from a free potential V_{NN} through the Brueckner G_B matrix,

$$G_B(\omega) = V_{NN} + V_{NN}\,\frac{Q_p}{\omega - h_0^{(1)} - h_0^{(2)}}\,G_B(\omega)\,,$$

where h_0 is a single particle Hamiltonian incorporating the kinetic energy and a single particle potential for each of the interacting nucleons. The G_B matrix is a renormalized nucleon-nucleon interaction which, through the infinite sum of potential scatterings, eliminates the strong effects of the repulsive core in much the same manner as does the t matrix. The Pauli operator Q_p prevents the interacting nucleon pair from scattering into states which are already occupied by the remaining nucleons or which are treated explicitly in the dynamical framework (e.g., diagonalization). For all applications, Q_p must be specified in the selected basis representation. The two most common bases for nuclear problems are the harmonic oscillator and the plane wave representation.

Our development in this section uses K rather than G_B or G to avoid notational confusion. We develop a formal relationship between the T and K operator expansions of GSE. From Appendix B we know that given projectors[14] $P+Q=1$, the operator expression (5.1)

$$A = B + BCD \tag{5.1}$$

may be decomposed to yield

$$A = D + DCPA\,, \tag{5.2a}$$

where

$$D = B + BDQD\,. \tag{5.2b}$$

The general expressions (5.2a) and (5.2b) together possess the same content as (5.1) and may be used to generate specific operator separations. The subcluster transition operators $t^{(\nu)}_{(\mu)}$ of (3.10) are defined by operator expressions of the form of (5.1). Thus, if we define a set of projectors

$\{p^{(\nu)}_{(\mu)}, q^{(\nu)}_{(\mu)}\}$ to obey the projection rules of Appendix B, we may define a new set of operators $k^{(\nu)}_{(\mu)}$ by

$$k^{(\nu)}_{(\mu)} = V^{(\nu)}_{(\mu)} + V^{(\nu)}_{(\mu)}\tilde{g}^{(\nu)}_{(\mu)}q^{(\nu)}_{(\mu)}k^{(\nu)}_{(\mu)}, \tag{5.3}$$

which is related to $t^{(\nu)}_{(\mu)}$ by the expression

$$t^{(\nu)}_{(\mu)} = k^{\nu}_{\mu} + k^{(\nu)}_{(\mu)}\tilde{g}^{(\nu)}_{(\mu)}p^{(\nu)}_{(\mu)}t^{(\nu)}_{(\mu)}, \tag{5.4}$$

with the restriction that

$$p^{(\nu)}_{(\mu)} + q^{(\nu)}_{(\mu)} = 1(n). \tag{5.5}$$

The Eqs. (5.3) and (5.4) are formally equivalent to the expressions (3.10) for the subcluster transition operator.

We may define a total K operator as

$$K = \sum k^{i}_{\alpha} + \sum (k^{ij}_{\alpha} - k^{i}_{\alpha} - k^{j}_{\alpha}) + \cdots. \tag{5.6}$$

The physical content of (5.6) is manifest once the subcluster projectors $\{q^{(\nu)}_{(\mu)}\}$ have been specified. Two examples we consider in this section involve selections of $\{q^{(\nu)}_{(\mu)}\}$ such that K is an optical potential or such that $K = T$. This latter condition implies only that $q^{(A)}_{(B)} \equiv 1(N)$. The remaining $q^{(\nu)}_{(\mu)}$ are still at our disposal. Using the definitions of the Brueckner reaction matrix as a guide, we select $q^{(\nu)}_{(\mu)}$ to have the property that states occupied by the $|A+B-\nu-\mu|$ spectators are forbidden to the $|\nu+\mu|$ participants. In this case we call the $k^{(\nu)}_{(\mu)}$ the subcluster reaction operators.

In order to illustrate the physical content of the $k^{(\nu)}_{(\mu)}$, we specify a set of $q^{(\nu)}_{(\mu)}$ in the spectral representation of (4.14) or (4.12). We define the Pauli projector as

$$q^{(\nu)}_{(\mu)} = \sum_{\substack{(a)\ \text{Pauli-restricted} \\ \text{by } |A+B-\nu-\mu| \text{ spectators}}} |\phi^{(\nu)}_{(\mu)}(a)\rangle\langle\phi^{(\nu)}_{(\mu)}(a)|. \tag{5.7}$$

Note that for $(\nu) = (A)$ and $(\mu) = (B)$, $q^{(A)}_{(B)} = 1(N)$, since the absence of spectators eliminates the additional external Pauli restrictions upon (a). In fact, whatever contributions are not included in $q^{(\nu)}_{(\mu)}$ are implicitly contained in $p^{(\nu)}_{(\mu)}$. From a pragmatic viewpoint the content of $q^{(\nu)}_{(\mu)}$ may be enlarged or restricted in order to construct the model space in which $p^{(\nu)}_{(\mu)}$ is desired to operate. However, we must remark that while such modifications do not affect the validity of the expansion for K, the convergence of the lower-order terms may be affected.

