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Correction to the Nuclear-Matter Calculations of Second Order in v/c*

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A calculation of the correction, second order in v/c, to the binding energy of nuclear matter in the first-order Hartree-Fock approximation yields a result of +0.5 MeV per particle at $k_F = 1.36$. The correction becomes large very quickly as the Fermi momentum increases. These results are based on the nonlocal separable potential of Tabakin.

1. INTRODUCTION

Most of the calculations in nuclear theory are performed using a phenomenological N-N interaction, chosen to give agreement with the experimental aspects of the two-body problem. These calculations require a relativistic correction, since the potential used to represent this interaction is, in general, not a relativistic covariant quantity. Particularly in a tightly bound many-body system, where nucleons can have large kinetic energies and can approach nearer to each other, these corrections are expected to be important.

Though it is very difficult to estimate the relativistic corrections accurately, these are expected to be very small. In this paper, the investigation of these corrections is presented with the assumptions: the energy involved is relatively low so that the expansion in powers of p/m is still valid, and the number of particles is conserved. Under such circumstances, one can calculate first-order relativistic corrections by the natural extension of the concept of a potential in the nonrelativistic Schrödinger equation. For this purpose, the analogy with the force between charged particles suggests that an investigation of the Lorentz properties of interaction must be carried out. If it transforms like a part of a four-dimensional vector, the rest of this vector must be included in some such covariant manner just as ϕ and $e\vec{A}$ are replaced by $\vec{p} \to \vec{p} - e\vec{A}$ and $E \to E - e\phi$ in the electromagnetic interaction. If it is an *invariant* with respect to Lorentz transformations, it can be included as a part of the rest energy.

In the literature, it is the second approach which has been discussed extensively.[1-4] Here one identifies the Hilbert space of a relativistic system as a representation space of the inhomogeneous Lorentz group (IHLG); then the problem of finding a relativistic theory is equivalent to a search for a set of Hermitian operators satisfying the well-known commutation relations (C.R.) for IHLG:

$$[P_i, P_j] = 0, \quad [P_i, H] = 0, \quad [\vec{J}, H] = 0,$$

$$\begin{aligned} &[J_i, J_j] = +i\epsilon_{ijk}J_k, \quad [J_i, P_j] = i\epsilon_{ijk}P_k, \\ &[K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [H, K_j] = -iP_j, \\ &[J_i, K_j] = i\epsilon_{ijk}K_k, \quad [P_i, K_j] = i\delta_{ij}H, \end{aligned} \tag{1.1}$$

where $i, j, k = 1, 2, 3$, and H, $\vec{P}$, $\vec{J}$, and $\vec{K}$ are infinitesimal generators of time translation, space translation, space rotation, and pure Lorentz transformation, respectively. The infinitesimal element,

$$1 - i\delta\vec{\phi}\cdot\vec{J} - i\delta\vec{V}\cdot\vec{K} - i\delta\vec{x}\cdot\vec{P} - i\delta t\cdot H,$$

operating on a state vector corresponding to a given physical system at a time t gives a new state vector corresponding to the same physical system rotated by an angle $\delta\vec{\phi}$, given a velocity $\delta\vec{V}$ and displacement $\delta\vec{x}$ at a time $t + \delta t$. One constructs these operators at a common time, the *instant form of dynamics*,[5] by considering coordinates, momenta, and spins of individual particles as the basic variables. The theory should also incorporate the following requirements[6]:

(1). The interaction of two particles should be the same when they are alone or when other particles are present at a large distance.

(2). Since a particle system may break up into two or more noninteracting clusters, a dynamical description of the entire system must also contain the correct Lorentz-invariant description of each of the clusters. Coester[4] constructed such a two-particle interaction in the presence of a distant third particle incorporating the above-mentioned requirements.

In this paper, as our aim is to estimate the relativistic corrections to B.E./A of nuclear matter, we shall confine ourselves to the lowest order beyond the nonrelativistic case. In other words, the correction calculated by Foldy[2] and Shirokov[3] is the most suitable for our purpose. This approach has a further advantage that one can easily identify the phenomenological potential and lump together arbitrary functions of the internal variables resulting from intergration. The remainder is a correction of the next order to this phenomenological potential. For the sake of continuity we shall present in Sec. 2, briefly, the calculation of the correction as given in Refs. 2 and 3. The application of these corrections to nuclear-matter calculations is presented in Sec. 3. The last section contains results and discussion.

