# Nucleon-nucleon scattering above the pion production threshold

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Starting from a simple nonrelativistic field theory of pions and nucleons, we derive linear integral equations coupling amplitudes of baryon number equal two systems. These equations guarantee two- and three-body unitarity above the threshold for single pion production, and include explicitly the coupling of the $NN$ to (e.g.) $N\Delta$ and $\pi d$ channels. The relationship of this approach to earlier schemes is discussed.

[NUCLEAR REACTIONS Nucleon-nucleon scattering. Derived coupled equations for $NN$, $N\Delta$, $\pi d$ with heavy-boson exchange and $NN\pi$ vertex as input.]

## I. INTRODUCTION

There has been a keen interest in the $NN\pi$ system for the past few years. Much of this attention has been focused on the resonance region where the pion is relativistic but the nucleons may still be treated nonrelativistically. (For a recent survey see Ref. 1.) A major reason for this interest is the observation that, because of absorption channels, an accurate description of pion induced reactions on nuclei must eventually go beyond standard multiple scattering theory. The $NN\pi$ system is the simplest testing ground for any such theory.

Our main concern here is that the same theory which allows one to couple the $NN$ system to $NN\pi$ also implies a model for the $NN$ interaction. In this context, we mention first the work of Afnan and Thomas.[2] These authors studied a strict three-body model for $\pi d$ scattering, and included the coupling to the "$NN$" channel by allowing a bound state in the $\pi N$ $P_{11}$ channel. Thus, while one nucleon resulting from pion absorption on the deuteron was truly elementary, the other was composite (labeled $N'$). In spite of the unsymmetrical treatment of the two nucleons, the theory reproduced the peripheral $I=1$ $NN$ phase shifts quite well. Since the driving term included only one-pion exchange (OPE) no more was to be expected. A covariant version[3] of essentially the same model, still suffering from the mentioned defect, has been worked out by Kloet *et al.*[4]

A consistent model has been proposed by Mizutani,[5] and Mizutani and Koltun.[6] They treated the $NN\pi$ vertex explicitly, and in particular showed how to avoid double counting by removing the nucleon pole term from the $\pi N$ scattering amplitudes. Expressions for any desired amplitude could be given in perturbation theory. The Mizutani-Koltun model must then be solved in what is essentially the one-pion approximation.

Using reduction techniques,[7] and assuming the $NN$ interaction to be known, Rinat derived sets of coupled integral equations for all other relevant amplitudes (e.g., $NN\to\pi d$).[8] In this context we should also mention a Tamm-Dancoff approach to the two-nucleon system with only the $NN\pi$ vertex as input.[9] Finally, we mention a class of models which couple the $NN$ channel to other baryon number two channels in a potential-like picture.[10–12] Specification of the coupling matrix (e.g., OPE for the $NN\to N\Delta$ potential) defines these models. While the work of Weber *et al.*[12] showed the practical importance of including the nonstatic aspects of pion exchange in such potentials, we stress that the static approximation[10,11] misses important contributions to unitarity. This is particularly germane to the discussion of the inelasticity in $pp$ scattering, and to the analysis of possible resonances in the region of 600–800 MeV proton laboratory energy.[13,14]

In the following, we work out the consequences of the Mizutani-Koltun model[6] (which, incidentally, is closely related to the work of Ref. 14). Contrary to the approach used in Ref. 8, we now couple the $NN$ channel to the other $NN\pi$ channels in a consistent unitary fashion. In Sec. II we derive a set of Faddeev-type equations (very similar in appearance to those of Afnan and Thomas[2]) for the coupled $NN\to NN$, $NN\to N\Delta$, and $NN\to\pi d$ amplitudes, using graph summation techniques. In Sec. III the same set of equations is derived using reduction techniques.[7,8]

In Sec. IV we show how one can formally eliminate all but the $NN$ channel from the coupled integral equations and derive an effective $NN$ potential. The latter includes higher order pion exchanges which could be represented by a one-boson-exchange potential (OBEP), together with a part due to any number of nucleon-isobar rescatterings. Our derivation thus provides a theoretical basis for the model of Weber *et al.*,[12] as well as a formalism for more complete calculations.

Finally, we address ourselves to the question of isobars in nuclei and argue that meaningful statements can only be made within a specific model.

## II. DIAGRAMMATIC TECHNIQUE

In this section we use very general arguments, based only on the topological properties of graphs, to derive integral equations for $NN$ scattering above pion production threshold. While this technique is closely related to that of Taylor,[15] we work with a much simpler system—nonrelativistic nucleons and no antinucleons. The graphs we draw are of the old-fashioned time ordered type. This has the advantage, from the point of view of exact two- and three-body $s$-channel unitarity, that we always know how many particles are present. The technique has been used previously to resolve the question of double counting in the reaction $NN \to \pi d$,[14] and to isolate the effect of absorption on $\pi d$ elastic[16] scattering, with results in complete agreement with Mizutani and Koltun[6] and Rinat.[8]

The power of the diagrammatic method is that we do not need to specify exactly what Hamiltonian is involved.[14] (Certainly the Hamiltonian chosen by Mizutani and Koltun and used in Sec. III is one special case.) We simply assume that the renormalization procedure can be carried out completely for whatever Hamiltonian is chosen, and that all renormalized graphs leading from the initial to the final state could be written down. These will all be irreducible in the sense that no explicit self-energy diagrams or vertex modifications appear in the set.