As mentioned above we can also construct an optical potential through the judicious selection of P and Q. In order to maintain a close parallel to the work of ELMT and ST, we shall develop a generalization of the optical potential discussed by ST. We select $P+Q=1(N)$, where P is the projection onto the nuclear ground states of clusters A and B. Thus, for $\tilde{g}^{i}_{\alpha}$, $\tilde{g}^{ij}_{\alpha}, \ldots$, P and Q operate in a sequence of different basis representations but retain the same physical content. In order to emphasize this point we shall use P_{opt} and Q_{opt} throughout rather than $P_{\text{opt}}{}^{(\nu)}_{(\mu)}$ and $Q_{\text{opt}}{}^{(\nu)}_{(\mu)}$. From (3.10) we have

$$t^{(\nu)}_{(\mu)} = V^{(\nu)}_{(\mu)} + V^{(\nu)}_{(\mu)}g^{(\nu)}_{(\mu)}t^{(\nu)}_{(\mu)}, \tag{3.10$'$}$$

and in analogy with (5.3) and (5.4) we obtain

$$u^{(\nu)}_{(\mu)} = V^{(\nu)}_{(\mu)} + V^{(\nu)}_{(\mu)}\tilde{g}^{(\nu)}_{(\mu)}Q_{\text{opt}}u^{(\nu)}_{(\mu)} \tag{5.8a}$$

and

$$t^{(\nu)}_{(\mu)} = u^{(\nu)}_{(\mu)} + u^{(\nu)}_{(\mu)}\tilde{g}^{(\nu)}_{(\mu)}P_{\text{opt}}t^{(\nu)}_{(\mu)}. \tag{5.8b}$$

We may then write the total optical potential as

$$U_{\text{opt}} = \sum u^{i}_{\alpha} + \sum (u^{ij}_{\alpha} - u^{i}_{\alpha} - u^{j}_{\alpha}) + \sum (u^{i}_{\alpha\beta} - u^{i}_{\alpha} - u^{i}_{\beta}) + \cdots \tag{5.9a}$$

$$= \sum U^{i}_{\alpha} + \sum U^{ij}_{\alpha} + \sum U^{i}_{\alpha\beta} + \cdots. \tag{5.9b}$$

Recall from lemma A that $U_{\text{opt}} \equiv u^{(A)}_{(B)}$, and from (5.8a) we obtain

$$u^{(A)}_{(B)} = V^{(A)}_{(B)} + V^{(A)}_{(B)}G_0Q_{\text{opt}}u^{(A)}_{(B)},$$

which is the conventional definition of U_{opt}. In correspondence to Eq. (3.12) for $T(n)$, we now write

$$U_{\text{opt}}(2) \equiv \sum U^{i}_{\alpha}, \tag{5.10a}$$

$$U_{\text{opt}}(3) \equiv \sum U^{ij}_{\alpha} + \sum U^{i}_{\alpha\beta}, \tag{5.10b}$$

$$\vdots$$

$$U_{\text{opt}}(n) \equiv \sum_{l=1}^{n-1} \sum_{\substack{(\nu)\subset A \\ |\nu|=l \\ (\mu)\subset B \\ |\mu|=n-l}} U^{(\nu)}_{(\mu)}. \tag{5.10c}$$

Equations (5.10) let us write the total optical potential as

$$U_{\text{opt}} = \sum_{n=2}^{A+B} U_{\text{opt}}(n). \tag{5.11}$$

It is easy to recognize the physical content of the $U_{\text{opt}}(n)$ for this expansion. $U_{\text{opt}}(n)$ is the contribution to the optical potential of n particles scattering in all possible combinatorial patterns. Specifically, U(2) is the superposition of all pairwise scattering between a particle from A and a particle from B, in which both cluster A and B remain in their ground state configuration. For U(3), we have the sum of the correlated pairs from $A(B)$ scattering from a single particle from $B(A)$

and again both clusters A and B remain in their respective ground states, and so forth for $U(n)$.

We recognize that the exact content of $U_{opt}(n)$ depends upon the subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$. Thus, for different propagators, the exact contribution of $U_{opt}(n)$ to U_{opt} will change. Furthermore, while we have specifically retained the conventional definition of U_{opt} in regard to defining Q_{opt} and P_{opt}, we could in fact obtain a different set of $U'_{opt}(n)$ which give U_{opt} for an alternate view of Q and P. For example, if we define the set of $\{p^{(\nu)}_{(\mu)}\}$ as the projectors upon the ground states of the subclusters $(A-\nu)$ and $(B-\mu)$, then $U'_{opt}\equiv U_{opt}$, but in general $U'_{opt}(n)\neq U_{opt}(n)$.

VI. MATRIX ELEMENTS

The bulk of our discussion has been concerned with the development of formal operator expressions and with the selection of subcluster operators based upon physical arguments. In this section we are more directly concerned with the construction of matrix elements for these operators. We shall consider specifically the elastic scattering of two nuclei wherein we retain only two-body potentials.