2. $(v/c)^2$ CORRECTION TO PHENOMENOLOGICAL N-N INTERACTION

For a single particle characterized by mass m_1, spin $\vec{s}_1$, momentum $\vec{p}_1$, and position $\vec{r}_1$, the realization of the algebra of the IHLG (1.1) is achieved when the generators are identified with the Hermitian operators

$$\begin{aligned} &H \equiv H_1 = (m_1^2 + \vec{p}_1^{\,2})^{1/2} = E_1, \quad \vec{P} = \vec{p}_1 , \\ &\vec{J} \equiv \vec{J}_1 = \vec{r}_1 \times \vec{p}_1 + \vec{s}_1 = -i\vec{p}_1 \times \frac{\partial}{\partial \vec{p}_1} + \vec{s}_1 , \\ &\vec{K} \equiv \vec{K}_1 = i p_{01} \frac{\partial}{\partial \vec{p}_1} - \frac{\vec{s}_1 \times \vec{p}_1}{p_{01} + m_1} , \end{aligned} \tag{2.1}$$

where $p_{01} = E_1$. The generators for two noninteracting relativistic particles 1 and 2 are given by the sum of the individual generators of the form (2.1) in the direct-product Hilbert space. Thus, using the subscript 0 to denote operators refering to a system of free particles, we have

$$H_0 = \sum_{i=1}^{2} H_i , \quad \vec{P}_0 = \sum_{i=1}^{2} \vec{p}_i , \quad \vec{J}_0 = \sum_{i=1}^{2} \vec{J}_1 ,$$

and

$$\vec{K}_0 = \sum_{i=1}^{2} \vec{K}_i . \tag{2.2}$$

To introduce an interaction, one lets $H_0 \to H$ with

$$H = H_0 + V , \tag{2.3}$$

where V is a Hermitian operator. To maintain the relativistic covariance, one must supplant H with the operators $\vec{P}$, $\vec{J}$, and $\vec{K}$ such that the C.R. of the IHLG given in (1.1) are satisfied. If, as in the instant form of dynamics, one also demands $\vec{P} = \vec{P}_0$ and $\vec{J} = \vec{J}_0$; the C.R. (1.1) leads to

$$[V, \vec{P}] = 0 , \tag{2.4a}$$

$$[V, \vec{J}] = 0 , \tag{2.4b}$$

expressing the translational and the rotational invariance of V. In other words, if these commutation relations are valid, then parts of the C.R. in (1.1) involving H, $\vec{P}$, $\vec{J}$ are automatically satisfied, and only those parts which involve $\vec{K}$ need further attention. However, in view of the last commutation relation in (1.1), $\vec{K}$ cannot be identified with $\vec{K}_0$, but should be modified to $\vec{K} = \vec{K}_0 + \vec{U}$, where $\vec{U}$ is a vector function of the dynamical variables. The problem of the relativistic extension now reduces to a search for V and $\vec{U}$ such that C.R. (1.1) are satisfied. To solve the problem up to the order $(v/c)^2$, we begin with the expansibility assumption:

$$\begin{aligned} &H = M + H^G + H^r + \cdots , \\ &\vec{K} = \vec{K}^G + \vec{K}^r + \cdots , \end{aligned} \tag{2.5}$$

with $M = m_1 + m_2$. In lowest order, the Lorentz group reduces to the Galilean group, the generators of which satisfy the C.R. as given in (1.1) except

$$\begin{aligned} &[K_i^G, K_j^G] = 0 \\ &[K_j^G, P_i] = i\delta_{ij} M . \end{aligned} \tag{2.6}$$

Because of these commutation relations, Foldy[2] has shown that one can take, without loss of generality, $\vec{K}^G$ to have the same form as in the case of free particles, even though an interaction is present. Thus one has

$$\begin{aligned} &H^G = (1/m)(\tfrac{1}{4}\vec{P}^2 + \vec{p}^{\,2}) + V_G , \\ &\vec{K}^G = 2im\, \partial/\partial \vec{P} , \end{aligned} \tag{2.7}$$

where we have taken the equal-mass case $m_1 = m_2 = m$, $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$.

$$[V_G, \vec{K}^G] = 0 \tag{2.8}$$

and is independent of the total momentum of the system. On substituting (2.5) in C.R. (1.1) and using (2.6), (2.7), and (2.8) one gets in the next order:

(a) $[\vec{P}, H^r] = [\vec{J}, H^r] = 0$,

(b) $[J_i, K_j^r] = i\epsilon_{ijk} K_k^r$,

(c) $[K^G, H^r] = [H^G, \vec{K}^r]$,

(d) $[K_i^G, K_j^r] + [K_i^r, K_j^G] = -i\epsilon_{ijk} J_k$,

(e) $[P_i, K_j^r] = i\delta_{ij} H^G$.

