Orientations of the recoil nucleus in muon capture

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(Received 7 March 1979)

General expressions are derived for all possible even and odd order tensor orientations of recoil nuclei, recoiling into the forward (or backward) hemisphere in the reaction $\mu^- + A(J_i = 0) \rightarrow B(J_f \geq 1) + \nu_\mu$. Numerical results are presented for the case of muon capture by ^{16}O.

[NUCLEAR REACTIONS Muon capture, recoil nucleus, orientations, all orders.]

It was first pointed out in Ref. 1 that the polarization of the recoil nucleus in muon capture is insensitive to the nuclear structure and hence capable of yielding a better estimate of the coupling constants of the fundamental muon capture interaction. This view was subsequently emphasized by many others.[2-6] It is the purpose of this article to derive general expressions for all possible orientations of the final nucleus in the following muon capture process:

$$\mu^- + A(J_i = 0) \rightarrow B(J_f \geqslant 1) + \nu_\mu . \tag{1}$$

As in Refs. 2 and 3, it is assumed that the orientation of the nuclei recoiling into one of the hemispheres alone will be preserved, while the orientation of the nuclei recoiling into the other hemisphere will be completely destroyed.

The density matrix ρ_f of the final nucleus completely specifies its spin orientation which can be conveniently represented[7] by a set of parameters $\langle T_k^{m_k}\rangle$ defined by

$$\langle T_k^{m_k}\rangle = \mathrm{Tr}(T_k^{m_k}\rho_f)/\mathrm{Tr}\rho_f , \tag{2}$$

where $T_k^{m_k}$ denotes a spherical tensor operator of rank k and the matrix element of $T_k^{m_k}$ in the spin space of the final nucleus is given by[7]

$$\langle J_f M_f' | T_k^{m_k} | J_f M_f\rangle = (2k+1)^{1/2} C(J_f k J_f, M_f m_k M_f'). \tag{3}$$

For further details and notation, the reader is referred to Ref. 3 (hereafter referred to as SPD). Our purpose is to evaluate these orientation parameters $\langle T_k^{m_k}\rangle$ which will naturally depend on the direction of the recoil nucleus. If a complete integration is performed over the entire sphere, all higher order tensor orientations with $k \geqslant 2$ will vanish and only $\langle T_1^{m_k}\rangle$ will be nonvanishing. So it is to be emphasized here that to study higher order tensor orientations, one should know either the direction of the recoil nucleus or the hemisphere into which the nucleus recoils retaining its spin orientation. The latter design is incorporated in the Louvain-Saclay experimental proposal[8,9]; hence, it will be suitable for studying the higher order tensor orientations.

Let us choose a frame of reference for evaluating the tensor orientation parameters. The normal choice for studying the reaction (1) is the rest frame of the initial state with the z axis along the direction of muon polarization $\vec{P}_\mu$. This corresponds to the c.m. system of the final state with the recoil momentum $\vec{p} = -\vec{\nu}$. As pointed out by Bernabeu[4] the final state can be most conveniently described in the helicity basis by choosing a rotating frame of reference with the z' axis along the direction of the nuclear recoil momentum $\vec{p}$. The latter coordinate system (hereafter referred to as the helicity frame) is obtained from the former (hereafter referred to as the rest frame) by rotation through the Euler angles $(\phi, \theta, 0)$.

The rest frame is the fixed physical frame of reference and the orientation parameters have to be evaluated in it. Using Eq. (2), we can do so provided we find the density matrix of the final nucleus in the rest frame. The density matrix of the final nucleus in the helicity frame has a simple structure because of the fixed helicity of the neutrino. It has only four nonvanishing elements and they are given in Eqs. (17)–(20) of SPD. Given the density matrix ρ_f' in the helicity frame, it is a simple task to determine the density matrix ρ_f in the rest frame:

$$\rho_f = D^* \rho_f' D^T , \tag{4}$$

where D is the rotation matrix $D^{J_f}(\phi, \theta, 0)$ corresponding to the nuclear spin J_f of the final nucleus, D^* its complex conjugate, and D^T the transpose of D. With the density matrix so constructed, it is a straightforward procedure to evaluate the orientation parameters using Eqs. (2) and (3).

The method described above suffers from a serious drawback. The density matrix ρ_f has to be constructed every time a final nuclear state is chosen since it involves the rotation matrix $D^{J_f}(\phi, \theta, 0)$. An alternative procedure outlined in SPD is preferred since it is more elegant and yields the orientation parameters $\langle T_k^0\rangle$ in terms of the other observables such as the longitudinal polarization P_L and the average polarization P_{av}. The results are applicable for all nuclear states with different J_f values, and the need for repetitive calculation for each particular case is thereby avoided.