As has been emphasized by ST, it is only in the case where t^i_α is a two-body operator that the $T(2)$ matrix elements reduce to a simple form. In the following discussion we shall suppress the indices i and α where no confusion can arise. Furthermore, we restrict this matrix element development to the case where g^i_α is the sum of simple kinetic energy operators for the interacting nucleons as in (4.12). Thus, we write

$$\tilde{g}(\vec{p},\vec{k})=(z-\vec{p}^2/2m-\vec{k}^2/2m)^{-1}. \tag{6.1}$$

Here $\vec{p}(\vec{k})$ refers to the momentum vector of the nucleon from $A(B)$. We may now write for the two-body operator t of Eq. (3.10′) the expression

$$(\vec{p}'\vec{k}'|t|\vec{p}\vec{k})=(\vec{p}'\vec{k}'|v|\vec{p}\vec{k})+\int d\vec{p}''d\vec{k}''(\vec{p}'\vec{k}'|v|\vec{p}''\vec{k}'')\times\tilde{g}(\vec{p}'',\vec{k}'')(\vec{p}''\vec{k}''|t|\vec{p}\vec{k}). \tag{6.2}$$

Let $\vec{q}$ and $\vec{P}_c$ be defined as follows:

$$\vec{q}=(\vec{p}-\vec{k})/2, \tag{6.3a}$$

$$\vec{P}_c=\vec{p}+\vec{k}. \tag{6.3b}$$

Then t, for translationally invariant v, may be written as

$$(\vec{q}'|t|\vec{q})=(\vec{q}'|v|\vec{q})+\int d\vec{q}''(\vec{q}'|v|\vec{q}'')\times\frac{1}{z-\vec{P}_c^{\,2}/4m-\vec{q}''^2/m}(\vec{q}''|t|\vec{q}). \tag{6.4}$$

In order to connect the two-nucleon matrix elements to the scattering system, we shall define the relationship between the interacting pair and the remaining (passive) nucleons. Let $\vec{P}_{cm}$ and $\vec{P}_r$ be the total momentum and relative momentum of the A-B system. We define $\vec{P}_A(\vec{P}_B)$ as the total momentum of the cluster $A(B)$, and $\vec{P}_{Ai}(\vec{P}_{B\alpha})$ as the total momentum of the cluster $A(B)$ with the ith(αth) nucleon removed. We then have the following definitions:

$$\vec{P}_A=\sum_j^A\vec{p}^j=\vec{p}^i+\sum_{j\neq i}^A\vec{p}^j=\vec{p}^i+\vec{P}_{Ai}, \tag{6.5a}$$

$$\vec{P}_B=\sum_\alpha^B\vec{k}_\alpha=\vec{k}_\alpha+\sum_{\beta\neq\alpha}^B\vec{k}_\beta=\vec{k}_\alpha+\vec{P}_{B\alpha}, \tag{6.5b}$$

$$\gamma_A=M_A/(M_A+M_B), \tag{6.6a}$$

$$\gamma_B=M_B/(M_A+M_B), \tag{6.6b}$$

$$\vec{P}_{cm}=\vec{P}_A+\vec{P}_B, \tag{6.7a}$$

$$\vec{P}_r=\gamma_B\vec{P}_A-\gamma_A\vec{P}_B, \tag{6.7b}$$

$$\vec{P}_A=\gamma_A\vec{P}_{cm}+\vec{P}_r, \tag{6.8a}$$

$$\vec{P}_B=\gamma_B\vec{P}_{cm}-\vec{P}_r. \tag{6.8b}$$

Here $M_A(M_B)$ is the mass of the nucleus $A(B)$, and the $\vec{p}^j(\vec{k}_\beta)$ are the individual momentum vectors of the nucleons.

Consider an arbitrary two-body operator θ^i_α; in order to form the matrix elements of θ^i_α we must first define the transformation which connects the intrinsic wave functions (IWF) to the reduced intrinsic wave functions (RIWF). By definition we call $\phi_A(a)$ and $\phi_B(b)$ the intrinsic wave functions of clusters A and B respectively in the eigenstates a and b. Furthermore, we call $\Phi_{Ai}(a)$ and $\Phi_{B\alpha}(b)$ the RIWF of the A-i and B-α systems. The momentum space wave functions are then functions of the following coordinates:

$$\phi_A(a)=\phi_A[\vec{p}_{in}^{\,1},\vec{p}_{in}^{\,2},\ldots,\vec{p}_{in}^{\,A-1}](a),$$

$$\phi_B(b)=\phi_B[\vec{k}_{in}^{\,1},\vec{k}_{in}^{\,2},\ldots,\vec{k}_{in}^{\,B-1}](b),$$

$$\Phi_{Ai}(a)=\Phi_{Ai}[\vec{P}_{Ai};\vec{p}_{in}^{\,1'},\vec{p}_{in}^{\,2'},\ldots,\vec{p}_{in}^{\,A-2'}](a),$$

$$\Phi_{B\alpha}(b)=\Phi_{B\alpha}[\vec{P}_{B\alpha};\vec{k}_{in}^{\,1'},\vec{k}_{in}^{\,2'},\ldots,\vec{k}_{in}^{\,B-2'}](b).$$