The equation (d) can be satisfied by taking the form of $\vec{K}^G + \vec{K}^r$ as that of a free particle.[3]

$$\vec{K}^r_{\text{free}} = \frac{i}{m}\left[(\vec{p}^{\,2} + \tfrac{1}{4}\vec{P}^2) \frac{\partial}{\partial \vec{P}} + \frac{\vec{P}\cdot\vec{p}}{2} \frac{\partial}{\partial \vec{p}} + \frac{\vec{P}}{2} + \frac{i}{4}(\vec{\sigma}^1 - \vec{\sigma}^2) \times \vec{p} + \frac{i}{8}(\vec{\sigma}^1 + \vec{\sigma}^2) \times \vec{P} \right],$$

where $\vec{\sigma}^i$ is the Pauli spin matrix for particle i. But to satisfy expression (e), one should modify $\vec{K}^r_{\text{free}}$ to

$$\vec{K}^r = \vec{K}^r_{\text{free}} + iV_G\, \partial/\partial \vec{P} . \tag{2.9}$$

The vector nature of $\vec{K}^r$ implies the validity of (b); for (2.9).

To determine H^r, we note that Eq. (a) restricts

H^r to only those forms which are translationally and rotationally invariant, and conservation of total c.m. momentum and total angular momentum is assured. From Eq. (c) one gets

$$-2im\frac{\partial}{\partial\vec{P}}H^r(\vec{P},\vec{p},\vec{p}\,')=\langle\vec{P},\vec{p}|[H^G\vec{K}^r-\vec{K}^rH^G]|\vec{P},\vec{p}\,'\rangle\,. \tag{2.10}$$

Since H^r, in part, should contain the kinetic energy terms corresponding to those of the free particles; we write

$$H^r=\frac{-1}{8m^3}p_1{}^4-\frac{1}{8m^3}p_2{}^4+V_r\,. \tag{2.11}$$

Using Eqs. (2.11), (2.7), and (2.9), we get from (2.10)

$$\frac{\partial V_r(\vec{P},\vec{p},\vec{p}\,')}{\partial\vec{P}}=\frac{-1}{m^2}\left\{\frac{\vec{P}}{2}+\frac{1}{4}\left[(\vec{P}\cdot\vec{p})\frac{\partial}{\partial\vec{p}}+(\vec{P}\cdot\vec{p}\,')\frac{\partial}{\partial\vec{p}\,'}\right]\right\}V_0(\vec{p},\vec{p}\,')+\frac{1}{8m^2}\{-V_0(\vec{p},\vec{p}\,')[i(\vec{\sigma}^1-\vec{\sigma}^2)$$
$$\times\vec{p}\,'+\tfrac{1}{2}i(\vec{\sigma}^1+\vec{\sigma}^2)\times\vec{P}]+[i(\vec{\sigma}^1-\vec{\sigma}^2)\times\vec{p}+\tfrac{1}{2}i(\vec{\sigma}^1+\vec{\sigma}^2)\times\vec{P}]V_0(\vec{p},\vec{p}\,')\}\,. \tag{2.12}$$

To solve (2.12), $V_r(\vec{P},\vec{p},\vec{p}\,')$ can be split into two parts $V_r=V_{r1}+V_{r2}$, where V_{r1} does not depend on the total momentum $\vec{P}$, and V_{r2} reduces to zero for $P\to 0$.

The operator V_{r1} cannot be determined uniquely from (2.12); however, it satisfies the nonrelativistic condition (2.13). Hence we can always consider it to be included in V_G, which can then be identified with the phenomenological potential

$$V_{\rm Phen}=V_G+V_{r1}\,.$$

This identification is the result of our inability to have a theory which gives the shape and parameters of a two-body potential from first principles. At the present time, one only takes a particular shape with a certain number of parameters to fit the two-body data, which already include some relativistic effects. This justifies our inclusion of V_{r1} in V_G and lumping our ignorance in the phenomenological potential.

Once this identification is accepted, (2.12) gives a unique determination of V_{r2}. Since we are only interested in the corrections of order $(v/c)^2$, we can write V_{r2} as a polynomial of second order in $\vec{P}$:

$$V_{r2}=P_iA_i+P_iP_jB_{ij}\,. \tag{2.13}$$

Substituting (2.13) in (2.12), one can uniquely determine A_i and B_{ij} by using the fact that $V_{\rm Phen}$ and V_{r2} satisfy (2.4b), i.e., they are invariant under rotation in three-dimensional space. This leads to