It may be observed that $\mathrm{Tr}\rho_f$ is a scalar and hence invariant under rotation. $\mathrm{Tr}(T_k^{m_k}\rho_f)$ or alternatively the orientation parameter $\langle T_k^{m_k}\rangle$ is a tensor of rank k and hence transforms as a tensor of rank k under rotation. Since ρ_f has a simple structure in the helicity frame irrespective of the final nuclear spin, $\mathrm{Tr}(T_k^{m_k}\rho_f)$ is evaluated first in the helicity frame and then transformed to the rest frame:

$$[\mathrm{Tr}(T_k^{m_k}\rho_f)]_{\mathrm{rest}} = \sum_m D^{k\,*}_{m_k, m}(\phi, \theta, 0) \times [\mathrm{Tr}(T_k^{m}\rho_f)]_{\mathrm{helicity}}. \tag{5}$$

In SPD, the second order tensor orientation integrated over the forward hemisphere $\langle T_2^0\rangle_{\mathrm{fhs}}$ has been evaluated; below, we derive general expressions for orientations of any order.

For evaluating the orientations of the recoil nucleus, it is convenient to consider separately the odd order and the even order orientations. Before that, some general observations may be made. The matrix elements of T_k involve only the Clebsch-Gordan coefficients and they are independent of the angles. The only quantities that depend on the angles are the rotation matrices D^k and the density matrix ρ_f'. The rotation matrices are defined in terms of the unnormalized associated Legendre polynomials $P_{km}(\theta)$ given in Table I of Jacob and Wick,[10]

$$D^{k*}_{0,m}(\phi, \theta, 0) = d^k_{0,m}(\theta) = (-1)^m\left(\frac{4\pi}{2k+1}\right)^{1/2} P_{km}(\theta). \tag{6}$$

First we shall consider the odd order orientations ($k=2n+1$, n integer). The following results are obtained:

$$C(J_f\, 2n+1\, J_f, 000) = 0, \tag{7}$$

$$\int_0^{\pi/2} d^{2n+1}_{0,1}(\theta)\sin^2\theta\, d\theta = \sqrt{2}\,\delta_{n,0}/3, \tag{8}$$

$$\int_0^{\pi/2} d^{2n+1}_{0,0}(\theta)\cos\theta\sin\theta\, d\theta = \delta_{n,0}/3, \tag{9}$$

$$\int_0^{\pi/2} d^{2n+1}_{0,0}(\theta)\sin\theta\, d\theta = \frac{(-1)^n 2n!}{2^{2n+1}(n+1)!\,n!}. \tag{10}$$

Equations (7)–(10) are obtained using the standard properties[12-14] of the unnormalized associated Legendre polynomials $P_{km}(\theta)$.

The orientation $\langle T^0_{2n+1}\rangle_{\mathrm{fhs}}$ is obtained from Eq. (5) after performing the integration over the forward hemisphere (fhs),

$$\langle T^0_{2n+1}\rangle_{\mathrm{fhs}} = \frac{(-1)^{n+1}(4n+3)^{1/2}(2n)!}{2^{2n+2}(n+1)!\,n!} C(J_f 2n+1 J_f, -1\,0\,-1)P_L + \left(\frac{3}{4\eta}\right)^{1/2} P_H P_\mu \delta_{n,0}, \tag{11}$$

with $\eta = J_f(J_f+1)$. The quantities P_L and P_H are, respectively, the longitudinal polarization[11] of the recoil nucleus obtained with unpolarized muons and the average polarization obtained with completely polarized muons. In deriving Eq. (11), the following relations have been used:

$$P_L = -\frac{|M_{-1/2}|^2}{\sum_\lambda |M_\lambda|^2}, \tag{12}$$

$$\int_0^\pi \int_0^{2\pi} \{\mathrm{Tr}\rho_f\}\sin\theta\, d\theta\, d\phi = \sum_\lambda |M_\lambda|^2, \tag{13}$$

where $M_{1/2}$ and $M_{-1/2}$ are the helicity amplitudes given in Ref. 4.

Now we shall turn to the evaluation of the even order orientations ($k=2n, n$ integer ≥ 1). Some useful relations are

$$C(J_f\, 2nJ_f, -1\,1\,0) = -C(J_f\, 2nJ_f, 0\,-1\,-1) = -\left[\frac{n(2n+1)}{2\eta}\right]^{1/2} C(J_f\, 2nJ_f, 0\,0\,0), \tag{14}$$

$$C(J_f\, 2nJ_f, -1\,0\,-1) = \left[1 - \frac{n(2n+1)}{\eta}\right] \times C(J_f\, 2nJ_f, 0\,0\,0), \tag{15}$$

$$\int_0^{\pi/2} d^{2n}_{0,0}(\theta)\sin\theta\, d\theta = 0, \tag{16}$$

$$\int_0^{\pi/2} d^{2n}_{0,0}(\theta)\cos\theta\sin\theta\, d\theta = A, \tag{17}$$