Since we always have just two nucleons present, it is sufficient to specify the number of pions present at any time. Thus the sum of the contributions from all diagrams leading through some interaction from an initial to a final $NN$ state will be denoted $\langle 0|T|0\rangle$. This is the $t$ matrix for $NN$ scattering. Next we observe that the set of diagrams making up $\langle 0|T|0\rangle$ falls into one of two mutually exclusive subsets: class ($\alpha$) contains those diagrams where there is an intermediate state with no mesons (just two nucleons) and class ($\beta$) contains those diagrams in which every intermediate state has at least one meson. The sum of all diagrams in class $\beta$ will be written $\langle 0|T|0\rangle_1$, where the subscript indicates the minimum number of mesons in any intermediate state. If we now cut the diagrams in class $\alpha$ at the *last two-nucleon-only* intermediate state we find easily

$$\langle 0|T|0\rangle = \langle 0|T|0\rangle_1 + \langle 0|T|0\rangle_1 G_0 \langle 0|T|0\rangle\,, \qquad (2.1)$$

where $G_0$ describes the free propagation of the two nucleons. In a more familiar notation this would be written

$$T_{NN} = V + V G_0 T_{NN}\,, \qquad (2.2)$$

formally illustrated in Fig. 1.

The difference in our $V$ as opposed to that usually appearing in the Lippmann-Schwinger equation is that it includes the dynamical effects of pion production. Our next job is to find an equation for $V$. To do this we again subdivide the diagrams in $\langle 0|T|0\rangle_1$ into two disjoint subsets: class ($\gamma$) contains those diagrams containing at least one intermediate state with less than two pions (and therefore just one, from the definition of $\langle 0|T|0\rangle_1$) and class ($\delta$) contains those diagrams in which every intermediate state has at least two mesons.

This leads to the equation

$$\langle 0|T|0\rangle_1 = \langle 0|T|0\rangle_2 + \langle 0|T|1\rangle_2 G_0 \langle 1|T|0\rangle_1\,. \qquad (2.3)$$

Clearly $\langle 0|T|1\rangle_2$ and $\langle 1|T|0\rangle_1$ are new objects involving, respectively, the absorption and production of one pion with an appropriate restriction on the number of pions in any intermediate state. (Once again $G_0$ represents a free particle propagator—this time for two nucleons and a pion.)

While this hierarchy of equations could be continued indefinitely, we rely on the concept of two- and three-body unitarity as a guide. (Two-pion production is very small below about $T_p^{\text{lab}} = 800$ MeV.) Because *every* intermediate state in $\langle 0|T|0\rangle_2$ and $\langle 0|T|1\rangle_2$ has two or more pions we can approximate these objects with no effect on unitarity. The simplest approximation for $\langle 0|T|1\rangle_2$ is shown in Fig. 2(a), and, consistent with earlier work, we will retain only this. Alternatively, it will be called $g_N$, the $NN\pi$ vertex function. Notice also that while we began by considering a system of pions and nucleons with only $\pi N$ interactions, we can formally introduce $\rho$, $\omega$

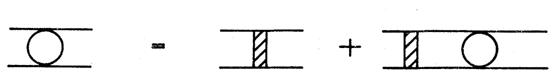

FIG. 1. Graphical representation of the $NN$ scattering equation (2.1). The shaded box represents the sum of all diagrams irreducible on two-nucleon lines.

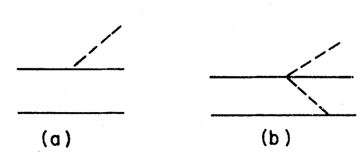


FIG. 2. The two lowest order contributions to $\langle 0|T|1\rangle_2$. Only (a) (called $g_N$) is retained in this work.

exchange and so on through $\langle 0|T|0\rangle_2$ (i.e., as the exchange of two or more correlated pions). To indicate this we shall henceforth label $\langle 0|T|0\rangle_2$ as $V_{\text{HBE}}$ (HBE = heavy-boson exchange), although its derivation was much more general than the OBE model. With these definitions, and the approximation

$$\langle 0|T|1\rangle_2 = g_N \tag{2.4}$$

(cf. Fig. 2), Eq. (2.3) can be written

$$V = V_{\text{HBE}} + g_n G_0 \langle 1|T|0\rangle_1 \,. \tag{2.5}$$

The scattering amplitude $\langle 1|T|0\rangle_1$ has been discussed in detail elsewhere.[14,16] It is what is left of the pion production amplitude $NN \to NN\pi$ after the initial state $NN$ interaction has been separated out. In fact, by making a similar separation to that used in ($\alpha$) and ($\beta$), and ($\gamma$) and ($\delta$) above, and also using the approximation (2.4), we find [cf. Eq. (F.9) of Ref. 16]

$$\langle 1|T|0\rangle_1 = g_N^* + \langle 1|T|1\rangle_1 G_0 g_N^* \,. \tag{2.6}$$

Now $\langle 1|T|1\rangle_1$ is the transition amplitude from an $NN\pi$ state to an $NN\pi$ state with at least one pion in every intermediate state. The latter condition guarantees that this amplitude has no contribution, due to what has been conventionally labeled "true pion absorption." The only exception to this would be crossed absorption contributions such as shown in Fig. 3, and these do not contribute to unitarity below the threshold for two-pion production.

It is therefore very natural to identify $\langle 1|T|1\rangle_1$ with the Faddeev amplitude for $\pi NN$ scattering using phenomenological $\pi N$ $t$ matrices and excluding pion absorption. The nucleon pole term of the $P_{11}$ interaction must be excluded. In the earlier work by Rinat[8] such purely multiple scattering amplitudes were labeled by a superscript zero. Consistent with that work and using separable $\pi N$ $t$ matrices, we find for the $3 \to 3$ amplitude

$$\langle 1|T|1\rangle_1 = \sum_\lambda g_\lambda G_\lambda g_\lambda + \sum_{\lambda\,\mu} g_\lambda G_\lambda T^{(0)}_{\lambda\mu} G_\mu g_\mu \,. \tag{2.7}$$

The sums over $\lambda$ and $\mu$ in Eq. (2.7) include all $\pi N$ and $NN$ channels interacting through the separable $t$ matrices $g_\lambda G_\lambda g_\lambda$. Using Eq. (2.7) in Eq. (2.6) leads to

$$\langle 1|T|0\rangle_1 = g_N^* + \sum_\lambda g_\lambda G_\lambda \left[ B_{\lambda N} + \left( \sum_\mu T^{(0)}_{\lambda\mu} G_\mu B_{\mu N} \right) \right] , \tag{2.8}$$

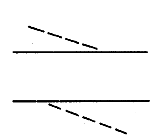

FIG. 3. Crossed absorption graph (with $n_\pi = 2$) omitted in the three-body approximation to $\langle 1|T|1\rangle_1$.

where we have defined

$$B_{\lambda N} = g_\lambda G_0 g_N^* \,, \tag{2.9}$$

as shown in Fig. 4.