$$V_{r2}(\vec{P},\vec{p},\vec{p}\,')=\frac{-1}{4m^2}\left\{\vec{P}^2+\tfrac{1}{2}\left[(\vec{P}\cdot\vec{p})\left(\vec{P}\cdot\frac{\partial}{\partial\vec{p}}\right)+(\vec{P}\cdot\vec{p}\,')\left(\vec{P}\cdot\frac{\partial}{\partial\vec{p}\,'}\right)\right]\right\}V_{\rm Phen}(\vec{p},\vec{p}\,')$$
$$+\frac{1}{8m^2}[V_{\rm Phen}(\vec{p},\vec{p}\,')i(\vec{\sigma}^1-\vec{\sigma}^2)\cdot(\vec{P}\times\vec{p}\,')-i(\vec{\sigma}^1-\vec{\sigma}^2)\cdot(\vec{P}\times\vec{p})V_{\rm Phen}(\vec{p},\vec{p}\,')]\,. \tag{2.14}$$

The operator V_{r2} does not satisfy the Galilean invariance and contains all the relativistic correction up to $(v/c)^2$ needed for the *phenomenological* potential.

3. CALCULATIONS OF CORRECTION TO B.E./A FOR NUCLEAR MATTER

From the analysis of the earlier section, we note that the correction term $\Delta V\equiv V_{r2}$ given by (2.14) is completely determined once the phenomenological potential $V\equiv V_{\rm Phen}$ is chosen. The correction term vanishes in the c.m. system of the two-body system ($\vec{P}=0$). However, it does not include all the corrections of order $(v/c)^2$, but those it does include are the only ones required for the phenomenological potential, which has already a part of the $(v/c)^2$ correction effectively taken care of by the very nature of the determination of parameters from the two-body data. This operator ΔV is a nonlocal integral operator. To apply this correction to estimate its effect on B.E./A of nuclear matter we use Tabakin's potential.

$$\langle\vec{p}|V|\vec{p}\,'\rangle=\frac{2\lambda}{\pi}\sum_{\substack{\alpha M\\ LL'}}f^{\alpha}_{LL'}(p|p')\mathcal{Y}^{\alpha}_{ML}(\hat{p})\mathcal{Y}^{\alpha *}_{ML'}(\hat{p}\,')\,, \tag{3.1}$$

and

$$f^{\alpha}_{LL}(p|p')=i^{L-L'}[-g_{\alpha L}(p)g_{\alpha L'}(p')+h_{\alpha L}(p)h_{\alpha L'}(p')]\,,$$

where $\lambda=h^2/m$, m is the nucleon mass, α denotes the quantum numbers JTS for the two-body system. Further, the function $\mathcal{Y}^{\alpha}_{ML}(\hat{p})$ is a normalized eigenstate of the total angular momentum $\vec{J}$ with z component M, and is

$$\mathcal{Y}^{\alpha}_{ML}(\hat{p}) \equiv \mathcal{Y}^{M}_{JLS}(\hat{p})P_T = \sum_{m_L} C(LSJ; m_L, M-m_L)Y_{Lm_L}(\hat{p})X_{Sm-m_L}P_T .$$

The function X_{SM-m_L} is a total spin state, and P_T is a projection operator for total isospin T and z component T_3 for the two-body system.

The total correction to the phenomenological Hamiltonian for the two-body system, up to second order in (v/c), is

$$\Delta H = \Delta T + \Delta V , \tag{3.2}$$

where the correction to the kinetic energy is

$$\Delta T = (-1/8m^3)(\vec{p}_1^{\,4} + \vec{p}_2^{\,4})$$

and to the potential energy in momentum space, using (2.14) and (3.1), is

$$\begin{aligned}\langle \vec{P}, \vec{p} | \Delta V | \vec{P}, \vec{p}' \rangle = \frac{-1}{4m^2}\Big\{ \Big[\vec{P}^2 + \tfrac{1}{2}(\vec{P}\cdot\vec{p})\Big(\vec{P}\cdot\frac{\partial}{\partial\vec{p}}\Big) \\ + \tfrac{1}{2}(\vec{P}\cdot\vec{p}')\Big(\vec{P}\cdot\frac{\partial}{\partial\vec{p}'}\Big)\Big] \langle \vec{p} | V | \vec{p}' \rangle \\ + \tfrac{1}{2}[\langle \vec{p} | V | \vec{p}' \rangle i(\vec{\sigma}^1 - \vec{\sigma}^2)\cdot(\vec{P}\times\vec{p}') \\ - i(\vec{\sigma}^1 - \vec{\sigma}^2)\cdot(\vec{P}\times\vec{p})\langle \vec{p} | V | \vec{p}' \rangle]\Big\} .\end{aligned} \tag{3.3}$$