$$\int_0^{\pi/2} d^{2n}_{0,1}(\theta)\sin^2\theta\, d\theta = A[2n(2n+1)]^{1/2}, \tag{18}$$

$$\int_0^{\pi/2} d^{2n}_{0,-1}(\theta)\sin^2\theta\, d\theta = -A[2n(2n+1)]^{1/2}, \tag{19}$$

with

$$A = (-1)^{n+1}\frac{(2n-2)!}{2^{2n}(n-1)!\,(n+1)!}. \tag{20}$$

TABLE I. Variation of B and C in units of 10^{-4} for $^{16}N(2^-)$ with the g_P/g_A ratio. $B=-\langle T_3^0\rangle_{\rm fhs} = \langle T_3^0\rangle_{\rm bhs}$, $C=-\langle T_4^0\rangle_{\rm fhs}/P_\mu = \langle T_4^0\rangle_{\rm bhs}/P_\mu$.

	model I (IPM)		model II (GV)		model III (EF)		model IV (Migdal)	
g_P/g_A	B	C	B	C	B	C	B	C
30.0	801	−281	807	−267	810	−261	814	−253
27.5	837	−192	841	−178	844	−170	848	−158
20.0	884	136	883	152	883	154	882	180
12.5	823	462	818	474	820	470	808	500
7.5	739	626	733	634	737	628	720	655
0.0	594	770	589	773	596	769	573	784
−10.0	425	822	422	822	430	822	408	823
−20.0	305	796	303	795	310	799	292	791
−30.0	224	746	223	745	229	750	215	739

Equations (14) and (15) are obtained from Eqs. (3.27) and (3.30) of Rose.[12] Equations (16)–(19) are derived by using the standard properties[13,14] of the unnormalized associated Legendre polynomials $P_{k,m}(\theta)$. An expression for the even order tensor orientation after integration over the forward hemisphere is obtained as

$$\langle T^0_{2n}\rangle_{\rm fhs} = [(4n+1)^{1/2}A/2\eta]\, C(J_f\, 2n J_f,\, 0\,0\,0) \times[\eta+2\eta P_L+3n(2n+1)P_H]P_\mu, \qquad (21)$$

where $1\leq n\leq J_f$. It can be easily seen that Eq. (21) reduces to Eq. (34) of SPD when $n=1$. Also,

$$\langle T^0_k\rangle_{\rm fhs}+\langle T^0_k\rangle_{\rm bhs}=0, \quad k\geq 2. \qquad (22)$$

The relations (11) and (21) are independent of nuclear structure and the dynamics of the muon capture interaction. Only the principle of rotational invariance and the negative helicity of the neutrino have been used in obtaining them.

For the sake of illustration, we have calculated $\langle T^0_3\rangle_{\rm fhs}$ and $\langle T^0_4\rangle_{\rm fhs}$ for the reaction

$$\mu^- + {}^{16}O(0^+) \to {}^{16}N(2^-) + \nu_\mu,$$

and the results are presented in Table I. The values of P_L and P_H calculated earlier[3] using the four different nuclear models—independent particle model (IPM), Gillet and Vinh Mau wave functions (GV), Elliot and Flowers (EF) wave functions, and Migdal wave functions—have been used in the present study. In the calculation of P_L and P_H, the Fujii-Primakoff effective Hamiltonian,[1,15,16] which includes the momentum dependent terms of order $1/M$, has been used. Also, the complete set of nuclear tensor operators has been taken into account. The values of the coupling constants used for the muon capture interaction are exactly those given in Refs. 2, 3, and 17, and a value $b=1.76$ fm has been chosen for the oscillator length parameter for ^{16}O.

The following salient features are observed. Orientations are almost insensitive to nuclear models but sensitive to the g_P/g_A ratio. Even order tensor orientations exist only if the muons are polarized, whereas odd order tensor orientations exist irrespective of muon polarization.

The recent experimental value[18] of

$$g_P/g_A=7.1\pm2.7$$

(assuming $g_T=0$) yields

$$\langle T^0_3\rangle_{\rm bhs}=-\langle T^0_3\rangle_{\rm fhs}=0.071181^{+0.005\,08}_{-0.005\,30},$$

$$\langle T^0_4\rangle_{\rm bhs}=-\langle T^0_4\rangle_{\rm fhs}=(0.06644^{+0.005\,73}_{-0.007\,37})P_\mu.$$

In conclusion, we wish to emphasize the usefulness of the helicity formalism in arriving at interesting relations between the different observables. These relations are independent of nuclear structure, the coupling constants, and the dynamics of the muon capture interaction. The determinations of $\langle T^0_1\rangle_{\rm fhs}$ and $\langle T^0_2\rangle_{\rm fhs}$ have already been done.[2,3] The present study aims at generalization and thus completes the evaluation of all possible tensor orientations of recoil nuclei in muon capture.

One of us (V.D.) would like to thank Professor Bruce H.J. McKellar for warm hospitality at the University of Melbourne.

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