The transformation which connects the IWF and the RIWF is defined by the following expression:

$$|\vec{P}_A\phi_A(a)\rangle=\int d\vec{p}^{\,i}\delta(\vec{p}^{\,i}+\vec{P}_{Ai}-\vec{P}_A)|\vec{p}^{\,i}\Phi_{Ai}(a)\rangle. \tag{6.9}$$

An analogous expression exists for cluster B. This equation defines the meaning of the RIWF $\Phi_{Ai}(a)$. We define the single-body densities of the RIWF, such that the bra-ket of the IWF yields $\delta(\vec{P}_A-\vec{P}'_A)\delta_{aa'}$,

$$\rho_{Ai}^{(1)}(\vec{P}_A - \vec{p}^i)_{aa'} = \int \Phi_{Ai}^*[\vec{P}_{Ai}; \vec{p}_{in}^{\,1}, \vec{p}_{in}^{\,2}, \ldots, \vec{p}_{in}^{\,A-2}](a)\delta(\vec{P}_{Ai} + \vec{p}^i - \vec{p}_A) \times \Phi_{Ai}[\vec{P}_{Ai}; \vec{p}_{in}^{\,1}, \vec{p}_{in}^{\,2}, \ldots, \vec{p}_{in}^{\,A-2}](a') d\vec{P}_{Ai} d\vec{p}_{in}^{\,1} d\vec{p}_{in}^{\,2} \cdots d\vec{p}_{in}^{\,A-2}, \tag{6.10a}$$

where the integration of this density is

$$\int \rho_{Ai}^{(1)}(\vec{P})_{aa'} d\vec{P} = \delta_{aa'}. \tag{6.10b}$$

Having made these definitions and restrictions we can write the transformation for the A-B system in the following manner:

$$|\vec{P}_{cm}\vec{P}_r \phi_A(0)\phi_B(0)\rangle = \int d\vec{p}^i \delta(\vec{p}^i + \vec{P}_{Ai} - \gamma_A \vec{P}_{cm} - \vec{P}_r) d\vec{k}_\alpha \delta(\vec{k}_\alpha + \vec{P}_{B\alpha} - \gamma_B \vec{P}_{cm} + \vec{P}_r) |\vec{p}^i \vec{k}_\alpha \Phi_{Ai}(0)\Phi_{B\alpha}(0)\rangle. \tag{6.11}$$

Since we anticipate treating antisymmetrization through effective multi-body interactions[4] we work in the simple product representation. Thus, the matrix elements of the two-body operator θ_α^i may now be written as

$$\langle \vec{P}_{cm}^F \vec{P}_r^F \phi_A(0)\phi_B(0)|\theta_\alpha^i|\vec{P}_{cm}^I \vec{P}_r^I \phi_A(0)\phi_B(0)\rangle = \int \langle \vec{p}^{iF} \vec{k}_\alpha^F \Phi_{Ai}(0)\Phi_{B\alpha}(0)| d\vec{p}^{iF} \delta(\vec{p}^{iF} + \vec{P}_{Ai} - \gamma_A \vec{P}_{cm}^F - \vec{P}_r^F) \times d\vec{k}_\alpha^F \delta(\vec{k}_\alpha^F + \vec{P}_{B\alpha} - \gamma_B \vec{P}_{cm}^F + \vec{P}_r^F)\theta_\alpha^i d\vec{p}^{iI} \delta(\vec{p}^{iI} + \vec{P}_{Ai} - \gamma_A \vec{P}_{cm}^I - \vec{P}_r^I) \times d\vec{k}_\alpha^I \delta(\vec{k}_\alpha^I + \vec{P}_{B\alpha} - \gamma_B \vec{P}_{cm}^I + \vec{P}_r^I)|\vec{p}^{iI}\vec{k}_\alpha^I \Phi_{Ai}(0)\Phi_{B\alpha}(0)\rangle. \tag{6.12}$$

This expression reduces to

$$\langle \theta_\alpha^i \rangle = \int d\vec{p}^{iI} d\vec{k}_\alpha^I (\vec{p}^{iF} \vec{k}_\alpha^F |\theta_\alpha^i| \vec{p}^{iI} \vec{k}_\alpha^I) \rho_{Ai}^{(1)}(\vec{p}^{iI} - \gamma_A \vec{P}_{cm}^I - \vec{P}_r^I) \rho_{B\alpha}^{(1)}(\vec{k}_\alpha^I - \gamma_B \vec{P}_{cm}^I + \vec{P}_r^I), \tag{6.13}$$

where

$$\vec{p}^{iF} = \vec{p}^{iI} + \gamma_A(\vec{P}_{cm}^F - \vec{P}_{cm}^I) + (\vec{P}_r^F - \vec{P}_r^I), \tag{6.14a}$$

$$\vec{k}_\alpha^F = \vec{k}_\alpha^I + \gamma_B(\vec{P}_{cm}^F - \vec{P}_{cm}^I) - (\vec{P}_r^F - \vec{P}_r^I), \tag{6.14b}$$

and the densities are those defined by (6.10a), (6.10b), and their equivalent for B. The arguments in the one-body densities of the spectators indicate their role in absorbing recoil momenta in the interaction of the participating nucleon pair.