Without being too unreasonable, we expect this correction to be small enough, and treat it perturbatively in the first-order Hartree-Fock approximation. With this assumption, the contribution of (3.2) to B.E./A of nuclear matter is given by

$$\frac{\Delta E}{A} = -\frac{3}{56}k_F^{\,4} + \tfrac{1}{2}A^{-1}\sum_{\mu\nu<F}\langle \mu\nu | \Delta V | \mu\nu - \nu\mu \rangle . \tag{3.4}$$

The symbols μ, ν denote the single-particle states below the Fermi sea with Fermi momentum k_F. The states are characterized by the product of plane wave, spin, and isospin states for infinite matter. The number of nucleons A in the normalizing volume Ω is given by $(2k_F^{\,3}/3\pi^2)\Omega$. The first term in (3.4) arises from the contribution of the kinetic energy correction ΔT in (3.2), while the second is due to ΔV. In Eq. (3.4), we have used $\hbar = m = 1$. Therefore, $(\hbar^4/m^3c^2) = 1.835356$ MeV fm^4 is the required factor which should multiply (3.4) to get the energy per particle in MeV instead of the usual factor 41.497.

The second term on the right-hand side (R.H.S.) in the expression (3.4) can be rewritten.

$$\frac{3}{2^5\pi k_F^{\,3}}\sum_{(\mu\nu)}\langle X_\mu X_\nu | \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3p\, d^3P \times \langle \vec{P}, \vec{p} | \Delta V(1-\rho^p\rho^\sigma\rho^\tau) | \vec{P}, \vec{p} \rangle | X_\mu X_\nu \rangle , \tag{3.5}$$

where the summation over $(\mu\nu)$ implies summation over spin and isospin states only, and ρ^p, ρ^σ, ρ^τ are the exchange operators in ordinary, spin, and isospin space, respectively. The reduction (3.5), is achieved by making use of the fact $\delta(\vec{P}-\vec{P}') \to \Omega/(2\pi)^3$ and $\sum_\mu \to \Omega/(2\pi)^3 \int d^3p_\mu \sum_{(\mu)}$. At this stage we first perform the $\vec{P}$ integration in (3.5), the details of which are given in the Appendix. These integrals are

$$\begin{aligned} I_1 &= \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P\, \vec{P}^2 = 2^3\pi k_F^{\,3}\langle \vec{P}^2\rangle_{\rm av} , \\ I_2 &= \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P (\vec{P}\cdot\vec{p})\Big(\vec{P}\cdot\frac{\partial}{\partial\vec{p}}\Big) \\ &= 2^3\pi k_F^{\,3}\langle(\vec{P}\cdot\hat{p})^2\rangle_{\rm av}\Big(\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\Big) , \\ I_3 &= \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P\, \sigma\cdot(\vec{P}\times\vec{p}) = 0 , \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \langle \vec{P}^2\rangle_{\rm av} &= \frac{2^4}{5}k_F^{\,2}\Big(1-\frac{p}{k_F}\Big)^3\Big(1+\frac{p}{2k_F}+\frac{p^2}{6k_F^{\,2}}\Big) , \\ \langle(\vec{P}\cdot\hat{p})^2\rangle_{\rm av} &= \frac{2^4}{15}k_F^{\,2}\Big(1-\frac{p}{k_F}\Big)^4\Big(1+\frac{p}{4k_F}\Big) . \end{aligned} \tag{3.7}$$

Thus, the integration over $\vec{P}$ leaves us with the terms which, at the most, involve the derivative with respect to the magnitude of p, and no spin-dependent terms survive. The exchange term, therefore, gives as usual the same contribution to the summation in (3.5) as the direct term. Following a similar procedure to that in the work of Tabakin[7] we finally get

$$\frac{\Delta E}{A} = \frac{-3}{56}k_F^{\,4} + \sum_{JSTL}(2J+1)(2T+1)I_{JSTL} , \tag{3.8}$$

$$I_{JSTL} = \frac{-3}{4\pi}\int_0^{k_F} dp\, p^2 [\langle \vec{P}^2\rangle_{\rm av} f^{JTS}_{LL}(p|p) + p\langle(\vec{P}\cdot\vec{p})^2\rangle_{\rm av}\frac{d}{dp}f^{JTS}_{LL}(p|p')|_{p'=p}] .$$

4. RESULTS AND CONCLUSIONS

To calculate $\Delta E/A$ as given in (3.8) we have taken only two-body forces in S, P, D, waves only. The correction to B.E./A for the saturation value of the Fermi momentum $k_F = 1.36$ fm^{-1} is found to be +0.50 MeV. This seems to decrease B.E./A of nuclear matter in the first-order Hartree-Fock approximation. Further, the saturation value of B.E./A to first order becomes −7.1 MeV instead