Next we use Eq. (4.12a) of Rinat,[8]

$$T_{\alpha N} = \left( B_{\alpha N} + \sum_\beta T^{(0)}_{\alpha\beta} G_\beta B_{\beta N} \right) \omega_{NN} \,, \tag{2.10}$$

to establish from Eq. (2.8) that

$$\langle 1|T|0\rangle_1 = g_N^* + \sum_\lambda g_\lambda G_\lambda T_{\lambda,N} \omega_{NN}^{-1} \,. \tag{2.11}$$

Here $\omega_{NN}$ is the Möller operator corresponding to $T_{NN}$, and the sum over $\lambda$ does not include $N$, of course. We can now use Eq. (2.11) with Eq. (2.5) to obtain an equation for the $NN$ potential,

$$V = V_{\text{HBE}} + g_N G_0 g_N^* + \sum_\lambda B_{N\lambda} G_\lambda T_{\lambda N} \omega_{NN}^{-1} \,. \tag{2.12$'$}$$

The second term on the right is just the nonstatic one-pion exchange $NN$ interaction, so that we could label the first two terms on the right $V_{\text{OBE}}$

$$V_{\text{OBE}} = V_{\text{HBE}} + g_N G_0 g_N^* \,. \tag{2.12}$$

Finally, we can substitute Eq. (2.12), which we have derived for the $NN$ potential including pion production, into the original scattering equations (2.1) and (2.2):

$$T_{NN} = V_{\text{OBE}} + \sum_\lambda B_{N\lambda} G_\lambda T_{\lambda N} \omega_{NN}^{-1} + V_{\text{OBE}}\, G_0 T_{NN} + \sum_\lambda B_{N\lambda} G_\lambda T_{\lambda N} \omega_{NN}^{-1} G_0 T_{NN} \,. \tag{2.13}$$

However,

$$\omega_{NN}^{-1} = (1 + G_0 T_{NN})^{-1} \tag{2.14}$$

can be used to simplify the second and fourth terms in Eq. (2.13) to yield

$$T_{NN} = V_{\text{OBE}} + V_{\text{OBE}} G_0 T_{NN} + \sum_\lambda B_{N\lambda} G_\lambda T_{\lambda N} \,. \tag{2.15a}$$

For completeness we record the equations satisfied by $T_{\lambda N} (\lambda \neq N)$, viz. Eq. (4.8) of Rinat,

$$T_{\lambda N} = B'_{\lambda N} + B_{\lambda N} G_0 T_{NN} + \sum_\mu B_{\lambda\mu} G_\mu T_{\mu N} \,. \tag{2.15b}$$

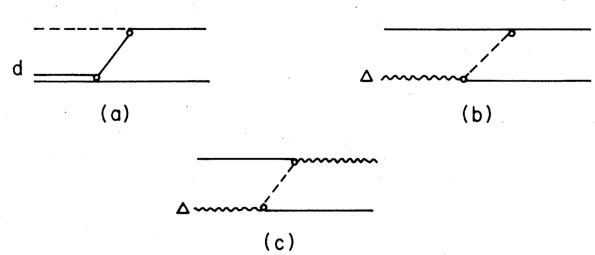


FIG. 4. Illustration of some single-particle exchange amplitudes. (a) $B_{d,N}$, (b) $B_{\Delta N}$, (c) $B_{\Delta\Delta}$. The first two appear in Eqs. (2.15) and (3.22), the latter for instance in $V$, Eq. (4.1).

Equations (2.15) constitute the end of our search. They constitute a closed set of linear integral equations coupling the reactions

$$NN \to NN\,,\quad NN \to N\Delta\,,\quad NN \to \pi d \quad (\text{or } NN \to N\beta \text{ in general})\,. \tag{2.16}$$

These equations guarantee two- and three-body unitarity and, as we shall explain in Sec. IV, should therefore provide a reliable framework for the description of $NN$ scattering above pion production threshold. So far we have ignored numerical factors arising from antisymmetrization. While these could be determined in the preceding formalism, they fall very naturally out of the reduction techniques of Sec. III, and we have included them explicitly only in that section. In concluding this section, we note the formal similarity of Eqs. (2.15) to those introduced *ad hoc* by Afnan and Thomas[2] several years ago.

## III. DERIVATION OF $T_{NN}$ USING A REDUCTION TECHNIQUE

We now proceed with an alternative derivation of the results, Eq. (2.15), exploiting a reduction technique used in Ref. 8. In order to avoid lengthy derivations we shall frequently cite results from there (I), referred to by an equation number.

Our starting point is the Mizutani-Koltun Hamiltonian[5,6]

$$H = H_0 + H'\,, \tag{3.1a}$$

where $H_0 = \sum_{\vec k} k^0 a_{\vec k}^\dagger a_{\vec k} + \sum_{\vec q} q^0 b_{\vec q}^\dagger b_{\vec q}$ is the energy of free nucleons and pions. In that model (which has no antinucleons) only nucleons undergo mass renormalization, $k^0 = (\vec k^2 + M^2)^{1/2}$ is the nucleon energy after renormalization, and $q^0 = (\vec q^2 + m_\pi^2)^{1/2}$.