We can now write for the matrix elements of $T(2)$ the expression

$$\langle T(2)\rangle = AB \int d\vec{p}\, d\vec{k} (\vec{p}'\vec{k}'|t|\vec{p}\vec{k}) \times \rho_A^{(1)}(\vec{p} - \gamma_A \vec{P}_{cm}^I - \vec{P}_r^I) \times \rho_B^{(1)}(\vec{k} - \gamma_B \vec{P}_{cm}^I + \vec{P}_r^I), \tag{6.15}$$

where we have taken advantage of the indistinguishability of the nucleons to obtain the result in terms of one-body densities independent of the particle labels. If we look at the limiting case of $A \to 1$ and $B \to 1$, we find that (6.15) reduces exactly to the two-body t matrix for the free scattering of two nucleons.

VII. CONCLUDING REMARKS

Beginning with a Hamiltonian framework we have described the transition operator T for the interaction of two composite systems in which arbitrary multi-body potentials are permitted to occur. We have shown how few-body mean fields may be ascribed to the averaging over the many-body interactions, or alternatively, how auxiliary potentials can be introduced in the formal structure. In Appendix A we have proved an algebraic identity and a functional theorem with corollaries which facilitate the reduction of N-body operators to finite sums of arbitrary two-body, three-body, ..., N-body operators. We have shown how the application to the transition operator T for cluster-cluster interactions yields a generalized correlation expansion, the generalized spectator expansion. Our multiple scattering theory reduces in the appropriate limits to the correlation expansion of ELMT or to the spectator expansion of ST, and as well, contains as a specialization the familiar multiple scattering theory of Watson. The GSE includes the full extent of subcluster-correlations in a physically transparent style. We delineate the flexibility inherent in these operators and obtain a result which is not a perturbation series but an exact systematic decoupling of the operator T into more manageable subpieces.

The flexibility of the GSE makes the connection between the closure approximation,[7] impulse approximation,[15] fixed scatterer, etc., to the exact T operator evident, and places them in the proper perspective as leading-order terms in exact decomposition of T. Physically motivated arguments have been used to select the apposite definitions of propagators and the other operators arising in selected applications. Emphasis has been placed upon the connection of such arbitrary quantities with the Hamiltonian. The nature of the GSE casts a more fundamental aspect upon reactions which have hitherto been addressed primarily by phenomenology. We note especially the correlated cluster treatment of high energy back-angle proton scattering by Fujita and Hüfner.[12] Also, collective effects such as alpha clustering in even-even nuclei may be addressed by this framework.

In illustration of the adduced flexibility we have defined a reaction operator K, which makes provision for mean-field effects and Pauli restrictions on intermediate state scattering (of fermions). In the two-body pieces, the methodology resembles that of the Brueckner reaction matrix for nuclear structure calculations. We have also delineated cluster-cluster optical potentials through the use of projection operators. Finally, we sketched the formation of the matrix elements for two-body operators and demonstrated the embedding of these matrix elements in the local one-body densities of the interacting clusters.

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APPENDIX A.

In this appendix we prove a lemma and one theorem with two specialized corollaries, which allow us considerable flexibility in the construction of operator expansions. When applied to nucleus-nucleus operator expansions, this lemma is seen to be a generalization of one proved by Siciliano and Thaler. It reduces to their result in the particle-nucleus limit. The basic building blocks are purely algebraic expressions having no commutativity or inversion restrictions.

Lemma A: The identity expansion

Given an arbitrary set of quantities $\{\phi^{(\nu)}_{(\mu)}\}$, where $(\nu)\subset A$, and $(\mu)\subset B$, with $A>0$, $B>0$, the finite series expressed by