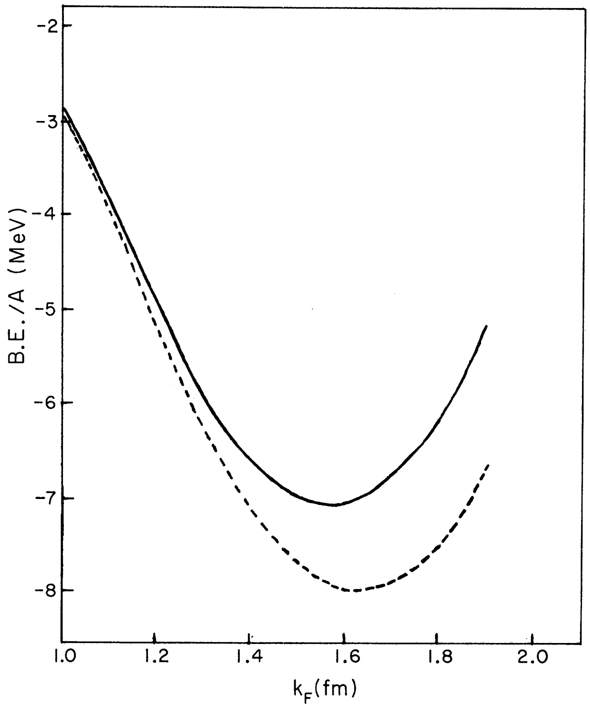

FIG. 1. B.E./A (in MeV) versus k_F (fm). The continuous curve represents present results, and the dashed curve represents Tabakin's first-order results.

of −8 MeV as quoted in Ref. 4. The continuous curve in the figure shows the new results for B.E/A versus the Fermi momentum, and reflects the importance of the correction for higher values of the Fermi momentum. Most of the correction from the potential term arises from $\langle \vec{P}^2 \rangle_{av}$, which accounts for 95% of the total correction. The sign of the numerical result would not have been possible to guess *a priori*, as the contributions from ΔT and the second term in (3.3) are opposite in sign to that of $\vec{P}^2$. The correction considered here is completely different from the one calculated by Brown, Jackson, and Kuo.[8] As a matter of fact, one should perform a self-consistent calculation using Brueckner-Bethe-Goldstone[9] theory. But these calculations will involve enormous amounts of algebra, since in the second-order calculation, there will arise a tremendous amount of coupling between different partial waves because of the presence of σ and derivative terms in (3.3). But a crude calculation, taking average values of the total momentum and assuming that the $\langle \vec{P}^2 \rangle_{av}$ term is dominant, gives

$$\Delta E/A \approx -0.336 - \tfrac{1}{4}\langle \vec{P}^2 \rangle_{av} \times \tfrac{1.835}{41.5} \times (-36) \approx +0.52 \text{ MeV}.$$

Finally, we cannot guess the sign of the application of this correction to other potential models involving a hard core.

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APPENDIX

To evaluate the integrals

$$I_1 = \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P\,\vec{P}^2, \quad I_2 = \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P(\vec{P}\cdot\vec{p})\left(\vec{P}\cdot\frac{\partial}{\partial\vec{p}}\right), \quad I_3 = \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P\,\vec{\sigma}\cdot(\vec{P}\times\vec{p}), \tag{A.1}$$

we shall take $\vec{p}$ along the z axis and make use of the property of the step function.

$$\theta(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\alpha\tau}}{\tau - i\epsilon}d\tau = \begin{cases} 1 \text{ for } \alpha > 0 \\ 0 \text{ for } \alpha < 0 \end{cases}. \tag{A.2}$$

This allows us to write

$$\int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P \to \int_0^\infty dP\,P^2 \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\frac{d\omega_1}{(\omega_1 - i\epsilon)}\int_{-\infty}^{\infty}\frac{d\omega_2}{\omega_2 - i\epsilon}\int_0^\pi d(\cos\theta_{Pp})\int_0^{2\pi}d\phi_P$$

$$\times \exp\{i\omega_1[k_F{}^2 - (\vec{p}+\tfrac{1}{2}\vec{P})^2] + i\omega_2[k_F{}^2 - (\vec{p}-\tfrac{1}{2}\vec{P})^2]\}, \tag{A.3}$$