Using a summation convention for notational simplicity, one writes the interaction $H'$ as

$$H' = \tfrac14 \sum_{\vec k_i \vec k_i'} \langle \vec k_1' \vec k_2' | v' | \vec k_1 \vec k_2\rangle a_{\vec k_1'}^\dagger a_{\vec k_2'}^\dagger a_{\vec k_2} a_{\vec k_1} + \sum_{\substack{\vec k \vec k' \\ \vec q \vec q'}} \langle \vec k' \vec q' | w' | \vec k \vec q\rangle a_{\vec k'}^\dagger b_{\vec q'}^\dagger a_{\vec k} b_{\vec q} + \sum_{\vec k \vec k' \vec q} [g_N^*(\vec k', \vec k \vec q)\, a_{\vec k'}^\dagger a_{\vec k} b_{\vec q} + \text{H.c.}] + H_{\text{CT}}\,; \tag{3.1b}$$

$H'$ contains an $NN\pi$ vertex part, an $NN$, and a $\pi N$ interaction. From the latter we remove those parts which can be generated by the $NN\pi$ vertex. Thus, one associates with $v' \to v_{\text{HBE}}$, the heavy-boson exchange portion of $v_{NN}$. Finally, $H_{\text{CT}}$ contains counter terms. The truncation of the basic interaction appears at first sight artificial, but is actually necessary in order to avoid overcounting.[6]

We start with an exercise which we shall frequently use and which is illustrated for elastic $\pi N$ scattering. For $\langle \vec q' \vec k' | \omega_{\pi N} | \vec q \vec k\rangle = \langle \vec q' \vec k' | \vec q \vec k \text{ in}\rangle$ one may write

$$\begin{aligned}\langle \vec q' \vec k' | \vec q \vec k \text{ in}\rangle &= \delta^{(3)}(\vec k - \vec k')\delta^{(3)}(\vec q - \vec q') + G_0(q^0 + k^0)\langle \vec q' \vec k' | t_{\pi N} | \vec q \vec k\rangle \\ &= \delta^{(3)}(\vec k - \vec k')\delta^{(3)}(\vec q - \vec q') + \langle \vec q' \vec k' | G(k^0 + q^0) J_{\vec k}^\dagger | \vec q\rangle\,.\end{aligned} \tag{3.2}$$

The first line [cf. I (2.14)] is a standard result, the second one is derived, reducing out of the scattering state $|\vec q \vec k \text{ in}\rangle$ the nucleon with momentum $\vec k$, thus [cf. I (2.11)]

$$|\vec q \vec k \text{ in}\rangle = \omega_{\pi N} |\vec q \vec k\rangle = a_{\vec k}^\dagger |\vec q\rangle + G J_{\vec k}^\dagger |\vec q\rangle\,, \tag{3.3a}$$

where as usual we denote by

$$G(z) = (z - H)^{-1} \tag{3.3b}$$

the propagator for the fully interacting system. $J_{\vec k}^\dagger$ in Eq. (3.2) is the Hermitian conjugate of the $N$ current operator [I (2.2)]

$$\begin{aligned} J_{\vec k}^\dagger(t) &= -i\dot a_{\vec k}(t)^\dagger = -[H', a_{\vec k}^\dagger(t)] \\ &= \tfrac12 \sum_{\vec k_1 \vec k_i'} \langle \vec k_1' \vec k_2' | v' | \vec k_1 \vec k\rangle a_{\vec k_1'}^\dagger a_{\vec k_2'}^\dagger a_{\vec k_1} + \sum_{\substack{\vec k' \\ \vec q \vec q'}} \langle \vec k' \vec q' | w' | \vec k \vec q\rangle a_{\vec k'}^\dagger b_{\vec q'}^\dagger b_{\vec q} + \sum_{\vec k' \vec q'} [g^*(\vec k' \vec q', \vec k) a_{\vec k'}^\dagger b_{\vec q'}^\dagger + g_N(\vec k', \vec k \vec q') a_{\vec k'}^\dagger b_{\vec q'}]\end{aligned} \tag{3.4}$$

(when no time label is mentioned, $t = 0$ is implied). We also cite the following relations, which are readily derived from the expression for $J_{\vec k}^\dagger$ above:

$$J_{\vec k_1}^\dagger |\vec q_1\rangle = w' |\vec k_1 \vec q_1\rangle + \sum_{\vec k_1'} g(\vec k_1', \vec k_1 \vec q_1) |\vec k_1'\rangle + \sum_{\vec k_1' \vec q_1'} g^*(\vec k_1' \vec q_1', \vec k_1) |\vec k_1' \vec q_1 \vec q_1'\rangle\,, \tag{3.5a}$$

$$J_{\vec k_1}^\dagger |\vec k_2\rangle = v' |\vec k_1 \vec k_2\rangle + \sum_{\vec k_1' \vec q'} g^*(\vec k_1' \vec q'; \vec k_1) |\vec k_1' \vec k_2 \vec q'\rangle\,. \tag{3.5b}$$

Now, comparing Eqs. (3.2) and (3.3) one concludes that

$$\langle \vec{q}'\vec{k}' | G(k^0+q^0) J^\dagger_{\vec{k}} | \vec{q}\rangle = G_0(k^0 q^0)\langle \vec{k}\vec{q} | t_{\pi N}(k^0+q^0) | \vec{k}'\vec{q}'\rangle \tag{3.6}$$

(where we use the convention of Refs. 7 and 8 where in current and scattering matrix elements initial and final states have different orderings) and a similar result holds for any elastic scattering amplitude.