$$\begin{aligned}\theta = &\sum_{i=1}^{A}\sum_{\alpha=1}^{B}\phi^{i}_{\alpha} + \sum_{i<j}^{A}\sum_{\alpha=1}^{B}(\phi^{ij}_{\alpha}-\phi^{i}_{\alpha}-\phi^{j}_{\alpha}) + \sum_{i=1}^{A}\sum_{\alpha<\beta}^{B}(\phi^{i}_{\alpha\beta}-\phi^{i}_{\alpha}-\phi^{i}_{\beta})\\
&+\sum_{i<j}^{A}\sum_{\alpha<\beta}^{B}(\phi^{ij}_{\alpha\beta}-\phi^{ij}_{\alpha}-\phi^{ij}_{\beta}-\phi^{i}_{\alpha\beta}-\phi^{j}_{\alpha\beta}+\phi^{i}_{\alpha}+\phi^{j}_{\alpha}+\phi^{i}_{\beta}+\phi^{j}_{\beta})\\
&+\sum_{i<j<k}^{A}\sum_{\alpha=1}^{B}(\phi^{ijk}_{\alpha}-\phi^{ij}_{\alpha}-\phi^{ik}_{\alpha}-\phi^{jk}_{\alpha}+\phi^{i}_{\alpha}+\phi^{j}_{\alpha}+\phi^{k}_{\alpha})\\
&+\sum_{i=1}^{A}\sum_{\alpha<\beta<\gamma}^{B}(\phi^{i}_{\alpha\beta\gamma}-\phi^{i}_{\alpha\beta}-\phi^{i}_{\alpha\gamma}-\phi^{i}_{\beta\gamma}+\phi^{i}_{\alpha}+\phi^{i}_{\beta}+\phi^{i}_{\gamma})\\
&+\sum_{i<j<k}^{A}\sum_{\alpha<\beta}^{B}(\phi^{ijk}_{\alpha\beta}-\phi^{ij}_{\alpha\beta}-\phi^{ik}_{\alpha\beta}-\phi^{jk}_{\alpha\beta}-\phi^{ijk}_{\alpha}-\phi^{ijk}_{\beta}\\
&\qquad+\phi^{ij}_{\alpha}+\phi^{ik}_{\alpha}+\phi^{jk}_{\alpha}+\phi^{ij}_{\beta}+\phi^{ik}_{\beta}+\phi^{jk}_{\beta}-\phi^{i}_{\alpha}-\phi^{i}_{\beta}-\phi^{j}_{\alpha}-\phi^{j}_{\beta}-\phi^{k}_{\alpha}-\phi^{k}_{\beta})\\
&+\cdots\\
&+(\phi^{12\cdots A}_{12\cdots B}-\cdots)\end{aligned} \tag{A1}$$

is exact for all arbitrary $\{\phi^{(\nu)}_{(\mu)}\}$, provided only that

$$\theta\equiv\phi^{12\cdots A}_{12\cdots B}=\phi^{(A)}_{(B)}\,. \tag{A2}$$

Proof. The proof follows the same procedure ST used in demonstrating a more restricted version of this lemma. We rewrite (A1) as

$$\theta=\lim_{x\to1}\Big[\sum_{i,\alpha}^{A,B}\phi_\alpha^i+\sum_{i<j,\alpha}^{A,B}(\phi_\alpha^{ij}-x\phi_\alpha^i-x\phi_\alpha^j)+\sum_{i,\alpha<\beta}^{A,B}(\phi_{\alpha\beta}^i-x\phi_\alpha^i-x\phi_\beta^i)$$
$$+\sum_{\substack{i<j\\ \alpha<\beta}}^{A,B}(\phi_{\alpha\beta}^{ij}-x\phi_\alpha^{ij}-x\phi_\beta^{ij}-x\phi_{\alpha\beta}^i-x\phi_{\alpha\beta}^j+x^2\phi_\alpha^i+x^2\phi_\alpha^j+x^2\phi_\beta^i+x^2\phi_\beta^j)$$
$$+\cdots\Big]. \tag{A3}$$

In order to facilitate conciseness of expression and abbreviate the labor involved in demonstrating various properties of the proof, we will use the following summation convention:

$$\{\text{algebraic expression, containing the ordered superscripts } [(i_1,i_2,i_3,\ldots,i_j),\ j<A] \text{ and the ordered subscripts } [(\alpha_1,\alpha_2,\ldots,\alpha_\beta),\ \beta<B]\}=\sum_{i_1<i_2<\cdots<i_j}^{A}\ \sum_{\alpha_1<\alpha_2<\cdots<\alpha_\beta}^{B}[\text{algebraic expression}].$$

Thus, after performing certain summations, we obtain from (A1)