where θ_{Pp} is the angle between $\vec{P}$ and $\vec{p}$. Thus, we have

$$I_1 = \int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\le k_F} d^3P\,P^2 = \int_{\substack{\text{over}\\\text{whole}\\\text{space}}} d^3P\,P^2 \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}\exp\{i\omega_1[k_F{}^2 - (\vec{p}+\tfrac{1}{2}\vec{P})^2] + i\omega_2[k_F{}^2 - (\vec{p}-\tfrac{1}{2}\vec{P})^2]\},$$

$$= 2\pi\int_0^\infty dP\frac{P^4}{(2\pi i)^2}\int_{-\infty}^{\infty}\frac{d\omega_1 d\omega_2}{\omega_1\omega_2}\int_{-1}^{1}dy\,\exp[i(\omega_2-\omega_1)Ppy]\exp\{i(\omega_1+\omega_2)[k_F{}^2 - \tfrac{1}{4}P^2 - p^2]\}$$

$$=\frac{2\pi}{p}\int_0^\infty dP\,P^3\Big(\{[k_F{}^2-(p-\tfrac{1}{2}P)^2]-[k_F{}^2-(p+\tfrac{1}{2}P)^2]\}\theta\big(k_F{}^2-(p-\tfrac{1}{2}P)^2\big)\theta\big(k_F{}^2-(p+\tfrac{1}{2}P)^2\big)$$

$$+2[k_F{}^2-(p^2+\tfrac{1}{4}P^2)]\theta(k_F{}^2-p^2-\tfrac{1}{4}P^2)[\theta(k_F{}^2-(p-\tfrac{1}{2}P)^2)-\theta(k_F{}^2-(p+\tfrac{1}{2}P)^2)\Big). \tag{A.4}$$

Thus if we define $\alpha=k_F{}^2-(p-\frac{1}{2}P)^2$ and $\beta=k_F{}^2-(p+\frac{1}{2}P)^2$, we have

$$I_1=4\pi\int_0^\infty dP\,P^4\left\{\theta(\alpha)\theta(\beta)+\frac{(k_F{}^2-p^2-\frac{1}{4}P^2)}{Pp}\theta\left(\frac{(\alpha+\beta)}{2}\right)[\theta(\alpha)-\theta(\beta)]\right\}. \tag{A.5}$$

Now we shall make use of the property of the θ function which yields

$$\theta(\alpha)=\theta\big((k_F-p+\tfrac{1}{2}P)(k_F+p-\tfrac{1}{2}P)\big)=1$$

when both $(k_F-p+\frac{1}{2}P)$ and $(k_F+p-\frac{1}{2}P)$ are positive. This is true for $k_F+p\geqslant\frac{1}{2}P$ because the other factor $(k_F-p+\frac{1}{2}P)$ is always positive as $k_F>p$. Similarly, we have

$$\theta(\beta)=\theta\big((k_F-p-\tfrac{1}{2}P)(k_F+p+\tfrac{1}{2}P)\big)=1 \quad \text{for } k_F-p\geqslant\tfrac{1}{2}P,$$

as the other factor is always positive. This leads to

$$\int_0^\infty dP\,\theta(\alpha)\theta(\beta)\to\int_0^{2(k_F-p)}dP$$

because the arguments of both θ functions are positive in this domain. Further

$$\theta(k_F{}^2-p^2-\tfrac{1}{4}P^2)=\theta\big([(k_F{}^2-p^2)^{1/2}-\tfrac{1}{2}P][(k_F{}^2-p^2)^{1/2}+\tfrac{1}{2}P]\big)=1 \quad \text{for } (k_F{}^2-p^2)^{1/2}\geqslant\tfrac{1}{2}P,$$

as $k_F>p$ makes $(k_F{}^2-p^2)^{1/2}+\frac{1}{2}P$ always positive. Therefore the remaining two integrals are

$$\int_0^\infty dP\,\theta\left(\frac{\alpha+\beta}{2}\right)\theta(\alpha)\to\int_0^{2(k_F{}^2-p^2)^{1/2}}dP,$$

as α is always positive in this domain; and

$$\int_0^\infty dP\,\theta\left(\frac{\alpha+\beta}{2}\right)\theta(\beta)\to\int_0^{2(k_F-p)}dP,$$

since the first θ function requires $(k_F{}^2-p^2)^{1/2}\geqslant\frac{1}{2}P$, whereas the second requires $k_F-p\geqslant\frac{1}{2}P$ and $(k_F{}^2-p^2)^{1/2}>k_F-p$ because $k_F>p$. Finally we have

$$I_1=4\pi\left[\int_0^{2(k_F-p)}dP\,P^4+\int_0^{2(k_F{}^2-p^2)^{1/2}}dP\,P^4\left(\frac{k_F{}^2-p^2-\frac{1}{4}P^2}{Pp}\right)-\int_0^{2(k_F-p)}dP\,P^4\left(\frac{k_F{}^2-p^2-\frac{1}{4}P^2}{Pp}\right)\right]$$