Our next step is the evaluation in (3.2) of $GJ^\dagger_{\vec{k}}$ using

$$G = G_0 + GH'G_0\,, \tag{3.7}$$

inserting a selected set of intermediate states. Limiting ourselves in the case of elastic $\pi N$ scattering to $n_\pi \leq 2$ and using Eqs. (3.5) one finds

$$\begin{aligned}\langle \vec{k}'\vec{q}' | GJ^\dagger_{\vec{k}} | \vec{q}\rangle &= \langle \vec{k}'\vec{q}' | G \,[\, |\vec{k}''\rangle\langle \vec{k}''| + |\vec{k}''\vec{q}_1\rangle\langle \vec{k}''\vec{q}_1| + |\vec{k}''\vec{q}_1''\vec{q}_2''\rangle\langle \vec{k}''\vec{q}_1''\vec{q}_2''|\,] J^\dagger_{\vec{k}} | \vec{q}\rangle \\ &= \sum \langle \vec{k}'\vec{q}' | G | \vec{k}''\rangle g(\vec{k}'', \vec{k}\vec{q}) + \sum \langle \vec{k}'\vec{q}' | G | \vec{k}''\vec{q}_1''\rangle\langle \vec{q}_1''\vec{k}_1'' | w' | \vec{k}\vec{q}\rangle + \sum \langle \vec{k}'\vec{q}' | G | \vec{k}''\vec{q}\vec{q}_2''\rangle g^*(\vec{k}''\vec{q}_2'', \vec{k})\,.\end{aligned} \tag{3.8}$$

When substituting (3.3a) for $G$ only $GH'G_0$ is effective in the first and third term on the right-hand side of Eq. (3.8). We shall now approximate $H'$ there by the vertex part in (3.2) and then reach

$$\langle \vec{k}'\vec{q}' | GJ^\dagger_{\vec{k}} | \vec{q}\rangle = \langle \vec{k}'\vec{q}' | Gw' | \vec{k}\vec{q}\rangle + \langle \vec{k}'\vec{q}' | G | \vec{k}_1\vec{q}_1\rangle[g^*_N(\vec{k}_1\vec{q}_1\vec{k}'')G_0 g_N(\vec{k}'', \vec{k}\vec{q})] + \sum \langle \vec{k}'\vec{q}' | G | \vec{k}_1\vec{q}_1\rangle[g_N(\vec{k}_1, \vec{k}''\vec{q})G_0 g^*_N(\vec{k}''\vec{q}_1, \vec{k})]\,. \tag{3.9}$$

For the approximation just shown, Eq. (3.9) clearly defines the total (or effective) $\pi N$ interaction. Notice that the bracketed forms in the second and third term correspond to the direct and the crossed nuclear pole potentials. The former acts only in the $P_{11}$ partial wave, whereas the latter has projections in any partial wave. Both terms ideally complement $w'_{\pi N}$ to its full strength $w_{\pi N}$ (cf. Ref. 6). In practice one uses for $P_{11}$ only the pole term[3] and represents for the other partial waves [in particular for the dominant $P_{33}$ ("$\Delta$")] the sum of $w'$ and the projected crossed pole by a separable interaction $w_\Delta = g_\Delta \lambda_\Delta g_\Delta$. The corresponding $t$ matrix then reads in standard fashion

$$t_\Delta = g_\Delta G_\Delta g_\Delta\,. \tag{3.10}$$

Except for the crossed $N$ pole contribution to $t_{\pi N}$ which has $n_\pi = 2$, we shall, from now on, restrict intermediate states to $n_\pi \leq 1$.

Consider now the elastic $NN$ amplitude ($E = k_1^0 + k_2^0 = k_1'^0 + k_2'^0$)

$$\langle \vec{k}_1\vec{k}_2 | T_{NN}(E) | \vec{k}_1'\vec{k}_2'\rangle = \langle \text{out}\, \vec{k}_1'\vec{k}_2' | J^\dagger_{\vec{k}_1} | \vec{k}_2\rangle\,. \tag{3.11}$$

Next, we substitute into (3.11) Eqs. (3.5b) and the equivalent of (3.2) for $J^\dagger_{\vec{k}_1}|\vec{k}_2\rangle$ and $|\vec{k}_1'\vec{k}_2'\text{ out}\rangle$, respectively. Insertion of intermediate states with $n_\pi \leq 1$ then leads to

$$\begin{aligned}\langle \vec{k}_1\vec{k}_2 | T_{NN} | \vec{k}_1'\vec{k}_2'\rangle = \langle \vec{k}_1\vec{k}_2 | v' | [1 + G_0 T_{NN}(E)] | \vec{k}_1'\vec{k}_2'\rangle \\ + \sum_{\vec{k}_1''\vec{q}''} \langle \text{out}\, \vec{k}_1'\vec{k}_2' | a^\dagger_{\vec{q}''} | \vec{k}_1''\vec{k}_2\rangle \\ \times g^*_N(\vec{k}_1''\vec{q}'', \vec{k}_1)\,.\end{aligned} \tag{3.12}$$

We now use [cf. I (2.11)]

$$a^{\dagger\text{out}}_{\vec{k}_1'} = a^\dagger_{\vec{k}'} + i\int_0^\infty J^\dagger_{\vec{k}_1'}(t)\,dt\,, \tag{3.13}$$

and may then rewrite the second term in (3.12) as

$$\Sigma = \sum \langle \text{out}\, \vec{k}_1'\vec{k}_2' | J^\dagger_{\vec{q}''} G(E - q''^0) | \vec{k}_1''\vec{k}_2\rangle g^*_N(\vec{k}_1''\vec{q}'', \vec{k}_1)\,. \tag{3.14}$$

Next, one substitutes (3.7) for $G(E)$ and when using (3.5), one readily derives for $\Sigma = \Sigma_1 + \Sigma_2$

$$\Sigma_1 = \sum \langle \text{out}\, \vec{k}_1'\vec{k}_2' | J^\dagger_{\vec{q}''} | \vec{k}_1''\vec{k}_2\rangle G_0 g^*_N\,, \tag{3.15a}$$

$$\begin{aligned}\Sigma_2 = \sum \langle \text{out}\, \vec{k}_1'\vec{k}_2' | J^\dagger_{\vec{q}''} G(E - q''^0)\{[H', a^\dagger_{\vec{k}_1''}] \\ + a^\dagger_{\vec{k}_1''} H'\} | \vec{k}_2''\rangle G_0 g^*_N\,.\end{aligned} \tag{3.15b}$$