$$\theta=\lim_{x\to1}\Big\{\Big(\sum\phi_\alpha^i\Big)+\Big[\Big(\sum\phi_\alpha^{ij}\Big)-x(A-1)\Big(\sum\phi_\alpha^i\Big)\Big]+\Big[\Big(\sum\phi_{\alpha\beta}^i\Big)-x(B-1)\Big(\sum\phi_\alpha^i\Big)\Big]$$
$$+\Big[\Big(\sum\phi_{\alpha\beta}^{ij}\Big)-x(B-1)\Big(\sum\phi_\alpha^{ij}\Big)-x(A-1)\Big(\sum\phi_{\alpha\beta}^i\Big)+x^2(A-1)(B-1)\Big(\sum\phi_\alpha^i\Big)\Big]$$
$$+\Big[\Big(\sum\phi_\alpha^{ijk}\Big)-x(A-2)\Big(\sum\phi_\alpha^{ij}\Big)+x^2\frac{(A-1)(A-2)}{2}\Big(\sum\phi_\alpha^i\Big)\Big]$$
$$+\Big[\Big(\sum\phi_{\alpha\beta\gamma}^i\Big)-x(B-2)\Big(\sum\phi_{\alpha\beta}^i\Big)+x^2\frac{(B-1)(B-2)}{2}\Big(\sum\phi_\alpha^i\Big)\Big]$$
$$+\Big[\Big(\sum\phi_{\alpha\beta}^{ijk}\Big)-x(A-2)\Big(\sum\phi_{\alpha\beta}^{ij}\Big)-x(B-1)\Big(\sum\phi_\alpha^{ijk}\Big)$$
$$+x^2\frac{(A-2)(A-1)}{2}\Big(\sum\phi_{\alpha\beta}^i\Big)+x^2(B-1)(A-2)\Big(\sum\phi_\alpha^{ij}\Big)-x^3\frac{(A-1)(A-2)}{2}(B-1)\Big(\sum\phi_\alpha^i\Big)$$
$$+\cdots$$
$$+(\phi_{12\cdots B}^{12\cdots A}-x\cdots)\Big]\Big\}. \tag{A4}$$

We now regroup the terms of (A4) in the manner

$$\theta=\lim_{x\to1}\Big\{\Big(\sum\phi_\alpha^i\Big)\Big[1-(A-1)x-(B-1)x+(A-1)(B-1)x^2+\frac{(A-1)(A-2)x^2}{2}+\frac{(B-1)(B-2)x^2}{2}$$
$$-\frac{(A-1)(A-2)}{2}(B-1)x^3-\frac{(A-1)(B-1)(B-2)}{2}x^3+\frac{(A-1)(A-2)}{2}\frac{(B-1)(B-2)}{2}x^4+\cdots\Big]$$
$$+\sum\phi_\alpha^{ij}\Big[[1-(B-1)x-(A-2)x+(B-1)(A-2)x^2+\frac{(B-1)(B-2)}{2}x^2+\cdots]\Big]$$
$$+\sum\phi_{\alpha\beta}^i\Big[[1-(A-1)x-(B-2)x+(A-1)(B-2)x^2+\frac{(A-1)(A-2)}{2}x^2+\cdots]\Big]$$
$$+\sum\phi_{\alpha\beta}^{ij}\Big[[1-(A-2)x-(B-2)x+\frac{(A-2)(A-3)}{2}x^2+\cdots]\Big]$$
$$+\cdots$$
$$+(\phi_{12\cdots B}^{12\cdots A})\Big\}, \tag{A5}$$

and this becomes

$$\theta = \lim_{x\to 1}\Big[\Big(\sum \phi_\alpha^i\Big)(1-x)^{A-1}(1-x)^{B-1} + \Big(\sum \phi_\alpha^{ij}\Big)(1-x)^{A-2}(1-x)^{B-1} + \Big(\sum \phi_{\alpha\beta}^i\Big)(1-x)^{A-1}(1-x)^{B-2} + \Big(\sum \phi_{\alpha\beta}^{ij}\Big)(1-x)^{A-2}(1-x)^{B-2} + \cdots + (\phi_{12\cdots B}^{12\cdots A})\Big]. \tag{A6}$$

Taking the limit $x \to 1$, then yields

$$\theta \equiv \phi_{12\cdots B}^{12\cdots A} = \phi_{(B)}^{(A)}. \tag{A7}$$

Q.E.D. Having shown that lemma A is true, we may now demonstrate a theorem on functional forms and specify two useful corollaries.

Theorem A: The functional identity

Let X and Y have sets $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$ respectively, which satisfy lemma A. Then for Z defined by

$$Z \equiv F(X, Y), \tag{A8}$$

if

$$f_{(B)}^{(A)} \equiv F(X_{(B)}^{(A)};\ Y_{(B)}^{(A)}), \tag{A9}$$

the set $\{Z_{(\mu)}^{(\nu)}\}$ described by the otherwise arbitrary set of functionals $\{f_{(\mu)}^{(\nu)}\}$ as

$$Z_{(\mu)}^{(\nu)} \equiv f_{(\mu)}^{(\nu)}(\{X_{(\mu')}^{(\nu')}\};\ \{Y_{(\mu')}^{(\nu')}\}) \tag{A10}$$

satisfies lemma A for Z.

Proof.

(i) $X_{(B)}^{(A)} \equiv X;\ Y_{(B)}^{(A)} \equiv Y$ by lemma A,

(ii) $f_{(B)}^{(A)} = F(X, Y)$ by (A9) and (i),

(iii) $Z_{(B)}^{(A)} = F(X, Y)$ by (A10) and (ii),

(iv) $Z_{(B)}^{(A)} = Z$ by (A8) and (iii).