$$=4\pi\left[\int_0^{2(k_F-p)}dP\,P^4+\int_{2(k_F-p)}^{2(k_F{}^2-p^2)^{1/2}}dP\,P^4\left(\frac{k_F{}^2-p^2-\frac{1}{4}P^2}{Pp}\right)\right]=\frac{2^7}{5}\pi k_F{}^5\left(1-\frac{p}{k_F}\right)^3\left(1+\frac{1}{2}\frac{p}{k_F}+\frac{1}{6}\frac{p^2}{k_F{}^2}\right). \tag{A.6}$$

The expression (A.6) can be recognized as the one often used for an angle average approximation in the nuclear matter calculations. However, the technique employed permits easy extension to the cases which involve complicated dependence on the magnitude as well as direction of the c.m. momentum. For example, the I_2 integral in (A.1) can be evaluated by writing

$$I_2=\int d^3P\,(\vec{P}\cdot\vec{p})\left(\vec{P}\cdot\frac{\partial}{\partial\vec{p}}\right)=\left[\int_{|\vec{p}\pm\frac{1}{2}\vec{P}|\leq k_F}d^3P(\vec{P}\cdot\vec{p})^2\right]\left(\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\right)\Big/p^2. \tag{A.7}$$

This reduction was possible because we have taken $\vec{p}$ along the z axis and φ_p integration yields the result. Thus

$$I_2=\left\{2\pi\int_0^\infty dP\,P^2\frac{1}{(2\pi i)^2}\int\frac{d\omega_1d\omega_2}{\omega_1\omega_2}\exp[i(\omega_1+\omega_2)(k_F{}^2-p^2-\tfrac{1}{4}P^2)]\int_{-1}^1dy\,Ppy\,\exp[i(\omega_2-\omega_1)Ppy]\right\}\left(\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\right)\Big/p^2.$$

Performing the integration over y, and then over ω_1, ω_2 we get

$$I_2=4\pi\int_0^\infty dP\,P^4\Big\{\theta(\alpha)\theta(\beta)\left[\frac{\beta-\alpha}{2Pp}-\frac{1}{2}\frac{\alpha^2+\beta^2}{(Pp)^3}-\frac{1}{3}\frac{\alpha^3-\beta^3}{(Pp)^3}\right]$$
$$+\theta(\alpha+\beta)[\theta(\beta)-\theta(\alpha)]\left[\frac{\alpha+\beta}{2Pp}+\frac{1}{2}\frac{\alpha^2-\beta^2}{(Pp)^2}-\frac{1}{3}\frac{(\alpha+\beta)(\alpha^2+\beta^2+\frac{1}{2}\alpha\beta)}{(Pp)^3}\right]\Big\}\left(\vec{p}\circ\frac{\partial}{\partial\vec{p}}\right)$$
$$=\frac{2^7}{15}\pi k_F{}^5\left(1-\frac{p}{k_F}\right)^4\left(1+\frac{1}{4}\frac{p}{k_F}\right)\left(\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\right). \tag{A.8}$$

Similarly, one now sees that the third integral (A.1) is zero.

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Electron Scattering from Light, Deformed, Oriented Nuclei

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Electron scattering experiments from oriented nuclei have so far been carried out for one heavy nucleus only, that of holmium. Scattering from light nuclei has the advantages that the elastic scattering is unencumbered by unresolved low-level excitations and that the small distortion of the electron wave function renders orientation effects relatively quite pronounced despite small quadrupole moments, thus permitting the use of the Born approximation. Assuming 100% alignment, we calculate large orientation effects for elastic and also for inelastic scattering from oriented ^{10}B, as well as for 180° magnetic elastic scattering; for more realistic alignments we predict easily measurable effects.

I. INTRODUCTION

The first experiments on elastic electron scattering from intrinsically deformed nuclei[1] (Hf, Ta, W, Th, and U) indicated a departure from the usual diffraction pattern. This departure was a filling in of the minima and was ascribed to the form factor of the quadrupole part of the charge distribution. A theoretical calculation[2] for ^{181}Ta confirmed this conjecture and, by fitting the experimental data, led to a value for the intrinsic quadrupole moment $Q_0 \approx 10\times10^{-24}$ cm^2, in rough agreement with spectroscopic and Coulomb excitation values. Characteristic features of this calculation were as follows: (a) Since the Born approximation was deemed too inaccurate for such heavy nuclei, a high-energy distorted-wave Born approximation had to be used; (b) these deformed heavy nuclei