Using first a version of [I (2.12), (2.8)] and then Eqs. (3.13), (3.10), and (3.11), Eq. (3.15a) can be rewritten as

$$\begin{aligned}\Sigma_1 &= \sum \langle \text{out}\, \vec{k}_1'\vec{k}_2' | a^\dagger_{\vec{k}_1''} J^\dagger_{\vec{k}_2} | \vec{q}''\rangle G_0 g^*_N \\ &= v_{\text{OPE}} + v_{\text{OPE}} G_0 T_{NN}(E) + B_{N\Delta} G_\Delta T_{\Delta N}\,.\end{aligned} \tag{3.16}$$

In Eq. (3.16)

$$\begin{aligned}&\langle \vec{k}_1\vec{k}_2 | v_{\text{OPE}} | \vec{k}_1'\vec{k}_2'\rangle \\ &\quad = [g_N(\vec{k}_1\vec{q}', \vec{k}_2')G_0 g^*_N(\vec{q}\vec{k}_1', \vec{k}_2) - \text{exchange}]\end{aligned} \tag{3.17}$$

is the properly antisymmetrized one-pion exchange potential, which appeared before in (2.12), while $B_{N\Delta}$ [cf. (2.9)] is the single-pion exchange amplitude for $NN \to N\Delta$. We now turn to (3.15b) which contains a commutator or $N$ current operator [cf. Eq. (3.4)]. One then expands $\Sigma_2$ as before and obtains three terms:

$$\Sigma_2 = \tfrac{1}{2}\sum \langle \text{out}\, \vec{k}_1'\vec{k}_2' | J^\dagger_{\vec{q}''} | [\,|\vec{k}_1\vec{k}_2\rangle\langle \vec{k}_1\vec{k}_2| + |\vec{k}_1\vec{k}_2\vec{q}\rangle\langle \vec{k}_1\vec{k}_2\vec{q}|\,]\, G(E - q''^0) J^\dagger_{\vec{k}_1''} | \vec{k}_2''\rangle G_0 g^*_N + \sum \langle \text{out}\, \vec{k}_1'\vec{k}_2' | J^\dagger_{\vec{q}''} G(E - q''^0) a^\dagger_{\vec{k}_1''} | \vec{k}\vec{q}\rangle g_N\, G_0 g^*_N\,. \tag{3.18}$$

The last two terms are seen to correspond to $n_\pi = 2$ states [with the pions not connected to the same $N$ as in (3.9)] and these will be disregarded. In the remainder, one meets

$$\langle \vec{k}_1 \vec{k}_2 \,|\, G(E - q''^0) J^{\dagger}_{\vec{k}_1''} \,|\, \vec{k}_2'' \rangle$$

$$= G_0(E - q''^0) \langle \vec{k}_1'' \vec{k}_2'' \,|\, T_{NN}(E - q''^0) \,|\, \vec{k}_1 \vec{k}_2 \rangle \,, \qquad (3.19)$$

which is an identity like (3.6) for the $NN$ scattering amplitude.

Notice that $T_{NN}$ in (3.19) represents $NN$ rescattering in the presence of a spectator pion with energy $q''^0$ and should be contrasted with an $NN$ amplitude in a pion free state. The former appears to be dominated by the ${}^3S_1 - {}^3D_1$ channel[17] and we shall assume the interaction in that channel to be separable. We thus have

$$t_d = g_d \, G_d \, g_d \,, \qquad (3.20)$$

which allows (3.18) to be rewritten as

$$\Sigma_2 = B_{Nd} \, G_d \, T_{dN} \,. \qquad (3.21)$$

Finally, combining Eqs. (3.14), (3.16), and (3.21) we can write

$$T_{NN} = v_{\mathrm{OBE}} + v_{\mathrm{OBE}} G_0 T_{NN} + B_{N\Delta} \, G_\Delta \, T_{\Delta N} + 2^{1/2} B_{Nd} \, G_d \, T_{dN} \,,$$

$$T_{\Delta N} = B_{\Delta N} + B_{\Delta N} \, G_0 \, T_{NN} + B_{\Delta\Delta} G_\Delta \, T_{\Delta N} + 2^{1/2} B_{\Delta d} \, G_d \, T_{dN} \,,$$

$$2^{-1/2} T_{dN} = B_{dN} + B_{dN} \, G_0 \, T_{NN} + B_{d\Delta} G_\Delta \, T_{\Delta N} \,, \qquad (3.22)$$

where [with $v_{\mathrm{HBE}}$, $v_{\mathrm{OPE}}$ as in Eq. (3.17)]

$$v_{\mathrm{OBE}} = v_{\mathrm{HBE}} + v_{\mathrm{OPE}} \,. \qquad (2.13)$$

For the origins of the factors $2^{1/2}$ see the final paragraphs of Sec. V in I. The result (3.22) has been obtained by means of reduction techniques applied to a solution of the Hamiltonian (3.1), (3.2) and is identical to Eq. (2.15a) derived in Sec. II by graph summation techniques. Notice that in both methods states with $n_\pi \geq 2$ have been disregarded.

## IV. CONCLUSION AND DISCUSSION

The set of integral equations (2.15) and (3.22), derived by means of a consistent dynamical theory, couples the $NN$ elastic amplitude to the amplitude for $NN \to \pi d$, and all isobar ($\beta$) excitation amplitudes $NN \to N\beta$.

To the extent that the intermediate states have no more than one pion this theory is unitary, and may, as described in Refs. 3, 4, 17, and 18, readily be cast in covariant form. It therefore constitutes the desired extension of the model of Ref. 8, where $t_{NN}$ was assumed to be known. In that theory, which has been applied to $\pi d$ scattering[17] and to $\pi d \to NN$,[10] it was implied that whenever $t_{NN}$ was required the $t$ matrix corresponding to a static $NN$ potential (e.g., Reid soft core[20]) could be used. This prescription violates self-consistency as well as strict unitarity above the pion production threshold.