Therefore, the condition necessary for the set $\{Z_{(\mu)}^{(\nu)}\}$ to satisfy lemma A is met. Q.E.D. We can now specify theorem A to yield two corollaries.

Corollary I: Addition corollary

Given X and Y, having sets $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$ respectively, which satisfy lemma A, then for

$$Z \equiv X + Y, \tag{A11}$$

the set $\{Z_{(\mu)}^{(\nu)}\}$ defined by

$$Z_{(\mu)}^{(\nu)} \equiv X_{(\mu)}^{(\nu)} + Y_{(\mu)}^{(\nu)} \tag{A12}$$

also satisfies lemma A.

Corollary II: Multiplication corollary

Given X and Y, having sets $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$ respectively, satisfying lemma A, then for

$$Z = XY \tag{A13}$$

the set $\{Z_{(\mu)}^{(\nu)}\}$ defined by

$$Z_{(\mu)}^{(\nu)} \equiv X_{(\mu)}^{(\nu)} Y_{(\mu)}^{(\nu)} \tag{A14}$$

also satisfies lemma A.

It is convenient at this junction to point out that corollaries I and II are more fundamental than their derivation implies. One can show trivially for corollary I that $X+Y$ in terms of the expansion factors yields identically the terms given by the definition (A12). Furthermore, while it is not as obvious, the expansion of $Z=XY$ in terms of $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$ permits a rearrangement which yields the identical terms of definition (A14). In this regard, we must note that theorem A provides a minimal condition upon the expansion factors of Z, whereas, having defined the expansions of X and Y, we obtain a specific expansion of Z in terms of the sets $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$. This facet of corollaries I and II removes a degree of the arbitrariness inherent in such expansions. Physically, this implies that an operator expression has a "natural" set of expansion factors whose form will be constrained by the form of the operator expression. For example, $Z=XY$ has the "natural" set of $\{Z_{(\mu)}^{(\nu)} = X_{(\mu)}^{(\nu)} Y_{(\mu)}^{(\nu)}\}$, in which the arbitrariness has been restricted to that of the X and Y expansion sets.

APPENDIX B

Consider a set of m operators P_i defined in the N-body space $V(N)$. By definition the P_i are idempotent projection operators in $V(N)$, provided that the following conditions are satisfied for all P_i in the set:

$$\sum_{i=1}^{m} P_i = 1(N), \tag{B1}$$

$$P_i P_j = \delta_{ij} P_i. \tag{B2}$$

Take an operator expression (B3) defined in $V(N)$, where

$$A = B + BCA\,. \tag{B3}$$

A, B, and C are unspecified N-body operators. It is possible to generate an equivalent set of operator expressions for (B3) by using (B1). Let $m=2$, and $P_1 = P$ and $P_2 = Q$. We may now rewrite (B3) as follows:

$$A = B + BCA = B + BC(P+Q)A\,,$$

$$A = B + BCQA + BCPA\,. \tag{B4}$$

Inserting the expression (B4) for A into the second term of (B4) yields

$$A = B + BCQB + BCQBCQA + BCPA + BCQBCPA\,. \tag{B5}$$

And again using (B4) we obtain

$$A = B + BCQB + BCQBCQB + BCQBCQBCQA + BCPA + BCQBCPA + BCQBCQBCPA\,. \tag{B6a}$$

We see that (B6a) may more concisely be written as

$$A = B\left[\sum_{n=0}^{2} (CQB)^n\right] + (BCQ)^3 A + B\left[\sum_{n=0}^{2} (CQB)^n\right] CPA\,. \tag{B6b}$$

Iteration of this procedure yields

$$A = B\left[\sum_{n=0}^{\infty} (CQB)^n\right] + B\left[\sum_{n=0}^{\infty} (CQB)^n\right] CPA\,. \tag{B7}$$

We may identify a new operator D defined by the expression

$$D \equiv B\left[\sum_{n=0}^{\infty} (CQB)^n\right]. \tag{B8}$$

An alternate form for D is seen to be

$$D = B + BCQD\,. \tag{B9}$$

Having defined D we now rewrite (B7) in a much simpler form as

$$A = D + DCPA\,. \tag{B10}$$

By using the projection operators P and Q we have separated the single operator expression (B3) into two expressions (B9) and (B10) having the same content.

Using the same procedures it is possible to obtain three expressions for the set of three projectors P, Q_1, Q_2. Let $Q = Q_1 + Q_2$. Replacing Q in (B9) then yields after some manipulation

$$E = B + BDQ_1E\,,$$

$$D = E + ECQ_2D\,.$$

We can generalize this procedure to m projectors. The result is a set of m equations containing various portions of the full content of (B3). The generalized set looks like

$$D_{m-l} = D_{m-l+1} + D_{m-l+1} CP_{m-l} D_{m-l}\,, \tag{B11}$$

where $l = 0, 1, 2, \cdots (m-1)$, and $D_{m+1} = B$ and $D_1 = A$.

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