We have already observed the formal similarity of our equations to those of Afnan and Thomas.[2] In a theory which treats the nucleons symmetrically we have obtained an $NN$ interaction much richer than just the OPE of Ref. 2. Quantitatively this means that now both time orderings for the OPEP are included, and it will therefore have its full strength. (This defect in the OPEP of Ref. 2 was probably compensated for, rather fortunately, by fitting to a $P_{11}$ scattering length, which in view of recent analyses, seems to have been too large.[21]) As one remaining open problem we mention that a consequence of the symmetrical treatment of the two nucleons is the use of $G_0$ (the free propagator) for two-nucleon intermediate states rather than the dressed propagator $\tau_N'$ of Refs. 24 and 3. This means that we omit the contribution to the $N$-$N$ inelasticity shown in Fig. 5. Fortunately this contribution to pion production in $NN$ collisions is known to be very small below 800 MeV.[22]

We now return to Eqs. (2.15) or (3.22). Instead of $t_{NN}$ or $v_{NN}$ as in the theory of Rinat[8], their solution requires as input $v_{\mathrm{HBE}}$, the $NN$ potential due to heavy ($\rho$, $\omega$, $\eta, \ldots$) boson exchange. For it, there is no self-consistency problem since the input contains only single $N$ and $\pi$, and not $\rho$, $\omega, \ldots$, exchange amplitudes. We shall return to this point below, but first we construct the full $NN$ interaction within the framework of the model.

In the following we shall for simplicity disregard $\pi N$ channels other than $\beta = \Delta$. It is then easy to eliminate from (3.22) $T_{\Delta N}$ and $T_{dN}$ with the result

$$T_{NN} = V_{NN} + V_{NN} \, G_0 \, T_{NN} \,,$$

$$V_{NN} = v_{\mathrm{OBE}} + 2 B_{Nd} \, G_d \, B_{dN} + (B_{N\Delta} + 2 B_{Nd} \, G_d \, B_{d\Delta})$$

$$\times G_\Delta (1 - B_{\Delta\Delta} \, G_\Delta - 2 B_{\Delta d} \, G_d \, B_{d\Delta} \, G_\Delta)^{-1}$$

$$\times (B_{\Delta N} + 2 B_{\Delta d} G_d B_{dN}) \,. \qquad (4.1)$$

Neglecting for a moment contributions due to ${}^3S_1 - {}^3D_1$ ("$d$") $NN$ rescattering (i.e., setting $B_{Nd} = B_{\Delta d} = 0$), the $NN$ potential is readily seen to equal the sum of one-boson-exchange potentials and—in the model—all irredicible $NN$ amplitudes which can apparently be generated by repeated

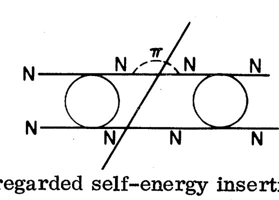


FIG. 5. Disregarded self-energy insertion on intermediate $N$ line in some contribution to $T_{NN}$ which would lead to an additional inelasticity.

single $\pi$ exchange between $N\Delta$ states [Fig. 4c]. (Notice that in including the $\pi d$ channel we addressed ourselves to the $I=1$ part of $V_{NN}$. Coupling of $N\beta$ with $I=\frac{1}{2}$ isobars will lead to a theory for the $I=0$ components of $V_{NN}$.) Likewise one may interpret additional contributions if the transition potentials $B_{Nd}$ and $B_{\Delta d}$ are included.

We now wish to compare our results. It can readily be checked that $V_{NN}$, Eq. (4.1), with $B_{\alpha d}=0$ is just the potential devised by Weber, Eisenberg, and Shuster.[12] Conversely, one can now precisely define the model which leads in a unique fashion to their $NN$ potential.

Consider next a situation wherein one wishes to couple the $NN$ channel to nucleon-isobar ($N\beta$) channels, with no more than one isobar $\beta$ per $\pi N$ partial wave. Such a model is defined if the potential coupling matrices are given. If one assumes that all coupling potentials are mediated by single-pion exchange, except $V_{NN,NN}$ which contains, in addition, heavy-boson exchange, one has

$$\begin{aligned} V_{NN,NN}(=v_{\mathrm{OBE}}) &= V^0_{NN}\,, \\ V_{N\Delta,NN} &= B^{\pi}_{N\Delta,NN} = B^{\pi}_{\Delta N}\,, \\ V_{N\Delta,N\Delta} &= B^{\pi}_{N\Delta,N\Delta} = B^{\pi}_{\Delta\Delta}\,. \end{aligned} \tag{4.2}$$

Specifically excluded are couplings shown in Fig. 6. The assumptions just expressed are those of the Jena-Kisslinger model or of the Helsinki-Mainz group.[11]

The coupled channel Schrödinger equation corresponding to the potential matrix (4.2) is of the form

$$\begin{aligned} (E-M_{NN}-V^0_{NN})u_N &= \sum_{\beta} B_{N\beta}u_\beta\,, \\ (E-M_{\beta\beta}-B_{\beta\beta})u_\beta &= V_{\beta N}u_N + \sum_{\beta'} B_{\beta\beta'}u_{\beta'}\,, \end{aligned} \tag{4.3}$$

with $M$ the mass matrix. The formal solution of (4.3) can readily be constructed, and, limiting ourselves again to $\beta=\Delta$, one finds for the effective $NN$ potential

$$V_{NN}=v_{\mathrm{OBE}}+B_{N\Delta}\,G^0_\Delta\,(1-B_{\Delta\Delta}\,G^0_\Delta)^{-1}B_{\Delta N}\,. \tag{4.4}$$

This potential $V_{NN}$ is obviously close to our solution (4.1) and differs in the propagator $(G^0_\Delta)^{-1} = s-M_{\Delta\Delta}{}^2+\frac{1}{2}iM_\Delta\Gamma_\Delta$ as opposed to our propagator [cf. Eq. (3.10)]

$$G_\Delta(s)=s-(M^0_{\Delta\Delta})^2+I_\Delta(s)\,,$$

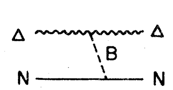


FIG. 6. Example of a $\Delta N$-$\Delta N$ coupling which when included in (4.2) would lead to contributions absent in (4.1).

$$I_\Delta(s)=\frac{1}{16\pi^3}\int\frac{E_N(p)+E_\pi(p)}{E_N(p)E_\pi(p)}\,[g_\Delta(p)]^2G_0(s_+,p)d\vec{p}\,. \tag{4.5}$$

$G^0_\Delta$ describes $\pi N$ propagation only on the mass shell, while $G_\Delta$ does the same for an interacting $\pi N$ pair also off the mass shell (in the restricted sense of a separable interaction).

An important remark is in order here. It has been shown by Aaron, Amado, and Young[3] that integral equations for coupled amplitudes of the form (3.22) embody a unitary theory (in the restricted sense used above, and below the threshold for the production of the lowest mass heavy boson) provided the form factors $g$ which build the driving terms $B_{\alpha\beta}=\langle g_\alpha|G_0|g_\beta\rangle$ are the same as those which describe the dressing $I_\Delta(s)$ of the interacting pair in Eq. (4.5). The model discussed here clearly provides a basis for the cited coupled-channel theories and points the way to a required unitarized version.

It is not possible to overemphasize the importance of unitarity if one wants to calculate the inelasticity in $NN$ scattering in a coupled-channels theory. In particular, using a static approximation for $G_0$ in the driving terms (potentials) $B_{\Delta,\Delta} \equiv \langle g_\Delta|G_0|g_\Delta\rangle$ means that contributions to $NN$ inelasticity of the type shown in Fig. 7(a) (See Ref. 3) are omitted completely. These are formally of the same order as those shown in Fig. 7(b) which are approximately included by giving the $\Delta$ a complex mass. We are forced to conclude (as is also implicit in Ref. 12) that static coupled-channels treatments may significantly underestimate the inelasticity in $NN$ scattering.

An obvious shortcoming of our model is the truncation of intermediate states, limiting those to contain at most one pion. It would be desirable indeed to include two-pion contributions both as

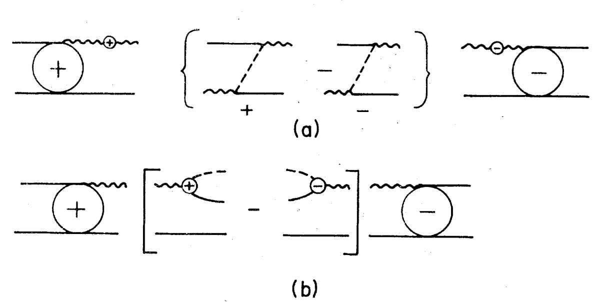


FIG. 7. (a) A three-body contribution to the imaginary part of the $NN$ scattering amplitude (i.e., $N$-$N$ inelasticity), which is absent in the static model (e.g., Refs. 10, 11, and 13). (b) That part of the $N$-$N$ inelasticity which is included (approximately) in the usual static models with a complex mass for the $\Delta$.

background (crossed pions, etc.), $\rho, \ldots$ contributions, and the like. The ensuing complications are not only of a technical nature. These relate to questions of how a $\Delta$ couples to $N\rho$ if the $\Delta$ is not elementary, but emerges as assumed [cf. Eq. (4.5)] from a model $\pi N$ interaction. Repeated use has recently been made of $NN\rho$ and $\Delta N\rho$ couplings in the description of the $\pi d \to NN$ process[24,25] and of the $\pi$-nucleus optical potential.[26] In these theories, one *formally* incorporates $\rho$ couplings, postulating the Born terms $B_{N\Delta}$, $B_{\Delta\Delta}$ to contain $\rho$ as well as $\pi$ exchange, i.e.,[26,27]

$$B_{N\Delta} \to B^{\pi}_{N\Delta} + B^{\rho}_{N\Delta}, \; B_{\Delta\Delta} \to B^{\pi}_{\Delta\Delta} + B^{\rho}_{\Delta\Delta}\,. \tag{4.6}$$

We prefer here a well-defined model rather than a possibly richer one, for which the question of consistency appears related to the concept of an elementary field.

In our final remark we wish to address the question of the presence of isobars in nuclei in general,[28] and in the deuteron in particular.[29,30] Can one, from an experiment which ideally measures a $d\Delta\Delta$ coupling (width),[30] relate that strength to a probability amplitude for the deuteron to contain those isobars?

By way of illustration we consider neutron spectroscopic factors, for instance as extracted from a $(d,p)$ reaction.[31] Even if the tool for analysis, the distorted wave first Born approximation, were perfect, it would still require a *model* for the initial state and the final one after neutron capture to extract a width. The various nuclear models (e.g., the shell model) chosen for analysis usually allocate a simple, yet nearly complete set of configurations to which initial and final states belong. Within that framework, one may then formulate sumrules (i.e., tests of unitarity) which have to be respected, and which ultimately allow extracted widths to be interpreted as probability amplitudes.

The example discussed emphasizes that the interpretation of, say, measured $d\Delta\Delta$ couplings, as probability amplitudes requires a model. The one presented above, which is based on the Hamiltonian (3.1) with given heavy-boson potentials and an elementary $NN\pi$ vertex, satisfies demands on consistency and unitarity within a given space.

Yet, regarding the question of the $\Delta\Delta$ content of the deuteron, our model has no answer because the necessary coupling (cf. Fig. 7) is absent. However, there is meaning to the question of how deuteron wave function spreads over $NN$ and $N\beta$ states. From the discussion it should be clear that the answer is dependent on, say, the one-boson content of $V_{\rm HBE}$ as are the neutron spectroscopic factors on configuration assignment and chosen single particle potentials.

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